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REAL AND IMAGINARY CLASSES IN THE ALTERNATIVE SET THEORY

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Abstract: This paper is meant as a contribution to the development of mathematics in alternative set theory. In the first section we shall introduce the concepts of real and imaginary classes. "Philosophical" reasons for this division are described. Some classes based on the axiom of choice and the axiom of cardinalities are proved to be imaginary. In the second section the notion of real equivalence and real subvalence are defined and investigated. The ordering by real subvalence is proved not to be linear.

Key words: Alternative set theory, real class, imaginary class, real equivalence, indiscernibility equivalence.

Classification: 02K10, 02K99

Various types of classes occur in the extended universe studied in alternative set theory. We shall introduce the concepts of real and imaginary classes (every class being of one of these two types). Real classes are those ones that may be seen when observing continuum. Imaginary classes are used mainly for calculations on classes. The first section is devoted to the fundamental properties of real and imaginary classes. It is proved e.g. that Ω and selectors are imaginary classes. The one-one mapping between two infinite sets having very different cardinalities is an imaginary class. On the other hand, set-theoretically definable classes, countable classes and classes definable with the help of real classes, are real.

When studying various finer types of cardinalities, we use equivalences based on one-one mappings of various types. Keeping accordance with this procedure we study so called real equivalence, i.e. the equivalence given by one-one mappings which are real classes, in the second section. It is proved e.g. that there are two real classes incomparable by real subvalence.

Our considerations follow those ones given in P. Vopěnka's book, Mathematics in alternative set theory. We use notions and notation used in this book and [V 1].

The work presented here has arisen in the Prague seminar of alternative set theory on the basis of discussions held between the authors.

§ 1. Basic properties of real and imaginary classes

Every our observation is characterized by an indiscernibility equivalence (see Ch. III [V]). The classes, we observe on the horizon of our observation abilities, are exactly the figures in the mentioned indiscernibility equivalence. These considerations lead to the following definitions.

A class from the extended universe is called real if there is an indiscernibility equivalence $\overset{+}{=}$ such that X is a figure in the equivalence $\overset{+}{=}$. If the class X is not real then X is called imaginary.

It is obvious that every set-theoretically definable class is real.

__Theorem.__ For any sequence $\{X_n; n \in FN\}$ of real classes there is a set u such that for every $n \in FN$ the class X_n is a figure in the equivalence $\overset{\circ}{\underset{\{u\}}{=}}$.

__Proof:__ Let $\{=_n; n \in FN\}$ be a sequence of indiscernibility equivalences such that for every n the class X_n is a figure in the indiscernibility equivalence $=_n$. Using [V 1], we can find a u such that $\overset{\circ}{\underset{\{u\}}{=}}$ is finer than $\bigcap\{=_n; n \in FN\}$.

The following two theorems are immediate consequences.

__Theorem.__ If X, Y are real classes then X-Y is real.

__Theorem.__ If $\{X_n; n \in FN\}$ is a sequence of real classes then $\bigcup\{X_n; n \in FN\}$ and $\bigcap\{X_n; n \in FN\}$ are real.

Especially, every countable class is real. Similarly any σ-class (π-class) is real.

__Theorem.__ Let F be an automorphism. If X is a real class then F"X is real.

__Proof:__ If X is a figure in the indiscernibility equivalence R then F"X is evidently a figure in the indiscernibility equivalence F"R.

__Theorem.__ If X is a figure in the indiscernibility equivalence $\overset{\circ}{\underset{\{u\}}{=}}$ and F is an automorphism such that F(u) = u then F"X=X.

__Proof:__ Obviously it is sufficient to prove the assertion for a monad in $\overset{\circ}{\underset{\{u\}}{=}}$. In this case there is a sequence $\{\varphi_n(x); n \in FN\}$ of set-formulas of the language $FL_{\{u\}}$ such that $X = \bigcap\{\{x; \varphi_n(x)\}; n \in FN\}$. If $x \in X$ then for any n the formula $\varphi_n(x)$ holds. As $F(u) = u$ ($F^{-1}(u) = u$) we have also $\varphi_n(F(x))$ ($\varphi_n(F^{-1}(x))$). Hence $F(x) \in X$ ($F^{-1}(x) \in X$) holds.

Theorem. If $\varphi(x,X_1,\ldots,X_n)$ is a formula of the language FL_V and $Y_1,\ldots,Y_n$ are real classes then the class $\{y;\ \varphi(y,Y_1,\ldots,Y_n)\}$ is real.

