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AN ELIMINATION OF INFINITELY SMALL QUANTITIES AND INFINITELY LARGE NUMBERS (WITHIN THE FRAMEWORK OF AST)

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Abstract: Let $\varphi(X)$ be a formula describing a property of parts of real numbers using infinitely small quantities or infinitely large numbers (i.e. $\varphi(X)$ is a formula in which only real numbers are quantified with one free variable X for parts of real numbers using the predicate "to be an infinitely large natural number"). In particular we deal with all properties of real functions of n real variables described in infinitesimal calculus of Leibniz type. Infinitely large natural numbers can be quantified on various places in φ. A procedure is given how to find a formula $\psi(X)$ describing the same property in which we use only auxiliary variables for real and natural numbers and we do not use infinitely large and infinitely small quantities.

Note that e.g. for the property $\lim_{x \to a} f(x)=b$ such a procedure is well known from the Cauchy times. By the way some interesting assertions concerning indiscernibility equivalences in the alternative set theory are given.

Key words: Infinitely large natural number, indiscernibility equivalence, endomorphic universe with standard extension, $\mathfrak{M}$-fin(x).

Classification: Primary 03H05
Secondary 03E70, 03H15

Introduction. In the paper we describe a procedure how to eliminate infinitely large natural numbers (thus also infinitely small quantities) from the definitions of analytical notions. As an example can serve the property "f is a function continuous in the point x" - $(\forall \delta_1)(\exists \delta_2)(f(x+\delta_1)=f(x)+\delta_2)$

(δ_1, δ_2 being infinitely small quantities). In the given example the formula in the definition is in the prenex form and in the prefix of the formula only one change of quantifiers occurs. For this case an easier procedure due to P. Vopěnka is described in the paper [Č] and let us note that only auxiliary variables for natural numbers are needed. For the case of two changes of quantifiers, A. Vencovská has found an eliminating procedure using auxiliary variables for natural numbers and one auxiliary variable for functions from natural numbers to natural numbers (real numbers). P. Vopěnka has proved that this sort of auxiliary variables (real numbers) cannot be omitted. If we admit auxiliary variables for parts of real numbers, we obtain a trivial equivalent "in the sense of ultraproduct is valid". On the other hand, in [S] A. Sochor proved that the predicate "to be a standard real number" cannot be eliminated in general nonstandard models. But a generalization of the given procedure eliminates this predicate in $\mathfrak{c}^+$-saturated models ($\mathfrak{c}^+$ being the successor of the cardinal number of continuum).

The eliminating procedure shows how the notions defined using infinitely small quantities can be defined not using these quantities. But it may happen that the obtained formula is so complicated that it is not understandable.

The eliminating procedure is quite general and can be used in various nonstandard models. It can be also adopted for the "generalized infinitely small" (to be a member of a monad of a filter). In the present paper we describe the procedure in the framework of the alternative set theory (AST) as there are available suitable technical means in this theory and the leading ideas of the procedure appears in a striking form. The author intends to write another paper in the language of nonstandard models, where the procedure will be adopted with all the necessary technicalities.

The procedure consists of four steps. In the first one we find to the formula φ a predicate P(t) and a formula ψ not using the predicate "to be infinitely large" such that $\varphi(X) \equiv (\exists t; P(t))\ \psi(t,X)$. We use finite semisets (parts of formally finite sets) to this purpose. In the second one we prove that $\{t; P(t)\}$ is a figure (figure being a nonstandard topological notion). The topology makes it possible to remain in continuum. In the third step we describe the relation to the classical topology. In the fourth step we give standard definitions of parameters defined nonstandardly. The fourth step is not contained in the paper, as we prove here only the existence of a part of continuum which is used in the procedure.

In the paper we use the first three chapters of [V],[V 1], [SV 1].

The work was referred on the Prague seminar on AST and the finished version uses many fruitful remarks and ideas from the members of that seminar. Especially, P. Vopěnka was pertinacious and the author (also on behalf of the readers) expresses here his thanks to him for this.

