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CARDINALITIES AND RANKS OF π- BASES IN TOPOLOGICAL SPACES

Petr A. BIRYUKOV

Abstract: In this paper, relations between ranks and cardinalities of π-bases are studied in certain classes of topological spaces (the concept of rank was introduced independently by Nagata and Archangel'skii). The existence of a dense subspace having a base of rank 1 in the Čech-Stone remainder of the integers is shown to be independent of ZFC. A new metrization theorem is given for dyadic spaces.

Key words and phrases: π-base, rank, compact space, P'-space.

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0. Preliminaries

0.1. For standard terminology and notation, the reader is referred to [E] and [J]. Recall that a family P of non-empty open sets in a space X is a π-base of X if every non-empty open set in X contains a member of P and π-weight of X is $\pi(X) = \min\{\text{Card } P: P \text{ is a } \pi\text{-base of } X\}$. The letters α and β below will denote ordinals and τ will denote an infinite cardinal. Cardinals are identified as usual with corresponding initial ordinals and the first infinite cardinal is denoted by ω .

0.2. A family F of sets is called an anti-chain if its members are mutually incomparable by inclusion. Let A be a family of subsets of a set X and $x \in X$, rank of A at x is the cardinal $r_x A = \sup\{\text{Card } F: F \text{ is an anti-chain in } A_x\}$, where $A_x = \{B \in A: x \in B\}$; and rank of A is the cardinal $rA = \sup\{r_x A: : x \in X\}$. In particular, A is a family of rank 1 iff for each pair B_1, B_2 of its members either B_1 and B_2 are comparable or $B_1 \cap B_2 = \emptyset$ and A has countable rank if $rA \leq \omega$. All spaces below are supposed to be T_1.

1. <u>Partial order arguments</u>. Let $\langle P, \leq \rangle$ be a partially ordered set. A subset $D \subset P$ is called dense in P if for every $p \in P$ there is an element $q \in D$ such that $q \leq p$. If A and B are two subsets of P such that for each $a \in A$ there exists an element $b \in B$ such that $a < b$, then A is said to strictly refine B. A subset $A \subset P$ is called Noetherian if every $B \subset A$ contains a maximal element. The following proposition is quite natural, but the author did not find it in the literature.

1.1. <u>Lemma</u>. Every partially ordered set $\langle P, \leq \rangle$ contains a dense Noetherian subset.

Proof. Let P_0 be a maximal anti-chain in P (recall that an anti-chain is a set of mutually incomparable elements). If we have defined $P_\beta \subset P$ for all $\beta < \alpha$, define P_α as a maximal anti-chain in P such that P_α strictly refines all P_β, $\beta < \alpha$. The construction terminates by an ordinal α^* for which $P_{\alpha^*} = \emptyset$. Let $P^* = \bigcup\{P_\alpha : \alpha < \alpha^*\}$. If $\alpha < \beta < \alpha^*$, $x \in P_\alpha$, $y \in P_\beta$, then $x \not< y$, otherwise for some $z \in P_\alpha$ we have $x < y < z$ and we get a contradiction, for P_α is an anti-chain. Let $A \subset P^*$, $\alpha_0 = \min\{\alpha : A \cap P_\alpha \neq \emptyset\}$ and $x \in A \cap P_{\alpha_0}$. By the previous observation x is a maximal element of A, hence P is Noetherian. For recognizing of P^* to be dense it suffices to prove that for each $x \in P \setminus P^*$ there exists an element $p \in P^*$ such that $p < x$. Put $\alpha_0 = \min\{\alpha : x \nleq y$ for each $y \in P_\alpha\}$. Then $\alpha_0 < \alpha^*$ because of $P_{\alpha^*} = \emptyset$. For each $\alpha < \alpha_0$ there exists $x_\alpha \in P_\alpha$ such that $x < x_\alpha$. In view of maximality of P_{α_0}, x must be comparable with some $p \in P_{\alpha_0}$, then $p < x$ by the definition of α_0.

1.2. Corollary. Every π-base of an arbitrary topological space contains a Noetherian π-base.

1.3. Lemma. If $\mathcal{P}$ is a π-base of a regular space X, then there exists a π-base $\mathcal{P}^* \subset \mathcal{P}$ such that $\mathcal{P}^* = \bigcup\{\mathcal{P}_\alpha : \alpha < \alpha^*\}$ and for each $\alpha < \beta < \alpha^*$ the family $\{c\ell\, U : U \in \mathcal{P}_\beta\}$ strictly refines $\mathcal{P}_\alpha$.

Proof. We say that two elements $U, V \in \mathcal{P}$ are equivalent if $c\ell\, U = c\ell\, V$. Let $\mathcal{P}'$ be a set of representatives for this equivalence relation. Since X is regular, $\mathcal{P}'$ is a π-base of X. It remains now to apply the construction of Lemma 1.1 to $\mathcal{P}'$, ordered by inclusion.

1.4. Lemma. If a space X contains a dense first category subset, then for every π-base $\mathcal{P}$ of X there exists a π-base $\mathcal{P}^* \subset \mathcal{P}$ representable by a union of countably many antichains. In particular, if X has a π-base of rank 1, then X has also a σ-disjoint π-base.