Proof: Let us choose u to have the following properties. All the constants occurring in $\varphi(x,X_1,\ldots,X_n)$ denote elements of the class $Def_{\{u\}}$. The classes $Y_1,\ldots,Y_n$ are figures in $\overset{\circ}{=}_{\{u\}}$. Let x be such that $\varphi(x,Y_1,\ldots,Y_n)$. If $y \overset{\circ}{=}_{\{u\}} x$ then there is an automorphism F such that $F(u) = u$ and $F(x) = y$ [V §1 Ch.V]. As $\varphi(x,Y_1,\ldots,Y_n)$ holds, we also have $\varphi(F(x),\ F''Y_1,\ldots,F''Y_n)$. Using the previous theorem we obtain $F''Y_1=Y_1,\ldots,F''Y_n=Y_n$ and thus we have $\varphi(y,Y_1,\ldots,Y_n)$.

Theorem. Let $\varphi(X_0,X_1,\ldots,X_n)$ be a formula of the language FL_V. Let $Y_1,\ldots,Y_n$ be real classes such that $(\exists!Z)\varphi(Z, Y_1,\ldots,Y_n)$. If Y is a class such that $\varphi(Y,Y_1,\ldots,Y_n)$ holds, then the class Y is real.

Proof: Obviously $Y = \{x;(\exists X_0)(\varphi(X_0,Y_1,\ldots,Y_n)\ \&\ x\in X_0)\}$. Now we use the previous theorem.

Remark. If $\{X_n; n\in FN\}$ is a sequence of real classes then $\{X_n\times\{n\}; n\in FN\}$ is also a sequence of real classes and $\bigcup\{X_n\times\{n\}; n\in FN\}$ is a real class. Hence a sequence of real classes can be understood as a real class.

Theorem. The class $\{F''X;\ F$ is an automorphism$\}$ is codable iff X is a real class.

Proof: Let X be real. Let u be a set such that X is a figure in the equivalence $\overset{\circ}{=}_{\{u\}}$. If F, G are automorphisms such that $F(u) = G(u)$ then $F''X=G''X$. (To prove it we note that $G(u)= =(G\circ F^{-1})F(u)$, $G''X=(G\circ F^{-1})''(F''X)$ and we can use the above theorems.) Let us put $X_V=\{y;(\exists F)(F$ is an automorphism &

$\& F(u)=v \& (\exists x \in X)(y=F(x))\}$. Obviously $\{X_v; v \in V\}$ is a codable class and $\{F''X;\ F$ is an automorphism$\}$ is a subclass of the mentioned class.

To prove the converse implication we suppose that X is imaginary. At first we prove that if $Y \neq X$ is an arbitrary uncountable class and if F_0 is an at most countable similarity then there are u, v such that $F_0 \cup \{\langle u,v \rangle\}$ is a similarity and $u \in X \equiv v \notin Y$ holds. Let us choose z such that $\mathrm{dom}(F_0) \subseteq \mathrm{Def}_{\{z\}}$. X is not a figure in the equivalence $\overset{\circ}{\underset{\{z\}}{=}}$ because X is imaginary. Hence there are u, $\bar{u}$ such that $u \in X$, $\bar{u} \notin X$ and $u \overset{\circ}{\underset{\{z\}}{=}} \bar{u}$. Let v be such that $F_0 \cup \{\langle v,u \rangle\}$ is a similarity. $F_0 \cup \{\langle v,\bar{u} \rangle\}$ is a similarity, too. Now $\langle v,u \rangle$ or $\langle v,\bar{u} \rangle$ has the needed property.

Let $\{X_\alpha; \alpha \in \Omega\}$ be an enumeration of the class $\{F''X;\ F$ is an automorphism$\}$ (bijection of Ω onto the class). Let $\{y_\alpha; \alpha \in \Omega\}$ be an enumeration of all sets. By the transfinite recursion we construct a sequence $\{G_\alpha; \alpha \in \Omega\}$ of at most countable similarities having the following properties. For every α we have $y_\alpha \in \mathrm{dom}(G_\alpha)$, $y_\alpha \in \mathrm{rng}(G_\alpha)$, $(\exists y \in \mathrm{dom}(G_\alpha))(y \in X \equiv G_\alpha(y) \notin X_\alpha)$ and $\beta \in \alpha \cap \Omega \Rightarrow G_\beta \subseteq G_\alpha$. Let us put $G = \bigcup\{G_\alpha; \alpha \in \Omega\}$. G is obviously an automorphism and for every α we have $G''X \neq X_\alpha$ - a contradiction.