§ 1. <u>$\mathfrak{M}$-finite sets and some properties of the indiscernibility equivalence</u> $\overset{\circ}{=}_{\{V\}}$. In the first section we do some preliminary considerations. **Remember that the formulas** of the language FL_C are all the formulas with the finite length using sets from C as parameters. V denotes the class of all sets.

1.1. Definition: A system of classes $\mathfrak{M}$ is said to be normally closed iff for every normal formula $\varphi(z,Z_1,\dots,Z_n)$ of the language FL_V and all classes $X_1,\dots,X_n$ from $\mathfrak{M}$ the class $\{x;\ \varphi(x,X_1,\dots,X_n)\}$ is in $\mathfrak{M}$.

1.2. Definition: Let $\mathfrak{M}$ be a normally closed system of classes. A set u is said to be $\mathfrak{M}$-finite ($\mathfrak{M}$-fin(u)) iff for every X from $\mathfrak{M}$, $u \cap X$ is a set.

Examples: 1) Let $\mathfrak{M}$ be the system of all classes. A set u is $\mathfrak{M}$-finite iff u is finite.

2) Let $\mathfrak{M}$ be the system of all set-theoretically definable classes. Every set u is $\mathfrak{M}$-finite.

3) Let A be an endomorphic universe with the standard extension. Let $\mathfrak{M}$ be the system of all the classes definable by normal formulas of the language FL_V from standard extensions of parts of A. A set u is $\mathfrak{M}$-finite iff $u \in Ex(V_{FN})$ in this case.

The assertion in the third example can be proved if we use the fact that Ex(FN) has the "same" properties as FN if only classes being extensions of parts of A and sets (V=Ex(A)) are taken into account (see [SV 1]).

1.3. Definition: Let $\vec{X} = \langle X_1,\dots,X_n \rangle$. We shall use $\vec{X}$-finite ($\vec{X}$-fin(u)) instead of $\mathfrak{M}$-fin(u), where $\mathfrak{M}$ is the system of all the classes definable by normal formulas of the language FL_V with the classes $\vec{X}$ as parameters.

1.4. Theorem: $\mathfrak{M}$-finite sets have the following properties.

1) The smaller $\mathfrak{M}$ is the larger $\mathfrak{M}$-finite set may be.

2) $\mathrm{Fin}(u) \Longrightarrow \mathfrak{M}\text{-fin}(u)$.

3) $\mathfrak{M}$-finite sets form an ideal.

4) For any set function f, $\mathfrak{M}\text{-fin}(u) \Longrightarrow \mathfrak{M}\text{-fin}(f''u)$.

5) $\mathfrak{M}\text{-fin}(u) \Longrightarrow \mathfrak{M}\text{-fin}(\mathcal{P}(u))$.

6) $\mathfrak{M}\text{-fin}(u)\ \&\ \mathfrak{M}\text{-fin}(v) \Longrightarrow \mathfrak{M}\text{-fin}(u \times v)$.

Proof: 1),2) are obvious. To prove 3) let us note that $X \cap (u \cup v) = (X \cap u) \cup (X \cap v)$. 4) is implied by the equality $(f''u) \cap X = f''(u \cap (f^{-1}{}''X))$. 5) It is sufficient to prove 5) only for natural numbers (we use 4)). Let $\mathfrak{M}\text{-fin}(\alpha)$ and let X be from $\mathfrak{M}$. We put $Y = \{\beta \in \alpha + 1;\ (\forall f, f : \mathcal{P}(\alpha) \longleftrightarrow \mathcal{P}(\alpha))\ (\mathcal{P}(\beta) \cap f''(X) \in V)$. It is sufficient to prove $\alpha \in Y$. Y is from $\mathfrak{M}$, $Y \subseteq \alpha + 1$, $\mathfrak{M}\text{-fin}(\alpha + 1)$ by 2),3),4). Thus Y is a set and has a maximal member $\bar{\alpha}$. We prove that the assumption $\bar{\alpha} \in \alpha$ contradicts the maximality of $\bar{\alpha}$. For $f = I \upharpoonright \mathcal{P}(\alpha)$ we prove that $\mathcal{P}(\bar{\alpha}+1) \cap f''X \in V$. For $f \neq I \upharpoonright \mathcal{P}(\alpha)$ the proof is analogous. Let $g : \mathcal{P}(\bar{\alpha}) \longleftrightarrow \mathcal{P}(\bar{\alpha}+1) - \mathcal{P}(\bar{\alpha})$. We put $\bar{f} = g \cup g^{-1} \cup I \upharpoonright (\mathcal{P}(\alpha) - \mathcal{P}(\bar{\alpha}+1))$. We have $\mathcal{P}(\bar{\alpha}+1) \cap X = (\mathcal{P}(\bar{\alpha}) \cap X) \cup \bar{f}''(\mathcal{P}(\bar{\alpha}) \cap \bar{f}''X)$ and both parts of the union are sets by $\bar{\alpha} \in Y$. The property 6) is a consequence of 3) and 5).

