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AN ELIMINATION OF THE PREDICATE „TO BE A STANDARD MEMBER" IN NONSTANDARD MODELS OF ARITHMETIC

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Abstract: In the paper, we are interested in the following problem: Let ${}^*\mathcal{N}$ be a nonstandard model of Peano arithmetic. Let $\mathcal{N}$ be the standard submodel of ${}^*\mathcal{N}$. Let us define a new external predicate P(x) in ${}^*\mathcal{N}$ using the predicate "to be a member of $\mathcal{N}$" and arithmetical (internal) means. We want to find a new definition of P(x) in which the external part and the internal part are separated. A method is described, how to solve this problem. Namely, the new definition is obtained by an algorithm which uses the syntactical form of the original definition.

Key words: Nonstandard model of Peano arithmetic , nonstandard model of ZF_{fin}, external, internal.

Classification: Primary 03H10
Secondary 03E70

Introduction. In the paper, a procedure is given how a new form of description of an external predicate can be found in any nonstandard model of Peano arithmetic . We suppose that only the predicate "to be a member of the standard submodel" and arithmetical (internal) means are used in the original description of the predicate. The external part and the internal part of the description are separated in the new description.

The main result of the paper is the following theorem: Let ${}^*\mathcal{N}$ be a nonstandard model of Peano arithmetic (we need the induction for all formulas). Let $\mathcal{N}$ be the standard submodel of ${}^*\mathcal{N}$. Let $St(x) \equiv x \in \mathcal{N}$. Let $P(x) \equiv$ $\equiv {}^*\mathcal{N} \models \varphi(x, a_1, \dots, a_n)$, where φ is a formula in which only the predicate St(x) and arithmetical means are used and $a_1, \dots, a_n \in {}^*\mathcal{N}$ (e.g. $\varphi \equiv (\forall z, St(z))(x < a_1 \cdot z \,\&\, (\exists t)(t = z + a_2 \equiv St(t)))$). A formula $\psi(t, x, a_1, \dots, a_n)$ of the language of Peano arithmetic and a set $\mathcal{K} \subseteq \{F; F : \mathcal{N} \to \mathcal{N}\}$ can be found such that $P(x) \equiv (\exists F \in \mathcal{K})(\forall n \in \mathcal{N})({}^*\mathcal{N} \models \psi(F(n), x, a_1, \dots, a_n))$. More then: The syntactical form of ψ can be found by an algorithm using the syntactical form of φ . $\mathcal{K}$ can be defined from the standard system $\mathcal{S}$ of the model ${}^*\mathcal{N}$ by a formula in which only the quantifications of natural numbers and members of $\mathcal{S}$ are used. The syntactical form of the formula χ defining $\mathcal{K}$ can be obtained by an algorithm using the syntactical form of φ.

Remember that the standard system $\mathcal{S}$ of the nonstandard model ${}^*\mathcal{N}$ is the system of parts X of $\mathcal{N}$, such that for some formula $\varphi(x, a_1, \dots, a_n)$ of the language of P.A. and some members $a_1, \dots, a_n \in {}^*\mathcal{N}$, $n \in X \equiv {}^*\mathcal{N} \models \varphi(n, a_1, \dots, a_n)$.

The paper is a free continuation of the paper [Č 2]. The facts contained in [Č 2] are used only in remarks con cerning the generalizations of the given procedure. The leading ideas of both the papers are the same but the technicalities connected with the work in nonstandard models of P.A. (or ZF_{fin} - Zermelo-Fraenkel set theory for finite sets) are not trivial (we do not require the model ${}^*\mathcal{N}$ to be ω_1-saturated). The procedure can be (using some technicalities) generalized for compact enlargements and the author intends to write another free continuation of these papers in the language of nonstandard analysis describing this generalization.

The set-theoretical language is mostly used in the paper. The usage of this language is correct as the reader will be able to prove the following fact after reading the first section of the first chapter of [V]. Fact: Let us define a new predicate in Peano arithmetic $a \,\bar{\in}\, b \equiv$ a-th member of the dyadic expression of b is $1 \equiv (\exists\, k,m,n)(b = k \cdot 2^a + m \,\&\, m < 2^a \,\&\, k = 2n + 1)$. With respect to this new predicate we obtain the Zermelo-Fraenkel set theory with the axiom of regularity and with the negation of the axiom of infinity (the cardinality of every set is a natural number).