<u>Remark</u>. Later we shall prove that there are imaginary classes. Hence we shall see that the class of all automorphisms is not codable.

The following theorems serve as criteria for the decision if a class is imaginary. Using these theorems we can prove for some frequently occurring classes that they are imaginary.

<u>Theorem</u>. If X is a real revealed class then X is a π-class.

<u>Proof</u>: Let $\stackrel{+}{=}$ be an indiscernibility equivalence such that X is a figure in $\stackrel{+}{=}$. In § 2 Ch. III [V] it is proved that X is a closed figure in $\stackrel{+}{=}$ and thus a π-class.

The last theorem is equivalent to the following assertion. If a class X is revealed and if it is no π-class then X is imaginary.

<u>Theorem</u>. If a real class $X \subseteq N$ has the property $(\forall \alpha)(\forall \beta)\ (\alpha \in X\ \&\ \beta \doteq \alpha \Rightarrow \beta \in X)$ then X is a σ-class or a π-class.

<u>Proof</u>: We must consider two cases.

a) For any countable class $Y \subseteq X$ there is a $\gamma \in X$ such that $Y \subseteq \gamma$. The class Y is revealed in this case and thus Y is a π-class.

b) There is a class $Y \subseteq X$ which is countable and such that for every $\gamma \in X$ there is a $\beta \in Y$ such that $\gamma \in \beta$. We have $X = \bigcup Y$ in this case and X is a σ-class.

<u>Theorem</u>. If X is a real class such that for any set x the intersection of x and X is a set then X is a set-theoretically definable class.

<u>Proof</u>: We prove at first that the class X is revealed. Let Y be a subclass of X which is at most countable. Let $Y \subseteq u$. Let us put $v = X \cap u$. Obviously we have $Y \subseteq v \subseteq X$. Thus X is a π-class (we use the last but one theorem). Using similar arguments we prove even that V-X is a π-class. Thus X is set-theoretically definable (see § 5 Ch. II [V]).

Theorem. Let X be a real class. Let $\{\varphi_n; n \in FN\}$ be a sequence of set-formulas of the language FL_V. If for every $u \subseteq X$ there is an n such that $\varphi_n(u)$ then there are sequences $\{X_n; n \in FN\}$ of set-theoretically definable classes and $\{k_n; n \in FN\}$ of finite natural numbers having the following properties.

1) $X \subseteq \bigcup\{X_n; n \in FN\}$.

2) For every $u \subseteq X_n$ there is an $m \in k_n$ such that $\varphi_m(u)$.

Proof: Let X be a figure in the equivalence $\{\overset{\circ}{\doteq}_V\}$. Remember that there are at most countably many clopen figures in $\{\overset{\circ}{\doteq}_V\}$. Thus there is an enumeration $\{X_n; n \in FN\}$ of clopen figures in $\{\overset{\circ}{\doteq}_V\}$ such that for every X_n there is a $k_n \in FN$ such that the property 2) holds. It is sufficient to prove that $X \subseteq \bigcup\{X_n; n \in FN\}$. Let $x \in X$, let $\{Y_n; n \in FN\}$ be a sequence of clopen figures in $\{\overset{\circ}{\doteq}_V\}$ such that $Y_{n+1} \subseteq Y_n$ and $Mon(x) = \bigcap\{Y_n; n \in FN\}$ hold. It is sufficient to prove that there is an $n_o \in FN$ such that for every $u \subseteq Y_{n_o}$ there is an $m \in n_o$ such that $\varphi_m(u)$ holds. If it is not this case then there is a sequence $\{u_n; n \in FN\}$ such that $u_n \subseteq Y_n$ and for every $k \in n$ the formula $\neg\varphi_k(u_n)$ holds. If we prolong the sequence $\{u_n; n \in FN\}$, then there is an $\alpha \in N - FN$ such that for every $n \in FN$ we have $u_\alpha \subseteq Y_n$ and $\neg\varphi_n(u_\alpha)$. Thus we have $u_\alpha \subseteq X$ and there is no $n \in FN$ such that $\varphi_n(u)$ – a contradiction.

The following theorem is a consequence of the last theorem.