Corollary: $\mathfrak{M}$-finite sets are closed on Goedelian operations.

Let us consider the indiscernibility equivalence $\overset{\circ}{=}_{\{v\}}$ (see [V 1]).

1.5. __Theorem__: $x \overset{\circ}{=}_{\{v\}} y \equiv \langle x,v \rangle \overset{\Omega}{=} \langle y,v \rangle$.

Proof: Let $\neg\, x \overset{\circ}{=}_{\{v\}} y$. There is a set-formula $\varphi(z_1, z_2)$ of the language FL such that $\varphi(x,v)\ \&\ \neg\, \varphi(y,v)$. Let $\psi(z)$ be the formula $(\exists z_1,z_2)(z=\langle z_1,z_2\rangle \& \varphi(z_1,z_2))$. $\psi(z)$ is a set-formula of the language FL and we have $\psi(\langle x,v\rangle) \& \neg\psi(\langle y,v\rangle)$. Hence $\neg\langle x,v\rangle \overset{\circ}{=} \langle y,v\rangle$. If we suppose (on the other hand) $\neg\langle x,v\rangle \overset{\circ}{=} \langle y,v\rangle$ then there is a set-formula $\psi(z)$ of the language FL such that $\psi(\langle x,v\rangle) \& \neg\psi(\langle y,v\rangle)$. Let $\varphi(z_1,z_2)$ be the formula $(\exists z)(z=\langle z_1,z_2\rangle \& \psi(z))$. $\varphi(z_1,z_2)$ is a set-formula of the language FL in this case and we have $\varphi(x,v) \&$ $\& \neg\varphi(y,v)$. Hence $\neg x \overset{\circ}{=}_{\{v\}} y$.

1.6. Corollary: If X is a figure in $\overset{\circ}{=}_{\{v\}}$ then $X =$ $= (\mathrm{Fig}_{\overset{\circ}{=}}(X\times\{v\}))''\{v\}$.

1.7. Theorem: 1) $\langle x_1,y_1\rangle \overset{\circ}{=}_{\{v\}} \langle x_2,y_2\rangle \Rightarrow x_1 \overset{\circ}{=}_{\{v\}} x_2 \&$ $\& y_1 \overset{\circ}{=}_{\{v\}} y_2$.

2) $x \overset{\circ}{=}_{\{v\}} y \Rightarrow \mathrm{Fig}_{\overset{\circ}{=}_{\{v\}}}(x) = \mathrm{Fig}_{\overset{\circ}{=}_{\{v\}}}(y)$.