We find the formula ψ and the system $\mathcal{K}$ in four steps.

1) Using the operations $\mathcal{P}, \times$, - and an arbitrary infinite natural number α as a parameter, we find an external set $\sigma \subseteq {}^*\mathcal{N}$ and a normal formula ψ_1 (only members of ${}^*\mathcal{N}$ are quantified) such that $\varphi(x,\vec{a}) \equiv (\exists\, t \in \sigma)\ \psi_1(t,x,\vec{a},\alpha)$.

2) We prove that σ is a figure in an indiscernibility relation. (A figure and an indiscernibility relation being nonstandard topological notions.)

3) We find a connection with a standard compact metric space, where $\mathcal{K}$ corresponds with a subset of this space connected with σ.

4) We find the definition of $\mathcal{K}$ from the standard system $\mathcal{S}$ of ${}^*\mathcal{N}$.

The numbering and contents of sections corresponds to the described division on steps. In the section 0 we translate our problem into the set-theoretical language.

The author believes that the paper is readable also without usage of references except of the given fact, another fact in § 0 and remarks concerning generalizations.

§ 0. We use the notion class for parts of $^*\mathcal{N}$ (external sets) and the notion set for members of $^*\mathcal{N}$. We identify the set a with the class $\tilde{a} = \{x; x \mathrel{\tilde{\in}} a\}$. Thus we use only $\in$ and not $\tilde{\in}$. For classes we usually use the capital Roman letters. For sets, we usually use the lower case Roman letters. The small Greek letters are used for subclasses of sets and natural numbers (finite or infinite). For finite natural numbers we use $n, m, k, \ldots$.

Attention: 1) The members of $^*\mathcal{N}$ are not called natural numbers from this moment. If we say "x is a natural number", then we mean by this that x is a natural number in the sense of the set theory (w.r.t. $\in$).

2) A subclass of a set is usually a set in the set theory. In our case, this assertion does not hold. We prove that the class FN (finite natural mumbers) of all natural numbers being members of a standard submodel is a subclass of a set not being a set.

Definition 0.1: 1) $N = \{\alpha;\ \alpha$ is a natural number$\}$.

2) $FN = \{\alpha \in N;\ \alpha \in \mathcal{N}\}$.

Lemma 0.2: 1) $a \in b \Rightarrow {}^*\mathcal{N} \models a < b$.

2) $a \in b \,\&\, b \in \mathcal{N} \Rightarrow a \in \mathcal{N}$.

Proof: 1) Look at the definition of $\tilde{\in}$ in the introduction.

2) As $\mathcal{N}$ is the standard submodel and $^*\mathcal{N} \models a < b$ we have $a \in \mathcal{N}$.

Lemma 0.3: (We use ZF_{fin} + reg.)

1) $\alpha \in N\text{-}FN \,\&\, n \in FN \Rightarrow n \in \alpha$. Thus $(\forall \alpha \in N\text{-}FN)(FN \subseteq \alpha)$.

2) For $\alpha \in N$ we define V_α by recursion. $V_0 = \{0\}$; $V_{\alpha+1} = \mathcal{P}(V_\alpha)$. For every $n \in FN$ we have $V_n \in \mathcal{N}$.

3) $(\exists \alpha)(\alpha \in N - FN)$.

4) $\neg(\exists a)(a = FN)$.

Proof: 1) If $\alpha \in n$ then ${}^*\mathcal{N} \models \alpha < n$ (see L0.1).n is a member of the standard submodel hence α is also a member of $\mathcal{N}$ - a contradiction.

2) If $a \in \mathcal{N}$ then $\mathcal{P}(a) \in \mathcal{N}$ as $\mathcal{P}(a)$ is definable from a. If $V_n \notin \mathcal{N}$ for some $n \in FN$, then there must be first such n (we use the fact that $\mathcal{N}$ is the standard submodel). But $V_{n-1} \in \in \mathcal{N}$ - a contradiction with $\mathcal{P}(V_{n-1}) \notin \mathcal{N}$.

3) Using the regularity axiom we have $(\forall a)(\exists \alpha \in N)(a \in \in V_\alpha)$. Let $a \in {}^*\mathcal{N} - \mathcal{N}$. Let α be such that $a \in V_\alpha$. If $\alpha \in FN$ then $V_\alpha \in \mathcal{N}$, thus $a \in \mathcal{N}$ (see L0.2) - a contradiction.

