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WHAT TO EMBED INTO A CARTESIAN CLOSED TOPOLOGICAL CATEGORY

(Preliminary communication)

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Abstract: Herrlich and Nel [4] ask whether every topological category is a finitely productive subcategory of a cartesian closed one. We answer this in the negative and we characterize all such subcategories by a "smallness" condition.

Key words: Initially complete, cartesian closed, topological.

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I. Characterization. All categories are here considered to be concrete with finite concrete products; all subcategories to be full, finitely productive and concrete. The underlying set of an object A is denoted by $|A|$; hom-sets in $\mathcal{K}$ are denoted by $\mathcal{K}(A,B)$ $(\subset |B|^{|A|})$.

Categories, used by topologists, have a lot of common properties. Several authors have introduced axioms for these categories; the first was Hušek in [5]. We shall use Herrlich's notion of topological category [3]: this is a category which has

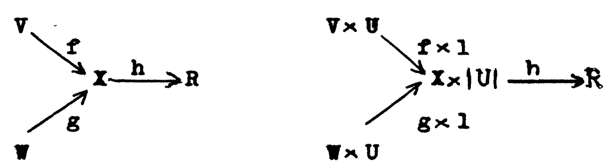
(i) projective generation [given objects A_i and maps $f_i: X \longrightarrow |A_i|$, $i \in I$, there exists an object A on X such that for each map h: $|B| \longrightarrow X$ we have: $h \in \mathcal{K}(B,A)$ iff $f_i.h \in \mathcal{K}(B,A_i)$ for each i] ;

(ii) small fibres [for every set X all objects A with $|A| = X$ form a small set] ;

(iii) constants [each $\mathcal{K}(A,B)$ contains all constant maps from $|A|$ to $|B|$].

While Antoine proves in [2] that every concrete category is a subcategory of a cartesian closed one, we are interested in subcategories of CCT (cartesian closed topological) categories. We find a necessary and sufficient condition for the existence of a CCT supercategory. By an important result of Herrlich and Nel [4] this is equivalent to the existence of a canonical (minimal) CCT supercategory, called CCT hull.

Let a category $\mathcal{K}$ be given. A <u>structured map</u> into a set X is a pair (f,V) consisting of an object V and a map f: $|V| \longrightarrow X$. Two such pairs (f,V) and (g,W) are <u>equivalent</u>



if for every map $h: X \longrightarrow |R|$, R an object, we have:

$$h.f \in \mathcal{K}(V,R) \text{ iff } h.g \in \mathcal{K}(W,R).$$

They are <u>productively equivalent</u> if for each object U the structured maps $(f\times 1, V\times U)$ and $(g\times 1, W\times U)$ are equivalent. Then we write $(f,V) \approx (g,W)$.

<u>Definition</u>. A category is <u>strictly small-fibred</u> if

for every set X there exists, up to $\approx$, only a set of structured maps onto X.

<u>Example</u>. Every small-fibred category with quotients which are finitely productive is strictly small-fibred.

<u>Theorem</u>. A topological category is isomorphic to a subcategory of a cartesian closed topological category iff it is strictly small-fibred.

<u>Counterexample</u>. The following category is topological but not strictly small-fibred.
<u>Objects</u>: pairs (X,H) where X is a set and H is a set of pairs (M.m) consisting of a subset $M \subset X$ and a power $m \leq \leq$ card M, subject to the condition:

$(\emptyset,0) \in H$ and $(\{x\},0)$, $(\{x\},1) \in H$ for each $x \in X$.
<u>Morphisms</u> from (X,H) to (Y,K): maps $f: X \longrightarrow Y$ such that $(M,m) \in H$ implies $(f(M),n) \in K$ where $n = \min(m, \text{card } f(M))$.

The proof of necessity in the above theorem is easy. Sufficiency is proved by the following construction.

II. <u>Construction</u>. Given a category $\mathcal{K}$ we define a new category $\mathcal{K}^*$.
<u>Objects</u> are pairs (X,A) where X is a set and A is a class of structured maps into X which is a union of a set (!) of equivalence classes of the productive equivalence $\approx$ (i.e., $A = \bar{A}_0$ for a set $A_0 \subset A$, where barr denotes the closure with respect to $\approx$).
<u>Morphisms</u> are defined inductively, forming a class $\bigcup \mathcal{K}_i^*$ (the union ranging over all ordinals i).

$\mathcal{K}^*_0$ consists of maps of the form $f.h:U \longrightarrow (Y,B)$, $U \in \mathcal{K}$, where $h \in \mathcal{K}(U,V)$ and $(f,V) \in B$.

$$\begin{array}{ccc} U & \longrightarrow & (Y,B) \\ & {\scriptstyle h} \searrow & \uparrow {\scriptstyle f} \\ & & V \end{array} \qquad \begin{array}{ccc} (X,\bar{A}_0) & \xrightarrow{\ p\ } & (Y,B) \\ {\scriptstyle f} \uparrow & & \\ V & & \end{array}$$

$\mathcal{K}^*_{i+1}$ is the least class, closed to composition, which contains maps $p:(X,\bar{A}_0) \longrightarrow (Y,B)$ such that $p.f:V \longrightarrow (Y,B)$ is in $\mathcal{K}^*_i$ for each $(f,V) \in A_0$.

$\mathcal{K}^*_\gamma = \bigcup_{i<\gamma} \mathcal{K}^*_i$ for each limit ordinal γ.

The category $\mathcal{K}^*$ has the following properties (of which only the first requires a somewhat technical proof).

1. $\mathcal{K}^*$ has finite products: $(X,A) \times (Y,B) = (X \times Y, \overline{A \dot{\times} B})$ where $A \dot{\times} B = \{(f \times g, V \times W);\ (f,V) \in A \text{ and } (g,W) \in B\}$.

2. $\mathcal{K}$ is a dense subcategory of $\mathcal{K}^*$ (full, finitely productive), closed to projective generation.

3. $\mathcal{K}^*$ is cocomplete and for each object (X,A) the endofunctor

$$(Y,B) \longmapsto (Y,B) \times (X,A)$$

preserves colimits.

4. If $\mathcal{K}$ is strictly small-fibred then $\mathcal{K}^*$ is cartesian closed and small-fibred and has projective generation.

5. If $\mathcal{K}$ is topological then $\mathcal{K}^*$ is CCT.

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