Proof: Let $\{R_n; n\in FN\}$ be a generating sequence of $\overset{\circ}{=}_{\{v\}}$ consisting of $Sd_{\{v\}}$ equivalences (see Ch. III [V]). Let $\{s_n \subset Def(\{v\}); n\in FN\}$ be a sequence of maximal R_n-nets. 1) If $\neg x_1 \overset{\circ}{=}_{\{v\}} x_2$ then there is an $n\in FN$ and a $t\in s_n$ such that $\langle x_1,t\rangle\in R_n \& \langle x_2,t\rangle\notin R_n$. Let $\varphi(z)$ be the formula $(\exists z_1,z_2)$ $(z=\langle z_1,z_2\rangle \& \langle z_1,t\rangle\in R_n)$. The formula φ is equivalent to a set-formula of the language $FL_{\{v\}}$ and $\varphi(\langle x_1,y_1\rangle) \&$ $\& \neg\varphi(\langle x_2,y_2\rangle)$ holds. Hence $\neg\langle x_1,y_1\rangle \overset{\circ}{=}_{\{v\}} \langle x_2,y_2\rangle$. In the case $\neg y_1 \overset{\circ}{=}_{\{v\}} y_2$ we proceed analogously. 2) Let $\mathrm{Fig}(x) \neq$ $\neq \mathrm{Fig}(y)$. Let $w\in x-\mathrm{Fig}(y)$ (in the case $x\subseteq \mathrm{Fig}(y)$ we choose $w\in y-\mathrm{Fig}(x)$ and proceed analogously). We have $\bigcap\{R''_n\{w\}\cap y;$ $n\in FN\}=0$ and hence there is an n such that $R''_n\{w\}\cap y=0$ (see § 4 Ch. I [V]). Let $t\in s_n\cap R''_n\{w\}$. Now we have $R''_n\{t\}\cap x\neq 0 \&$ $\& R''_n\{t\}\cap y=0$. Hence $\neg x \overset{\circ}{=}_{\{v\}} y$ as $t\in Def(\{v\})$.

Remark: The opposite implications do not hold.

1.8. Corollary: If X, Y are figures in $\overset{\circ}{=}_{\{v\}}$ then both $\mathcal{P}(X)$ and $X \times Y$ are figures in $\overset{\circ}{=}_{\{v\}}$.

Proof for $\mathcal{P}(X)$: $x \subseteq X \,\&\, x \overset{\circ}{=}_{\{v\}} y \Rightarrow \mathrm{Fig}(y)=\mathrm{Fig}(x) \subseteq X \Rightarrow$ $\Rightarrow y \subseteq X$.

1.9. Theorem: Let A be an endomorphic universe and let $v \in A$. If μ is a monad in $\overset{\circ}{=}_{\{v\}}$ then $\mu \cap A \neq 0$.

Proof: Let F be an endomorphism such that $F(v)=v$ and $A=F''V$. We have $F''\mu \subseteq \mu$ in this case (see [V 1]).

1.10. Corollary: Let A be an endomorphic universe and let $v \in A$. If $\mathcal{F}$ is a figure in $\overset{\circ}{=}_{\{v\}}$ then $\mathcal{F}=\mathrm{Fig}_{\overset{\circ}{=}_{\{v\}}}(A \cap \mathcal{F})$.

§ 2. The elimination

2.1. Lemma : Let $\vec{X}$-fin(u), $\sigma \subseteq u$ and let $\chi(w,\vec{Z},\vec{z})$ be a normal formula of the language FL. The following equivalence holds

$$(\forall t \in \sigma)\,\chi(t,\vec{X},\vec{x}) \equiv (\exists \bar{t} \in \mathcal{P}(u-\sigma))(\forall t \in u-\bar{t})\,\chi \quad .$$

The equivalence holds also for dual quantifiers.

Proof: Both sides of the equivalence are equivalent to the assertion $u-\bar{t}=\{t;\, \chi(t,\vec{X},\vec{x})\} \supseteq \sigma$. (We can use the notation $\bar{t}$ as $\vec{X}$-fin(u).)

Remark: The lemma shall be used for FN and semisets derived from FN.

2.2. Theorem: Let $\vec{X}$-fin($\vec{u}$), let $\vec{\sigma} \subseteq \vec{u}$ and let $\varphi(\vec{\xi},\vec{w},\vec{Z},\vec{z})$ be a normal formula of the language FL. We can find a normal formula $\psi(t,\vec{w},\vec{Z},\vec{z})$ of the language FL, a set u such that $\vec{X}$-fin(u) and a semiset $\sigma \subseteq u$ such that

$\varphi(\vec{\sigma},\vec{u},\vec{X},\vec{x}) \equiv (\exists t \in \sigma)\, \psi(t,\vec{u},\vec{X},\vec{x})$. The set u is definable from $\vec{u}$ using the operations $\mathcal{P}$, $\times$ and σ is definable from $\vec{u}$, $\vec{\sigma}$ using the operations $\mathcal{P}, \times, -$.