4) If $a = FN$ then $\max(a) \in FN$. Hence $\max(a) + 1 \in FN = a$ - a contradiction with the maximality of max(a). (Any subset of N must have a maximal element - we use ZF_{fin}.)

Definition 0.4: $V = \bigcup\{V_\alpha ; \alpha \in N\}$, $V_{FN} = \bigcup\{V_\alpha ; \alpha \in FN\}$.

<u>Theorem 0.5</u>: 1) $a \in {}^*\mathcal{N} \equiv a \in V$.

2) $a \in \mathcal{N} \equiv a \in V_{FN}$.

Proof: 1) We use the regularity axiom.

2) $V_{FN} \subseteq \mathcal{N}$ (see L0.3.2)). If $a \in \mathcal{N}$ then the first $\alpha \in \in N$ s.t. $a \in V_\alpha$ (the rank of a) is definable from a and hence α must be in $\mathcal{N}$.

Fact: A function G can be defined by the recursion such that $G : N \leftrightarrow V$ and $G : FN \leftrightarrow V_{FN}$.

Instead of the definition of G we find the value G(324).

(For the definition of G and the proof of the fact see [V].)
$324 = 2^8+2^6+2^2$; $8=2^3, 6=2^2+2^1, 2=2^1$; $3=2^1+2^0, 1=2^0$; $G(0)=0$;
$G(1)=\{0\}$, $G(2)=\{\{0\}\}$, $G(3)=\{\{0\},0\}$; $G(6)=\{\{\{0\}\},\{0\}\}$,
$G(8)=\{\{\{0\},0\}\}$; $G(324)=\{\{\{\{0\},0\}\}, \{\{\{0\}\},\{0\}\}, \{\{0\}\}\}$.
Let us note that the definition of $\bar{\epsilon}$ is connected with G.

Using the set-theoretical language we can consider the class $X=\{x; P(x)\}$ instead of the predicate $P(x)$. The fact that $P(x)$ is defined by an arithmetical formula with the predicate $St(x)$, and parameters $a_1,\dots,a_n$, can be expressed by the fact that $X=\{x; \varphi(x,a_1,\dots,a_n,V_{FN})\}$, where φ is a normal formula (only sets are quantified). To prove the equivalence of these two formulations it is sufficient to prove the following assertion.

Assertion: $({}^*\mathcal{N} \models a+b=c) \equiv G^{-1}(c)=G^{-1}(a)+G^{-1}(b)$,
$({}^*\mathcal{N} \models a\cdot b=c) \equiv G^{-1}(c)=G^{-1}(a)\cdot G^{-1}(b)$.

As the assertion concerns only the equivalence of the two formulations of the problem we give here only the principal mottos of the proof. 1) Let $a \circledless b \equiv ({}^*\mathcal{N} \models a<b)$, let $a \overset{G}{<} b \equiv$ $\equiv G^{-1}(a)< G^{-1}(b)$. In both the orderings we compare in the following manner: Order the members in the decreasing sequence and use the lexicographical ordering.

2) $a \circledless b \equiv a \overset{G}{<} b$. Let a be the $\circledless$ least member such that $\circledless$"$a \neq \overset{G}{<}$"a. Let b be the $\circledless$ predecessor of a. We have $1 \circledless b \circledless a \,\&\, a \overset{G}{<} b$. But this is a contradiction with 1) and with the fact that both a and b are sets of elements $\circledless$ less than b.

3) 2) implies $({}^*\mathcal{N} \models b=a+1) \equiv G^{-1}(b)=G^{-1}(a)+1$ and the required assertions we obtain by the induction.

As $G: FN \longleftrightarrow V_{FN}$ we can (using Th. 0.5.2)) reformulate the main theorem in the following form: Let $V, \in$ be a nonstandard model of ZF_{fin}+ reg, let FN be the class (external set) of standard natural numbers. An algorithm can be found which to any normal formula $\varphi(x,\vec{a},X)$ gives a normal formula $\psi(x,\vec{a},t)$ and a system of functions $\mathcal{K} \subseteq {}^{FN}V_{FN}$ such that $\varphi(x,\vec{a},FN) \equiv (\exists F \in \mathcal{K})(\forall n \in FN)(V \models \psi(x,\vec{a},F(n)))$. More then, $\mathcal{K}$ is found in the following form: Let $\mathcal{S}$ be the standard system of V ($\mathcal{S} = \{X \subseteq FN; (\exists a \in V)(X = FN \cap a)\}$). A formula $\overline{\psi}$ can be found in which only members of V_{FN} and $\mathcal{S}$ are quantified such that $\mathcal{K} = \{F \subseteq {}^{FN}V_{FN}; \overline{\psi}(F,\mathcal{S})\}$.