Theorem. For any uncountable real class X there is an infinite subset of X.

Proof: We prove that if a real class X has only finite subsets then X is at most countable. Let $\varphi_n(u)$ be the formula $u \hat{\approx} n$. Using the previous theorem we obtain sequences $\{X_n; n \in FN\}$ of set-theoretically definable classes and $\{k_n; n \in FN\}$ of finite natural numbers such that $X \subseteq \bigcup \{X_n; n \in FN\}$ and $(\forall u \subseteq X_n)(\exists m \leq k_n)(u \hat{\approx} m)$. In this case $X_n \hat{\precsim} k_n$ holds and thus X is at most countable.

<u>Theorem</u>. The class Ω is imaginary.

<u>Proof</u>: The class Ω is uncountable. If we suppose that Ω is real then using the previous theorem we obtain an infinite subset u of Ω. But $u \subseteq N$ and thus u is not wellordered by the relation $\{\langle \alpha, \beta \rangle; \alpha \in \beta \vee \alpha = \beta\}$ - a contradiction.

<u>Theorem</u>. Let $\doteq$ be a compact equivalence. Let $V/\doteq$ be an uncountable class. If X is a selector for $\doteq$ then X is imaginary.

<u>Proof</u>: The class X is uncountable. If X is real then there is an infinite subset u of X. As $\doteq$ is compact there are $x, y \in u$, $x \neq y$, $x \doteq y$. As $x, y \in X$ the class X is not a selector for $\doteq$ - a contradiction.

<u>Theorem</u>. Endomorphic universe is imaginary iff it is no σ-class.

<u>Proof</u>: Let A be a real endomorphic universe. A is obviously uncountable; thus it must have an infinite subset. In the paper [SV 1] it is proved that A is revealed in this case. Hence A is a σ-class following the first of our criteria.

Fact: The class of all automorphisms is not codable. In fact, we know that there are imaginary classes and thus using the above remark we obtain the assertion.

The following assertion is a special case of the above theorem.

Let X be a real class. Let $\varphi(x)$ be a set formula of the language FL_V. If $(\forall u \subseteq X)\varphi(u)$ holds then there is a sequence $\{X_n; n \in FN\}$ of set-theoretically definable classes such that $X \subseteq \bigcup\{X_n; n \in FN\}$ and for any finite natural number n we have $(\forall u \subseteq X_n)\varphi(u)$. (Put $\varphi_n(u) \equiv \varphi(u)$.) Especially, we obtain the following theorem.

<u>Theorem</u>. For any real function F there is a sequence $\{F_n; n \in FN\}$ of set-theoretically definable functions such that $F \subseteq \bigcup\{F_n; n \in FN\}$.

<u>Theorem</u>. Let α, γ be infinite natural numbers such that for every finite natural number n we have $n\alpha < \gamma$. If F is a one-one mapping of α onto γ then F is imaginary.

<u>Proof</u>: Suppose that F is real. As $F \subseteq \gamma \times \alpha$ there is a sequence of functions $\{f_n; n \in FN\}$ such that for every n we have $f_n \subseteq \gamma \times \alpha$ and $F \subseteq \bigcup\{f_n; n \in FN\}$. For every n we have $f_n''\alpha \precsim \alpha$. Obviously $\gamma = F''\alpha \subseteq \bigcup\{f_n''\alpha; n \in FN\}$. Thus there is an n_0 such that $\gamma = \bigcup\{f_n''\alpha; n \in n_0\}$. (See [V § 4 Ch.I].) But $\bigcup\{f_n''\alpha; n \in n_0\} \precsim n_0\alpha$ - a contradiction.

Especially we have: Every one-one mapping F of α onto α^2 is imaginary.

The proof of the following theorem is analogous. Hence it is left to the reader. We only advise the readed to use the properties of the geometric series in the proof.

<u>Theorem</u>. If α is an infinite natural number then there is no one-one real mapping of α onto $\{\gamma; (\forall n \in FN)$

$(n\gamma < \alpha)\}$.

<u>Theorem</u>. If F is an automorphism and F is not the identity mapping then F is imaginary.