Proof: We use the induction based on the complexity of the formula φ. We shall restrict ourselves on the case $0 \neq \sigma_i \subsetneqq u_i$ as the cases $\sigma_i=0$, $\sigma_i=u_i$ are obvious. 1) We have $x \in \sigma_i \equiv (\exists t \in \sigma_i)(x=t)$. Other cases of atomic formulas are obvious or can be substituted by the formulas using $x \in \sigma_i$ (e.g. $\sigma_i = \sigma_j \equiv (\forall x)(x \in \sigma_i \equiv x \in \sigma_j)$.

2) $(\exists t^1 \in \sigma^1)\, \psi^1(t^1,\vec{u},\vec{X},\vec{x}) \,\&\, (\exists t^2 \in \sigma^2)\, \psi^2(t^2,\vec{u},\vec{X},\vec{x}) \equiv$ $\equiv (\exists \bar{t} \in \sigma^1 \times \sigma^2)(\exists t^1,t^2)(\bar{t} = \langle t^1,t^2 \rangle \,\&\, \psi^1 \,\&\, \psi^2)$. And we put $\bar{\sigma} = \sigma^1 \times \sigma^2$ and $\bar{u} = u^1 \times u^2$.

3) $(\exists x)(\exists t \in \sigma)\, \psi(t,\vec{u},\vec{X},\vec{x},x) \equiv (\exists t \in \sigma)(\exists x)\psi$.

4) Let $\chi(u,\vec{u})$ be the definition of u from $\vec{u}$. $\neg(\exists t \in \sigma)\, \psi(t,\vec{u},\vec{X},\vec{x}) \equiv (\forall t \in \sigma) \neg \psi$. We put $\bar{\sigma} = \mathcal{P}(u-\sigma)$, $\bar{u} = \mathcal{P}(u)$. Using the lemma 2.1 we obtain the equivalent $(\exists \bar{t} \in \bar{\sigma})(\exists u, \chi(u,\vec{u}))(\forall t \in u-\bar{t}) \neg \psi$ which is of the form $(\exists \bar{t} \in \bar{\sigma})\, \bar{\psi}(\bar{t},\vec{u},\vec{X},\vec{x})$.

Remarks: 1) The theorem is the first step of the procedure.

2) If the semisets σ_i occur only in the prefix of the formula φ in the form $(\exists t \in \sigma_i)$, $(\forall t \in \sigma_i)$ then it is possible to modify only the prefix. This modification and the definition of σ, u from $\vec{\sigma}$, $\vec{u}$ is dependent only on the syntax of the prefix.

If σ_i, u_i are figures in $\{\overset{0}{=}\}$ then σ, u also are figures in $\{\overset{0}{=}\}$ and we can find another equivalent in this case.

Up to the end of this section we shall use $\{R_n;\, n \in FN\}$ for a generating sequence of $\overset{0}{=}$ consisting of Sd equivalences.

2.3. Theorem: Let $\psi(t,\vec{Z},\vec{z})$ be a normal formula of the language FL. Let σ, u be figures in $\frac{\circ}{\{v\}}$ and $\sigma \subseteq u$. Let $\vec{X}$-fin(u). If $Fig_{\underline{\circ}}(S) = Fig_{\underline{\circ}}(\sigma \times \{v\})$ then $(\exists t \in \sigma)\psi(t,\vec{X},\vec{x}) \equiv (\exists t_1 \in S)(\forall n \in FN)(\exists t \in (R_n''\{t_1\})''\{v\})\psi$.

Proof: At first we prove the following equivalence $(\exists t \in u)(\forall n \in FN)(t \in (R_n''\{t_1\})''\{v\} \,\&\, \psi) \equiv (\forall n \in FN)(\exists t \in u)(t \in R_n''\{t_1\})''\{v\} \,\&\, \psi)$.