§ 1

Lemma 1.1: 1) Let $\sigma \subseteq u$. If $\chi(t,\vec{z})$ is a normal formula then $(\forall t \in \sigma)\chi(t,\vec{z}) \equiv (\exists v \subseteq u)(v \supseteq \sigma \,\&\, (\forall t \in v)\chi(t,\vec{z}))$.

2) Especially $(\forall n \in FN)(\exists \alpha \in N, \alpha > n)\chi(t,\vec{z}) \equiv$
$\equiv (\exists \alpha \in N\text{-}FN)\chi(\alpha,\vec{z})$.

Proof: 1) $v = \{t \in u; \chi(t,z)\}$.

2) Let β be an arbitrary element of N-FN. Put $\overline{\chi}(\overline{\alpha},\vec{z},\beta) \equiv \overline{\alpha} \in \beta \,\&\, (\exists \alpha > \overline{\alpha})(\alpha \in N \,\&\, \chi(\alpha,\vec{z}))$. Use 1) for $\sigma = FN$, $u = \beta$ (cf. [Č 2]).

Lemma 1.2: Let $\sigma \subseteq u$ and let $\chi(w,\vec{z})$ be a normal formula. The following equivalence holds. $(\forall t \in \sigma)\chi(t,\vec{a}) \equiv$
$\equiv (\exists \overline{t} \in \mathcal{P}(u - \sigma))(\forall t \in u - \overline{t})\chi$. The equivalence holds also for dual quantifiers.

Proof: Use L.1.1.1).

Remarks: 1) The formulas on both the sides of the equivalence have a similar syntactic form - a quantification restricted to a class followed by a normal formula. The restricted quantifications are dual one to the other. This fact makes it possible to put the quantifiers restricted to classes to the beginning of the formula.

2) It is possible to generalize the lemma for classes $\vec{X}$ as parameters. We require in this case that no proper subclass of u can be defined by a normal formula using $\vec{X}$, $\vec{a}$ as parameters. (For more details see [Č 2].)

3) For the "dualisation" of quantifiers we do not need the whole powerset axiom (the whole induction schema). The following schema is sufficient. For any normal formula φ the following formula is an axiom $(\forall u)(\exists v)(\forall \vec{z})(\{t; \varphi(t,\vec{z})\ \&\ \& t \in u\} \in v)$. We can also do some hiearchy restriction on formulas in the schema if we want to use the "dualisation" only for hierarchy restricted formulas.

<u>Theorem 1.3</u>: Let $\alpha \in N\text{-}FN$. Let $\varphi(x, \xi, \vec{z})$ be a normal formula. A normal formula $\psi(x,y,\vec{z})$, a set u and a class $\sigma \subseteq u$ can be found such that $\varphi(t, FN, \vec{a}) \equiv (\exists \bar{t} \in \sigma)\psi(\bar{t}, t, \vec{a})$. More then: u is defined from α using the operations $\mathcal{P}, \times$ and σ is defined from α, FN using the operations $\mathcal{P}, \times, -$.

Proof: By the induction based on the complexity of the formula φ.

1) $x \in FN \equiv (\exists \bar{t} \in FN)(x = \bar{t})$ (we put $\sigma = FN$, $u = \alpha$). Other cases of atomary formulas are obvious (e.g. $x = FN \equiv x \neq x$).

2) $(\exists t^1 \in \sigma^1)\psi^1(t^1, t, \vec{a})\ \&\ (\exists t^2 \in \sigma^2)\psi^2(t^2, t, \vec{a}) \equiv$ $\equiv (\exists \bar{t} \in \sigma^1 \times \sigma^2)(\exists t^1, t^2)\ (\bar{t} = \langle t^1, t^2 \rangle\ \&\ \psi^1\ \&\ \psi^2)$. If $\sigma^1 \subseteq u^1\ \&\ \sigma^2 \subseteq u^2$ then we put $\sigma = \sigma^1 \times \sigma^2$ and $u = u^1 \times u^2$.

3) $(\exists x)(\exists \bar{t} \in \sigma)\psi(\bar{t}, t, \vec{a}, x) \equiv (\exists \bar{t} \in \sigma)(\exists x)\psi$.

