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THE SHAPE THEORY FOR UNIFORM SPACES AND THE SHAPE UNIFORM INVARIANTS

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Abstract: The purpose of this paper is to apply the notions of shape theory, shape category and shape invariant to the uniform spaces. As a result we obtain the new uniform shape categories and their homological, cohomological, homotopic and cohomotopic invariants (groups).

Key words: Homology, homotopy, pro-category, shape, uniformity.

AMS: 54E15, 54C55

At present there is a very intensive developed shape theory (with many different shape categories) for the topological spaces so the wish to have some good shape theory for the uniform spaces is quite natural. Anh Kiĕt [1] constructed a shape category for complete metric spaces. D. Doitchinov [2] obtained an other shape category for arbitrary metric spaces. We construct some shape category for arbitrary uniform spaces and receive for it some spectral homotopic, homological, cohomological and cohomotopic invariants (groups). Our method is very close to the general categorical method given by A. Deleanu and P. Hilton [3], which we got to know at Topological symposium, Prague 1976.

Let UH be the uniform homotopy category (all uniform spaces with uniform homotopy classes $[f]$ of all uniform mappings) and QH be the subcategory of UH, which consists of all uniform spaces with the uniform homotopy classes $[h]$ of all uniform <u>embeddings</u> h. We shall construct some <u>uniform shape theory</u> as the functor $F:\mathrm{UH} \longrightarrow \mathrm{pro\ QH} = \mathrm{pr\ QH}/\approx$ in the following manner.

Here the category pr QH consists of all inverse spectra $\underline{X} = \{X_\alpha, [p_{\alpha\alpha'}], A\}$, where A is an arbitrary directed set, and of all "promappings" $\underline{f} = \{[f_\beta], \varphi\} : \underline{X} \longrightarrow \underline{Y} = \{Y_\beta, [q_{\beta\beta'}], B\}$, where $\varphi : B \longrightarrow A$ and $f_\beta : X_{\varphi\beta} \longrightarrow Y_\beta$. The category pro QH is the category pr QH factorized by the following equivalence $\approx$ for "promappings":

$\underline{f} \approx \underline{g} = \{[g_\beta], \psi\} : \underline{X} \longrightarrow \underline{Y}$ iff the diagram is commutative for every $\beta \in B$ and some $\alpha \in A$ (see Grothendieck [4]).

$$\begin{tikzcd} & X_{\varphi\beta} \arrow[dr, "{[f.]}"] & \\ X_\alpha \arrow[ur, "{[p..]}"] \arrow[dr, "{[p..]}"'] & & Y_\beta \\ & X_{\psi\beta} \arrow[ur, "{[g.]}"'] & \end{tikzcd}$$

To construct the functor $F:\mathrm{UH} \longrightarrow \mathrm{pro\ QH}$ we fix some uniform embedding $X \subset M(X)$ for every object $X \in \mathrm{UH}$, where $M(X) \in \mathrm{ANRU}$ (absolutely neighborhood retracts for uniform spaces). It is possible because of one theorem given by Isbell [5] for separated uniform spaces, which as can easily be shown is true also for arbitrary uniform spaces. Therefore we fix the inverse spectrum $\underline{X}$ of all uniform neighborhoods U of X in M(X) and classes $[i_{UU'}]$ of natural embeddings $i_{UU'} : U' \hookrightarrow U$ for every object $X \in \mathrm{UH}$. Here the directed set A is the set of all these neighborhoods U with canonical order. So $\underline{X} = \{U, [i_{UU'}], A\}$. We put $F(X) = \underline{X}$ for every $X \in UH$.

Let $f: X \longrightarrow Y$ be some uniform mapping. Then for every uniform neighborhood V of Y in $N = M(Y)$ there exists some uniform neighborhood $U = \varphi(V)$ of X in $M = M(X)$ and some uniform extension $f_V: U \longrightarrow V$ of f. The received system $\underline{f} = \{[f_V], \varphi\}$ is some "promapping" $\underline{f}: \underline{X} \longrightarrow \underline{Y}$ from $\underline{X}$ to $\underline{Y} = \{V, [i_{VV'}], B\}$, where $B = \{V\}$. We fix for every morphism $[f] \in UH$ some received in this manner morphism $\underline{f} \in$ pr QH. We put $F([f]) = \{\underline{f}\}$, where $\{\underline{f}\}$ is the equivalence class of morphism f concerning the equivalence $\approx$.

It is not very difficult to prove the

Theorem 1. i) pro QH is a category,

ii) the correspondence $F: UH \longrightarrow$ pro QH is a (covariant) functor,

iii) any functor F' obtained by this construction with other selection of embeddings $X \subset M(X)$ and morphisms $\underline{f}$ is equivalent to the given functor F in the following sense:

*) There exist for every $X \in UH$ some isomorphism $\underline{j}_X: F(X) \rightarrow F'(X)$ such that the diagram is commutative for every mapping $f: X \longrightarrow Y$.

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle j_X}\downarrow & & \downarrow{\scriptstyle j_Y} \\ F'(X) & \xrightarrow{F'(f)} & F'(Y) \end{array}$$

The functor F generates the uniform shape category US in the following manner: the objects of US are the same of UH (i.e. the uniform spaces) and the morphisms $s: X \longrightarrow Y$ of US are the same of pro QH (i.e., the morphisms $\{\underline{f}\}: F(X) \rightarrow$

$\longrightarrow F(Y)$, which are to be understood as morphisms from X to Y naturally). The functor F generates the <u>uniform shape functor</u> $\underline{F}:UH \longrightarrow US$ also: namely $\underline{F}(X) = X$ for every $X \in UH$ and $\underline{F}([f] : X \longrightarrow Y) = F([f]) : X \longrightarrow Y$ for every $[f] \in UH$.