The implication $\Rightarrow$ is obvious. Let us prove $\Leftarrow$. We put $w_n = \{t \in u; t \in (R_n''\{t_1\})''\{v\} \,\&\, \psi\}$. $\{w_n; n \in FN\}$ is a not increasing sequence of non empty sets (we use $\vec{X}$-fin(u)) and thus $\bigcap\{w_n;\ n \in FN\} \neq 0$. This fact proves $\Leftarrow$. The theorem is an easy consequence of the proved equivalence and the formula $t \in \sigma \equiv (\exists t_1 \in S)(\forall n \in FN)(t \in (R_n''\{t_1\})''\{v\})$ being implied from the equality $\sigma = (Fig_{\underline{\circ}}(\sigma \times \{v\}))''\{v\}$ proved in § 1.

Now we use our knowledge for the elimination of infinitely large numbers or (equivalently) for the elimination of FN from inside of the formula φ .

2.4. Definition: Let X_1 be a transitive subclass of N, let $FN \subsetneqq X_1$. The numbers $\alpha \in X_1 - FN$ are called X_1-infinitely large. (We shall omit X_1 - hoping that confusion is not possible.)

2.5. Theorem: Let X_1 be a transitive subclass of N, let $FN \subsetneqq X_1 \,\&\, (\forall \alpha \in X_1)\vec{X}$-fin$(\alpha)$. Let $\varphi(\xi,\vec{Z},\vec{z})$ be a normal formula of the language FL. We can find a normal formula $\psi(w_1,w_2,\vec{Z},\vec{z})$ of the language FL and a figure $\mathcal{S}$ in $\underline{\circ}$ having the following property. For every class S such that $Fig_{\underline{\circ}}(S) = \mathcal{S}$ the formula $\varphi(FN,\vec{X},\vec{x}) \equiv (\exists t_1 \in S)(\forall n \in FN)(\exists \alpha \in X_1,\ n \in \alpha)(\exists t \in (R_n''\{t_1\})''\{\alpha\})\psi(t,\alpha,\vec{X},\vec{x})$ holds.

Remarks: 1) This theorem is the second step of the procedure.

2) If FN occurs only in the prefix of the formula φ then we can modify only the prefix of φ .

Proof: Let us choose an $\alpha \in X_1$-FN. We have FN $\subseteq \alpha$, $\vec{X}$-fin(α) and $Fig_{\underset{\{\alpha\}}{\underline{o}}}(\alpha)$ & $Fig_{\underset{\{\alpha\}}{\underline{o}}}$(FN). Using the previous results we can find a figure φ (in $\underline{o}$) and a normal formula ψ of the language FL having the following property. For every S such that $Fig_{\underline{o}}(S) = \mathcal{G}$ the formula $\varphi(FN, \vec{X}, \vec{x}) \equiv (\exists t_1 \in S)$ $(\forall n \in FN)(\exists t \in (R_n'' \{t_1\})'' \{\dot{\alpha}\}) \psi(t, \alpha, \vec{X}, \vec{x})$ holds. In the formula on the left hand side of the equivalence the symbol α does not occur. Using the law $\chi_1 \equiv \chi_2(\alpha) \vdash \chi_1 \equiv (\exists \alpha) \chi_2(\alpha)$ we obtain the equivalent $(\exists t_1)(\exists \alpha \in X_1 - FN)(\forall n \in FN)(\exists t \in (R_n'' \{t_1\})'' \{\alpha\}) \psi$.
To finish the proof it suffices to prove the implication
$(\forall n \in FN)(\exists \alpha \in X_1,\ n \in \alpha)(\exists t \in (R_n'' \{t_1\})'' \{\alpha\}) \psi \Longrightarrow$
$\Longrightarrow (\exists \alpha \in X_1 - FN)(\forall n \in FN)(\exists t \in (R_n'' \{t_1\})'' \{\alpha\}) \psi$.
Let us note at first that there is no increasing function F such that dom(F)=FN & $X_1 = \bigcup$ rng(F). (The existence of such a function contradicts $(\exists \alpha \in N-FN)\vec{X}$-fin($\alpha$).) Let us suppose that the left hand side holds. Let $\alpha_n \in X_1$ be such that $\alpha_n \ni$ $\ni n \& (\exists t \in (R_n'' \{t_1\})'' \{\alpha_n\}) \psi$. Let $\beta \in X_1$ be larger than every α_n. Let us put $w_n = \{\alpha < \beta;\ (\exists t \in (R_n'' \{t_1\})'' \{\alpha\}) \psi$. $\{w_n;$ $n \in FN\}$ is a not increasing sequence of nonempty sets and hence it has a nonempty intersection.
To prove the right hand side of the implication it suffices to choose α from this intersection.