<u>Proof</u>: If for every α the formula $F(\alpha)=\alpha$ holds, then F is the identity, as there is a one-one mapping of N onto V defined by a set-formula of the language FL. Thus there is an α such that $F(\alpha)\neq\alpha$. The automorphisms F and F^{-1} are both imaginary or both real and thus we can suppose $\alpha+1\leq\leq F(\alpha)$. Let us put $\gamma=F(\alpha)$. Obviously $F(\alpha^{\alpha})=\gamma^{\gamma}$ (F is an automorphism). If we suppose that there is an n such that $\gamma^{\gamma}< n\alpha^{\alpha}$ holds then $(\alpha+1)^{\alpha+1}\leq\gamma^{\gamma}< n\alpha^{\alpha}$. Thus we have $(\alpha+1)^{\alpha+1}< n\alpha^{\alpha}$ - a contradiction, because $\alpha\alpha^{\alpha}<<(\alpha+1)^{\alpha+1}$. These considerations prove that we can suppose that we have chosen α such that for every finite natural number n the formula $n\alpha<F(\alpha)=\gamma$ holds. If F is a real class, then $F\restriction\alpha$ is also a real class. But $F\restriction\alpha$ is a one-one mapping of α onto γ - a contradiction with the previous theorem.

§ 2. <u>Real equivalence and real subvalence</u>

The above considerations lead to the following notions.

Real classes X, Y are said to be really equivalent (we use the notation $X\overset{\vee}{\approx}Y$) iff there is a one-one function F such that X=dom(F), Y=rng(F) and F is a real class. Analogously we define $X\overset{\vee}{\precsim}Y$ iff there is a one-one real mapping F of X onto a subclass of Y. $X\overset{\vee}{\prec}Y$ iff $X\overset{\vee}{\precsim}Y$ and $\neg X\overset{\vee}{\approx}Y$.

There are plenty of obvious assertions holding for the real equivalence. We will not formulate such assertions here.

To show our approach we prove here only the Cantor-Bernstein theorem in the following version expressing the essence of the theorem.

Theorem. Let $X_0 \subseteq X_1 \subseteq X_2$ be real classes. If X_0 and X_2 are really equivalent then X_1, X_2 (and thus X_1, X_0) are really equivalent.

Proof: Remember that any class defined by a formula of the language FL_V with real parameters only is real. Cantor-Bernstein theorem is usually proved by the definition of the needed mapping. If we use only real classes in the definition, then the mapping is a real class. Especially, we construct the needed mapping in the following manner. Let F be a real one-one mapping of X_2 onto X_0. By induction we define a sequence $\{Y_n; n \in FN\}$ of real classes. We put $Y_0 = X_2 - X_1$, $Y_{n+1} = F''Y_n$. We put $G(x) = F(x)$ for $x \in \bigcup\{Y_n; n \in FN\}$, $G(x) = x$ for $x \in X_2 - \bigcup\{Y_n; n \in FN\}$. G is obviously a one-one mapping such that $dom(G) = X_2$, and G is a real class.

Theorem. A real class X is really equivalent with a real class $X \times FN$ iff there is a codable class $\mathfrak{M}$ having the following properties:

(1) $\mathfrak{M}$ is countable.

(2) $\bigcup \mathfrak{M} = X$

(3) $(\forall Y_1, Y_2 \in \mathfrak{M})(Y_1, Y_2$ are really equivalent real classes$)$.

(4) $(\forall Y_1, Y_2 \in \mathfrak{M})(Y_1 \neq Y_2 \Rightarrow Y_1 \cap Y_2 = \emptyset)$.

Proof: Let $\mathfrak{M}$ be a class having the mentioned properties. Let $\{X_n; n \in FN\}$ be an enumeration of $\mathfrak{M}$. Let $\{X^n_m; n, m \in FN\}$ be an enumeration of $\mathfrak{M}$ by the members of the class

FN^2. Let G^m_n be one-one function such that $dom(G^n_m)=X^n_m$, $rng\, G^n_m=(X_n\times\{m\})$. If $x\in X$ then there is exactly one X^n_m such that $x\in X^n_m$. In this case we put $G(x)=G^n_m(x)$. G is obviously a one-one function such that $dom(G)=X$, $rng(G)=X\times FN$ and G is a real class.

On the other hand, let G be a one-one real mapping of $X\times FN$ onto X. If we put $X_n=G''(X\times\{n\})$ and $\mathfrak{M}=\{X_n;n\in FN\}$ then $\mathfrak{M}$ has the needed properties.

<u>Theorem</u>. Every infinite set u is really equivalent to the class $u\times FN$.