4) $\neg(\exists t^1 \in \sigma^1)\psi^1(t^1, t, \vec{a}) \equiv (\forall t^1 \in \sigma^1)\neg\psi^1$, let

$\chi(u^1,\alpha)$ be the definition of u^1 from α. Let us put $u = \mathcal{P}(u^1)$, $\sigma = \mathcal{P}(u^1 - \sigma^1)$. Using L1.2 we obtain the equivalent $(\exists \bar{t} \in \sigma)(\exists u^1, \chi(u^1,\alpha))(\forall t^1 \in u^1 - t) \neg \psi^1$ having the required form.

Remarks: 1) The theorem can be generalized for several "small" classes (instead of FN) and "large" classes as parameters (see [Č 2]).

2) If FN occurs only in the prefix of φ then we can modify only the prefix. This modification and the definition of σ and u is dependent only on the syntactic form of the prefix of φ in this case.

§ 2

Definition 2.1: Let $\sim$ be an equivalence relation.

1) $\mathcal{F}ig_{\sim}(X) \equiv (\forall x,y)(x \in X \,\&\, y \sim x \Rightarrow y \in X)$, X is a figure in $\sim$.

2) $Fig_{\sim}(X) = \{y; (\exists x \in X)(y \sim x)\}$, the figure of X.

3) $\mu_{\sim}(x) = Fig_{\sim}(\{x\})$, the monad of x.

Fact: $\mathcal{F}ig(Fig(X))$.

Definition 2.2: 1) We use w for words defined by the following inductive definition: i) The empty word Λ is a word,

ii) if w_1, w_2 are words, then $(w_1 \times w_2)$ is a word,

iii) if w is a word, then $\mathcal{P}w$ is a word,

iv) any word is obtained by finitely many applications of ii) and iii) on the empty words.

2) For $\alpha \in N$ (finite or infinite) and for a word w we define a set u_{α}^{w} and an equivalence $\overset{w}{\underset{\alpha}{\cong}}$ on u_{α}^{w} by the recursion based on the complexity of w. i) $u_{\alpha}^{\Lambda} = \alpha$,

$\stackrel{A}{\overline{\overline{\alpha}}} = (Id/^{}FN) \cup ((\alpha - FN) \times (\alpha - FN))$, where Id is the identity mapping $Id(x) = x$.

ii) $u_\alpha^{(\mathscr{A}_1 \times \mathscr{A}_2)} = u_\alpha^{\mathscr{V}_1} \times u_\alpha^{\mathscr{V}_2}$, $\langle x_1, x_2 \rangle \stackrel{(\mathscr{V}_1 \times \mathscr{V}_2)}{\overline{\overline{\alpha}}} \langle y_1, y_2 \rangle \equiv$
$\equiv x_1 \stackrel{\mathscr{V}_1}{\overline{\overline{\alpha}}} y_1 \,\&\, x_2 \stackrel{\mathscr{V}_2}{\overline{\overline{\alpha}}} y_2$.

iii) $u_\alpha^{\mathscr{P}\mathscr{V}} = \mathscr{P}(u_\alpha^{\mathscr{V}})$, $x \stackrel{\mathscr{P}\mathscr{V}}{\overline{\overline{\alpha}}} y \equiv \mathrm{Fig}_{\stackrel{\mathscr{V}}{\overline{\overline{\alpha}}}}(x) = \mathrm{Fig}_{\stackrel{\mathscr{V}}{\overline{\overline{\alpha}}}}(y)$.

Remark: For $\alpha \in FN$ all the equivalences are identical with the equality.

Theorem 2.3: 1) $(\forall \alpha \in N\text{-}FN)(\mathscr{F}ig_{\stackrel{A}{\overline{\overline{\alpha}}}}(FN))$.

2) $\mathscr{F}ig_{\stackrel{\mathscr{V}}{\overline{\overline{\alpha}}}}(\sigma) \Longrightarrow \mathscr{F}ig_{\stackrel{\mathscr{V}}{\overline{\overline{\alpha}}}}(u_\alpha^{\mathscr{V}} - \sigma)$

3) $\mathscr{F}ig_{\stackrel{\mathscr{V}_1}{\overline{\overline{\alpha}}}}(\sigma_1) \,\&\, \mathscr{F}ig_{\stackrel{\mathscr{V}_2}{\overline{\overline{\alpha}}}}(\sigma_2) \Longrightarrow \mathscr{F}ig_{(\mathscr{V}_1 \stackrel{\times}{\overline{\overline{\alpha}}} \mathscr{V}_2)}$
$(\sigma_1 \times \sigma_2)$.