Proof. Let $X = c\ell\, D$, where $D = \bigcup\{D_n : n \in \omega\}$ and every D_n is nowhere dense in X. As in Lemma 1.1, we put $\mathcal{P}^* = \bigcup\{\mathcal{P}_\alpha : \alpha < \alpha^*\}$ with one more condition: $(\bigcup \mathcal{P}_n) \cap D_n = \emptyset$ for all $n \in \omega$. Since $\mathcal{P}_n$ remains maximal relative to the other conditions, we have as above that $\mathcal{P}^*$ is a π-base of X.

Obviously $\mathcal{P}_\omega = \emptyset$, hence $\mathcal{P}^* = \bigcup \{\mathcal{P}_n : n \in \omega\}$ and every $\mathcal{P}_n$ $n \in \omega$ is an anti-chain.

2. Rank and π-weight

2.1. Theorem. If X is a separable space with a π-base $\mathcal{P}$ of rank $\tau \geq \omega$, then $\pi(X) \leq \tau$.

Proof. Let D be a countable dense subset of X and let A be the set of all isolated points of X. If $c\ell\, A = X$, then $\{\{x\} : x \in A\}$ is a countable π-base of X. Otherwise let us consider the set $U = X \setminus c\ell\, A$. It suffices to prove that $\pi(U) \leq \tau$. It is clear that U has no isolated points, hence $C = D \cap U$ is a dense first category set in U. Applying Lemma 1.4 to $\mathcal{P}_U = \{P \in \mathcal{P} : P \subset U\}$, we obtain a π-base $\mathcal{P}^* = \bigcup \{\mathcal{P}_n : n \in \omega\}$ for U such that every $\mathcal{P}_n$, $n \in \omega$ is an anti-chain. Clearly Card $\mathcal{P}^* \leq \omega \cdot$ Card $C \cdot \tau = \tau$.

2.2. Theorem. If X is a regular space with a π-base $\mathcal{P}$ of rank $\tau \geq \omega$ and τ^+ is a calibre of X, then $\pi(X) \leq 2^\tau$.

Proof. Let $\mathcal{P}^* \subset \mathcal{P}$ be like in Lemma 1.3. If $\mathcal{A} \subset \mathcal{P}^*$ is an anti-chain, then Card $\mathcal{A} \leq \tau$ by the definition of rank and calibre. If $\mathcal{C} \subset \mathcal{P}^*$ is a chain, then, as it is easily seen, the reverse inclusion is a well-ordering on $\mathcal{C}$. For every $U \in \mathcal{C}$ denote by U^+ the successor of U in this well-ordering. Then $\mathcal{F} = \{U \setminus c\ell\, U^+ : U \in \mathcal{C}\}$ is a disjoint family of non-empty open sets such that Card $\mathcal{F}$ = Card $\mathcal{C}$. Hence, Card $\mathcal{C} \leq c(X) \leq \tau$ and Card $\mathcal{P}^* \leq 2^\tau$ is now a direct consequence of the combinatorial statement $(2^\tau)^+ \longrightarrow (\tau^+, \tau^+)^2$, which is a particular case of the Erdös-Rado theorem: $(2^\tau)^+ \longrightarrow (\tau^+)^2_\tau$ [J].

2.3. Corollary. A dyadic space X with a π-base of countable rank is metrizable.

Proof. Since $w(X) = \pi(X)$ [P] and by Theorem 2.2 $\pi(X) \leq 2^{\omega}$, X is a continuous image of the Cantor cube $D^{2^{\omega}}$ which is separable and so is X. By Theorem 2.1 $\pi(X) \leq \omega$ and hence X is second-countable.

Remark. As was noted in [GN], Kunen constructed a nonmetrizable compact space with a Noetherian base of countable rank.

3. P′-spaces. Recall that X is a P′-space if every non-empty G_δ-set in X has non-empty interior.

3.1. Definition [BPS]. An almost partition of X is a maximal disjoint family of open sets. A family of almost partitions is called a matrix. A matrix θ is called a refining if it is well ordered by the refineness relation.

3.2. Lemma. If X is a Baire P′-space, then for any countable matrix θ in X there exists an almost partition of X which refines every member of θ.

Proof. Let $\theta = \{\ _n : n \in \omega\}$, $G_n = \bigcup\ _n$. Then the set $G = \bigcap \{G_n : n \in \omega\}$ is dense in X. For every non-empty open set U in X one can find $V_n \in\ _n$ such that $U' = U \cap \bigcap \{V_n : n \in \omega\} \neq \emptyset$. Setting $U_0 = \operatorname{Int} U'$, we see that a maximal disjoint family of these U_0's is obviously an almost partition with the desired property.