<u>Proof</u>. It is sufficient to prove the assertion for every infinite natural number α. At first we prove that if β is an infinite natural number then there is a real class X such that $X\overset{\vee}{\lesssim}3\beta$ and $X\overset{\vee}{\approx}FN\times\beta$. To prove this we use the Vitali's idea of the construction of a nonmeasurable set. We put $Y=\{r\in BRN;\ (\exists\gamma\in N)(r=\gamma/\beta\ \vee\ r=-\gamma/\beta)\}$. We define an equivalence relation $\sim$ on Y in the following manner. $x\sim y\ \equiv(\exists r\in FRN)(x-y\doteq r)$ (where $\doteq$ is the usual indiscernibility equivalence $(\forall n\in FN)\ (|x-y|<1/n)$. Let Z be a selector for the equivalence $\sim$ such that $(\forall x\in Z)(0<x<1)$. Now the following properties hold

(a) $(\forall y\in Y)(\exists!r\in FRN)(\exists x\in Z)(y\doteq x+r)$.

(b) $(\forall y\in Y)(\forall r\in FRN)(\exists!x\in Y)(r\leq x-y<r+(1/\beta))\ \&$
$\&(\forall y\in Y)(\forall r\in FRN)(\exists!x\in Y)(r\leq y-x<r+(1/\beta)))$.

For $r\in FRN$ we put $X_r=\{y\in Y;\ (\exists x\in Z)(y\doteq x+r)\}$. The codable class $\{X_r;r\in FRN\}$ has the following properties.

(1) $r\neq s\Rightarrow X_r\cap X_s=\emptyset$.

(2) $Y=\cup\{X_r;r\in FRN\}$.

(3) $(\forall r,s \in FRN)(X_r$ and X_s are really equivalent).

The property (3) can be easily proved using the property (b). Let us put $X = \bigcup\{X_r; r \in FRN \& 0 \leq r \leq 1\}$. Now we have $(\forall y \in X)(-1 < y < 2)$. Thus $X \overset{\vee}{\precsim} 3\beta$. Obviously $Y \overset{\vee}{\approx} FN \times \beta$. Using the previous theorem we obtain $X \overset{\vee}{\approx} Y$.

Now let α be an infinite natural number. Let β be a natural number such that $3\beta \leq \alpha \leq 3\beta + 2$. We have $X \overset{\vee}{\precsim} 3\beta \overset{\vee}{\precsim} \alpha \overset{\vee}{\precsim} FN \times \beta \overset{\vee}{\precsim} \alpha \times FN \overset{\vee}{\precsim} \beta \times 4 \times FN \overset{\vee}{\approx} \beta \times FN \overset{\vee}{\approx} X$. Thus all classes are really equivalent.

The following theorem is an easy consequence.

<u>Theorem</u>. If α is an infinite natural number then the following properties hold.

(a) α and $\{\gamma; (\exists n)(\gamma < n\alpha)\}$ are really equivalent.

(b) If γ is a natural number such that $(\exists n)(\alpha/n < \gamma < n\alpha)$ then γ and α are really equivalent.

Remember that infinite natural numbers α, γ such that $(\forall n \in FN)(\alpha > n\gamma)$ are not really equivalent and $\{\gamma; (\forall n \in FN)(\gamma n < \alpha)\} \overset{\vee}{\prec} \alpha$.

The relation $\overset{\vee}{\precsim}$ is not a total ordering (as it is in the case of $\precsim$ and $\overset{\wedge}{\precsim}$). For any uncountable real class there are two uncountable real subclasses incomparable by $\overset{\vee}{\precsim}$.

As any real uncountable class has an infinite subset, we obtain the mentioned fact as a consequence of the following theorem. The property (3) of the theorem is not used in this paper but it is important for other purposes.

<u>Theorem</u>. If $\vartheta \in N - FN$ then there are real subclasses X, Y of ϑ having the following properties:

(1) $X \cap Y = \emptyset$.

(2) $\neg X \overset{\vee}{\lesssim} Y \;\&\; \neg Y \overset{\vee}{\lesssim} X$.

(3) $(\forall m \subseteq \vartheta)(m \supseteq X \Rightarrow \frac{\mathrm{card}(m)}{\vartheta} \doteq 1)$ and the same property for Y.