We give here two examples now. The first one is if we put X_1=N and $\vec{X}$ are set-theoretically definable classes. The second one is much more interesting. Let A be an endomorphic universe with the standard extension. We put $X_1 = Ex(FN)$ and let $\vec{X}$ be standard extensions of parts of A. Let $\mathcal{S}$ be the figure assured by the last theorem. We can put $S = \mathcal{S} \cap A$ as $\mathcal{S} = Fig_{\underline{\Omega}}(\mathcal{S} \cap A)$ (see § 1).

Using the elementary equivalence of A and V and admitting only parameters from A in φ we obtain an equivalent in the language of A. This situation is described in the following theorem.

2.6. <u>Theorem</u>: Let $\varphi(\xi, \vec{Z}, \vec{z})$ be a normal formula of the language FL. We can find a normal formula ψ of the language FL and a figure $\mathcal{S}$ in $\underline{\Omega}$ having the following property. For every endomorphic universe A with the standard extension, for every $\vec{X} \subseteq A$ and for every $\vec{x} \in A$ the following equivalence holds $\varphi(FN, Ex(\vec{X}), \vec{x}) \equiv (\exists t_1 \in \mathcal{S} \cap A)(\forall n \in FN)(\exists m, n \in m)((\exists t \in (R_n'' \{t_1\})'' \{m\}) \psi(t, m, \vec{X}, \vec{x}))^A$.

Proof: By the previous theorem we have $\varphi(FN, Ex(\vec{X}), \vec{x}) \equiv (\exists t_1 \in \mathcal{S} \cap A)(\forall n \in FN)(\exists \alpha \in Ex(FN), n \in \alpha)(\exists t \in (R_n'' \{t_1\})'' \{\alpha\}) \psi(t, \alpha, Ex(\vec{X}), \vec{x})$ (we use the equality $\mathcal{S} = Fig_{\underline{\Omega}}(\mathcal{S} \cap A)$ from § 1). The theorem follows from the equivalence $(\exists \alpha \in Ex(FN), n \in \alpha)(\exists t \in (R_n'' \{t_1\})'' \{\alpha\}) \psi(t, \alpha, Ex(\vec{X}), \vec{x}) \equiv (\exists m \in FN, n \in m)(\exists t \in (R_n'' \{t_1\})'' \{\alpha\} \cap A) \psi^A(t, m, \vec{X}, \vec{x})$ (see [SV 1]).

Remarks: 1) This theorem is the third step of the procedure.

2) The right hand side of the equivalence is a formula in the sense of A. Thus if we consider A as the "real" (standard) world and the standard extension of FN as a means, how

to obtain infinitely large natural numbers, then we have a way, how to go back from infinitely large natural numbers to the "real" world. Remember (on the other hand) that the formula ψ may be very complicated in comparison with the formula φ .

3) We can modify only the prefix if FN occurs only there.

4) The quantification of infinitely small quantities can be replaced by the quantification of infinitely large natural numbers.

5) Real numbers can be viewed as a part of A and hence parts of real numbers can be substituted to $\vec{X}$.

6) If we add before φ a prefix with quantification restricted to A we obtain an equivalent in the language of A. On the other hand, A. Sochor has found a formula of the type $(\forall \alpha \in \mathrm{Ex(FN)})(\exists x \in \mathrm{Real})\varphi$ not having an equivalent in the language of A (see [S]).

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