3.3. Theorem. Every Baire P′-space X with $\pi(X) = \omega_1$ has a π-base of rank 1.

Proof. Let $\{U_\alpha : \alpha < \omega_1\}$ be a π-base of X. In view of Lemma 3.2, one can define by transfinite induction a refining matrix $\theta = \{\mathcal{P}_\alpha : \alpha < \omega_1\}$ such that for every $\alpha < \omega_1$ $\mathcal{P}_\alpha$

refines $\{U_\alpha, \text{Int}\, X \setminus U_\alpha\}$. It is easily seen that $\mathcal{P} = \bigcup\{\mathcal{P}_\alpha : \alpha < \omega_1\}$ is a π-base of rank 1 for X.

3.4. <u>Theorem</u>. Every compact P′-space X of the weight ω_1 has a dense subspace with a base of rank 1.

Proof. Let $\{U_\alpha : \alpha < \omega_1\}$ be a base of X. Since X is regular, one can modify the construction above to obtain a refining matrix $\Theta = \{\mathcal{P}_\alpha : \alpha < \omega_1\}$ such that for every $\alpha < \omega_1$ the family $\{cl\, U : U \in \mathcal{P}_\alpha\}$ refines all $\mathcal{P}_\beta$ with $\beta < \alpha$. If $\xi = \{V_\alpha : \alpha < \omega_1\}$ is a decreasing sequence with $V_\alpha \in \mathcal{P}_\alpha$, then $\bigcap \xi \neq \emptyset$. If $x, y \in \bigcap \xi$ and $x \neq y$, then for some α we have $x \in U_\alpha$, $y \notin U_\alpha$. But $x \in V_\alpha \in \mathcal{P}_\alpha$ and $\mathcal{P}_\alpha$ refines $\{U_\alpha, \text{Int}\,(X \setminus U_\alpha)\}$, hence $V_\alpha \subset U_\alpha$ and $y \notin V_\alpha$ – a contradiction. Thus $\bigcap \xi$ is a singleton $\{x_\xi\}$. As above $\mathcal{P} = \bigcup\{\mathcal{P}_\alpha : \alpha < \omega_1\}$ is a π-base of X, so the set A of all x_ξ's is dense in X and therefore $\{U \cap A : U \in \mathcal{P}\}$ is a base of rank 1 for A.

3.5. <u>Corollary</u> (CH). If X is a zero-dimensional locally compact Lindelöf space of the weight 2^ω, then its Čech-Stone remainder $X^* = \beta X \setminus X$ has a dense subspace with a base of rank 1 (and, a fortiori, X^* has a π-base of rank 1).

Proof. It is easily seen that there are 2^ω clopen sets in X, so $w(X^*) = 2^\omega = \omega_1$. Since X^* is locally compact and realcompact [FG], X^* is a P′-space.

Let $N^* = \beta N \setminus N$ be the Čech-Stone remainder of the integers. Then N^* has a π-base of rank 1 [BPS]. It is quite remarkable that this fact is proved in ZFC, in view of the following result.

3.6. <u>Theorem</u>. The existence of a dense subspace with a base of rank 1 in N^* is independent of ZFC.

Proof. (a) Martin's Axiom implies: (A) A non-empty intersection of $< 2^{\omega}$ open sets in N^* has a non-empty interior [B].

Let $\{U_\alpha : \alpha < 2^{\omega}\}$ be a base of N^*. Assume that for some $\alpha < 2^{\omega}$ we have defined a refining matrix $\Theta = \{\mathcal{P}_\beta : \beta < \alpha\}$ such that every $\mathcal{P}_\beta$ consists of clopen sets and refines $\{U_\beta, N^* \setminus U_\beta\}$. If V is a non-empty clopen set in N^*, pick $V_\beta \in \mathcal{P}_\beta$ with $V_\beta \cap V \neq \emptyset$ for all $\beta < \alpha$. Then $\{V\} \cup \{V_\beta : \beta < \alpha\}$ is a centered family of clopen sets, hence $V' = V \cap \bigcap\{V_\beta : \beta < \alpha\} \neq \emptyset$. By (A) $\varphi(V) = \operatorname{Int} V' \neq \emptyset$. Define $\mathcal{P}_\alpha$ to be a maximal disjoint subfamily of $\{\varphi(V) : V \text{ is clopen in } N^*\}$ that refines $\{U_\alpha, N^* \setminus U_\alpha\}$. Now the argument of Theorem 3.4 proceeds a dense subspace with a base of rank 1.

(b) Let $D \subset N^*$ be a dense subspace with a base $\mathcal{B}$ of rank 1. Then for any subfamily $\mathcal{A} \subset \mathcal{B}$ either $\cap \mathcal{A}$ is open in D or $\mathcal{A}$ is a base of some point $x \in D$ (it is a general fact about bases of rank 1 [AF]). No point of D has a countable base, hence for every $x \in D$ and a countable $\mathcal{A} \subset \mathcal{B}_x$ $\cap \mathcal{A}$ is open in D, that is, all points of D are P-points in D, and so in N^*, since N^* is regular. However, the existence of a P-point in N^* is unprovable in ZFC (Shelah).

Question. Does every Baire (or even compact) P'-space of the π-weight 2^{ω} have a π-base of rank 1 (without any additional set-theoretic assumptions; cf. Theorem 3.3) ?

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