Proof: For $\beta_1, \beta_2 \in \vartheta$ let us put $\beta_1 \overset{*}{=} \beta_2 \equiv \frac{\beta_1}{\vartheta} \doteq \frac{\beta_2}{\vartheta}$. It is obvious that $\overset{*}{=}$ is an indiscernibility equivalence on ϑ. The class $Z \subseteq \vartheta$ is said to be a zero class iff for every $n \in FN$ there is a set v such that $v \lesssim \frac{\vartheta}{n} \;\&\; Z \subseteq v$. Any monad in the equivalence $\overset{*}{=}$ is obviously a zero class. If $\{Z_n; n \in FN\}$ is a sequence of zero classes, then $\bigcup\{Z_n; n \in FN\}$ is a zero class. If f is a one-one mapping and if Z is a zero class, then $f''Z \cap \vartheta$ is a zero class. At last, if $m \subseteq \vartheta$ and $\neg \frac{\mathrm{card}(m)}{\vartheta} \doteq 0$, then m is no zero class. Let $\{\{f_n^{\alpha}; n \in FN\}; \alpha \in \Omega\}$ be an enumeration of all sequences of one-one functions and let $\{m_{\alpha}; \alpha \in \Omega\}$ be an enumeration of all subsets of ϑ not being zero classes. We construct sequences $\{X_{\alpha}; \alpha \in \Omega\}$, $\{Y_{\alpha}; \alpha \in \Omega\}$ of zero classes by transfinite recursion in the following manner. We put $X_0 = Y_0 = \emptyset$. We put $\overline{X}_{\alpha} = \bigcup(\{X_{\beta} \cup Y_{\beta}; \beta \in \alpha \cap \Omega\} \cup \{((f_n^{\alpha})^{-1})''Y_{\beta}; n \in FN, \beta \in \alpha \cap \Omega\})$. The class $\overline{X}_{\alpha} \cap \vartheta$ is obviously a zero class. Let x_{α} be the least element of $m_{\alpha} - \overline{X}_{\alpha}$ in a fixed ordering of V of the type Ω. Let us put $X_{\alpha} = \bigcup\{X_{\beta}; \beta \in \alpha \cap \Omega\} \cup \mathrm{mon}(x_{\alpha}) \cup \mathrm{Fig}(\{f_n^{\alpha}(x_{\alpha}); n \in FN\})$. Analogously we put $\overline{Y}_{\alpha} = \overline{X}_{\alpha} \cup X_{\alpha} \cup \bigcup\{((f_n^{\alpha})^{-1})''(\overline{X}_{\alpha} \cup X_{\alpha}); n \in FN\}$, let y_{α} be the least member of the class $m_{\alpha} - \overline{Y}_{\alpha}$ and $Y_{\alpha} = \bigcup\{Y_{\beta}; \beta \in \alpha \cap \Omega\} \cup \mathrm{mon}(y_{\alpha}) \cup \mathrm{Fig}(\{f_n^{\alpha}(y_{\alpha}); n \in FN\})$. It is obvious that for every α we have $X_{\alpha} \cap Y_{\alpha} = \emptyset$; X_{α}, Y_{α} are figures in the equivalence $\overset{*}{=}$ and $\beta \in \alpha \cap \Omega \Rightarrow (X_{\beta} \subseteq X_{\alpha}) \;\&\; (Y_{\beta} \subseteq Y_{\alpha})$. Let us put $X = \bigcup\{X_{\alpha}; \alpha \in \Omega\}$, $Y = \bigcup\{Y_{\alpha}; \alpha \in \Omega\}$. It is evident that $X, Y \subseteq \vartheta$, X, Y are real classes and $X \cap Y = \emptyset$. If F is a one-one real mapping of X into Y then using § 1 we obtain the existence of an α such that $F \subseteq \bigcup \{f_n; n \in FN\}$. But we have $x_\alpha \in X$, $f_n^\alpha(x_\alpha) \in X$ for every $n \in FN$. Hence we have $F(x_\alpha) \in X$ - a contradiction with $F(x_\alpha) \in Y$ and $X \cap Y = \emptyset$. The proof of $\neg Y \overset{\vee}{\precsim} X$ is analogous. If m is a subset of ϑ such that $m \supseteq X \,\&\, \neg \frac{\mathrm{card}(m)}{\vartheta} \doteq 1$ then there is an α such that $m_\alpha = \vartheta - m$. But we have $x_\alpha \in X \cap m_\alpha$, hence $x_\alpha \in m$ - a contradiction with $m_\alpha = \vartheta - m$. The proof of the last property for Y is analogous.

R e f e r e n c e s

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