It is not difficult to prove the

<u>Theorem 2</u>. i) The class US is a category and the correspondence $\underline{F}$ is the functor,

ii) the uniform shape functors $\underline{F}$ and $\underline{F}'$ obtained by different selections of embeddings $X \subset M(X)$ and morphisms $\underline{f}$ are equivalent in the following sense:

**) There exists some functorial isomorphism $J:US \longrightarrow US'$, such that the diagram is commutative.

$$\begin{tikzcd} & US \arrow[d, "J"] \\ UH \arrow[ur, "\underline{F}"] \arrow[r, "\underline{F}'"'] & US \end{tikzcd}$$

iii) Every uniform shape functor $\underline{F}$ constructed here, is equivalent in the above sense to the uniform shape functor $F_A:UH \longrightarrow US_A$, where US_A and F_A are the uniform shape category and the corresponding uniform shape functor, which had been constructed by V. Agaronian [6].

It is clear that every functor $F:U \longrightarrow \text{pro } Q$ may be called a <u>general shape theory</u>. This definition is equivalent to the definition given by A. Deleanu and P. Hilton [3].

<u>Lemma A.</u> If two general shape theories $F, F':U \longrightarrow \text{pro } Q$ are equivalent, then the general shape categories S, S', and the general shape functors $\underline{F}:U \longrightarrow S$, $F':U \longrightarrow S'$, generated by F and F′ accordingly are equivalent, too.

Let $I:Q \longrightarrow G$ be some functor ("invariant" for objects of a given category Q) from Q to an arbitrary full category

G. Here under the fullness we understand that some limit functors LIM: pr $G \longrightarrow G$ and COLIM: copr $G \longrightarrow G$ are defined, i.e. every inverse and every direct spectrum has some limit and accordingly colimit. In this case we receive some limit functor LIM if we fix some LIM $\underline{G}$ for every inverse spectrum $\underline{G}$ and construct the morphism Lim $\underline{f}$: LIM $\underline{G} \longrightarrow$ $\longrightarrow$ LIM $\underline{H}$ for every morphism $\underline{f}:\underline{G} \longrightarrow \underline{H}$ of category pr G (this morphism LIM $\underline{f}$ is defined by usual categorical way and is unique for every $\underline{f}$).

Lemma B. The correspondence LIM, obtained in this manner, is functor and two functors LIM and LIM$'$, obtained by different selections of limit objects for all spectra G are equivalent.
This is true also for cofunctor (= contravariant functor) COLIM.

Lemma C. If $\underline{f} \approx \underline{g}$, then LIM $\underline{f}$ = LIM $\underline{g}$. Consequently every limit functor LIM: pr $G \longrightarrow G$ generates the limit functor $\widetilde{\mathrm{LIM}}$: pro $G \longrightarrow G$.
Therefore a given functor $I:Q \longrightarrow G$ generates for the given above general shape theory $F:U \longrightarrow$ pro Q the functor pro I: :pro $Q \longrightarrow$ pro G and the functor $\tilde{I} = \widetilde{\mathrm{LIM}} \circ \text{pro } I \circ F:U \longrightarrow G$. Finally we define some Čech or spectral functor $\check{I}:S \longrightarrow G$, generated by a given functor I for the given general shape theory F in the following manner: we put $\check{I}(X) = \tilde{I}(X)$ for every object X of the shape category S, obtained by the shape theory F, and $\check{I}(s) = \widetilde{\mathrm{LIM}}$ (pro I(s)) for every morphism $s \in S$.

It is not difficult to prove the

Theorem 3. i) The correspondence $\check{I}:S \to G$ is a functor,

ii) the functor I' obtained by other selection of limit functor LIM: pr $G \to G$ and for other but equivalent to F general shape theory F' is equivalent to $\check{I}$, too.

We received analogically for cofunctor $I:Q \to G$ the Čech or spectral cofunctor $\check{I}:S \to G$ (more precisely the class of equivalent spectral cofunctors $\check{I}$). For example if we take the singular homological, cohomological, homotopic and cohomotopic functors H_n, H^n, π_n and π^n for the category QH then we get for the constructed above uniform shape category US the Čech or spectral uniform homological, cohomological, homotopic and cohomotopic functors $\check{H}_n, \check{H}^n, \check{\pi}_n$ and $\check{\pi}^n$ accordingly. Consequently we shall have the Čech or spectral uniform homological, cohomological, homotopic and cohomotopic groups $\check{H}_n(X)$, $\check{H}^n(X)$, $\check{\pi}_n(X)$ and $\check{\pi}^n(X)$ for uniform spaces X. It is not difficult to prove the

Theorem 4. The groups $\check{H}_n(X)$, $\check{H}^n(X)$, $\check{\pi}_n(X)$ and $\check{\pi}^n(X)$ are invariant for isomorphisms of uniform shape category US and consequently for uniform homeomorphisms.

It can be proved by means of these uniform shape invariants that the uniform shape category constructed here is not equivalent to the uniform shape category given by Anh Kiět [1] and to that given by D. Doitchinov [2], too. In general case the groups $\check{H}^n(X)$ are not coinciding with the cohomological uniform groups constructed by V. Kuzminov and I. Švedov [7] as was noted by S. Bogaty.

In this way it is possible to get some generalizations of the known theorems, for example the Hurewicz's and Whitehead's theorems. Details will be published in Izvestija Akad. Nauk Arm. SSR.

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