4) $\mathscr{F}ig_{\stackrel{\mathscr{V}}{\overline{\overline{\alpha}}}}(\sigma) \Longrightarrow \mathscr{F}ig_{\stackrel{\mathscr{P}\mathscr{V}}{\overline{\overline{\alpha}}}}(\mathscr{P}(\sigma))$.

Proof: Only 4) is not obvious. Let us prove 4). We have to prove that $x \subseteq \sigma \,\&\, y \stackrel{\mathscr{P}\mathscr{V}}{\overline{\overline{\alpha}}} x \Longrightarrow y \subseteq \sigma$. We have $y \subseteq \mathrm{Fig}_{\stackrel{\mathscr{V}}{\overline{\overline{\alpha}}}}(y) =$
$= \mathrm{Fig}_{\stackrel{\mathscr{V}}{\overline{\overline{\alpha}}}}(x) \subseteq \sigma$ as σ is a figure.

Corollary 2.4: The set u from the theorem 1.2 is $u_\alpha^{\mathscr{V}}$ for a suitable $\mathscr{V}$, α and the class σ from this theorem is a figure in $\stackrel{\mathscr{V}}{\overline{\overline{\alpha}}}$.

Remark: The given step can be done also for several "input" classes, if we suppose that they are figures in suitable equivalences.

§ 3

Theorem 3.1: If $\beta \in \alpha \in N$ then $u_\beta^{\mathscr{V}} \subseteq u_\alpha^{\mathscr{V}}$ and $(\forall x, y \in u_\beta^{\mathscr{V}})(x \stackrel{\mathscr{V}}{\overline{\overline{\beta}}} y \equiv x \stackrel{\mathscr{V}}{\overline{\overline{\alpha}}} y)$.

Proof: By the induction based on the complexity of $\mathscr{V}$. Only the step for $\mathscr{P}\mathscr{V}$ is not obvious. Let us prove this step.

Let $x, y \in u_\beta^{\mathcal{P}w}$, let $x \overset{\mathcal{P}w}{\underset{\alpha}{\equiv}} y$ and let $t \in x$. There is a $s \in y$ s.t. $s \overset{w}{\underset{\alpha}{\equiv}} t$. As $x, y \in u_\beta^{\mathcal{P}w}$ we have $s, t \in u_\beta^{w}$. Using the induction assumption we obtain $s \overset{w}{\underset{\beta}{\equiv}} t$. The proof of the assertion with x, y changed and the proof of $\Longrightarrow$ are analogous.

Definition 3.2: For $\alpha, \beta \in N$ s.t. $\beta \in \alpha$ and a word w let us define the function ${}_\alpha f_\beta^{w}: u_\alpha^{w} \xrightarrow{\text{on}} u_\beta^{w}$. We proceed by the recursion based on the complexity of w.

1) ${}_\alpha f_\beta^{\Lambda}(\gamma) = \gamma$ for $\gamma \in \beta$,

$= \beta - 1$ for $\gamma \in \alpha - \beta$.

ii) ${}_\alpha f_\beta^{(w_1 \times w_2)}(\langle x_1, x_2 \rangle) = \langle {}_\alpha f_\beta^{w_1}(x_1), {}_\alpha f_\beta^{w_2}(x_2) \rangle$.

iii) ${}_\alpha f_\beta^{\mathcal{P}w}(x) = ({}_\alpha f_\beta^{w})''x$.

Lemma 3.3: 1) ${}_\alpha f_\beta^{w}$ is described by a set-formula with parameters α, β, w.

2) For $x \in u_\beta^{w}$ we have ${}_\alpha f_\beta^{w}(x) = x$.

3) If $\alpha \leq \beta \leq \gamma$ then ${}_\beta f_\alpha^{w} \circ {}_\gamma f_\beta^{w} = {}_\gamma f_\alpha^{w}$.

Proof: By the induction based on the complexity of w.

<u>Theorem 3.4</u>: 1) For any $\alpha, \beta \in N$, $\beta \leq \alpha$, any w and any $x, y \in u_\alpha^{w}$ the following implication holds:

$x \overset{w}{\underset{\alpha}{\equiv}} y \Longrightarrow {}_\alpha f_\beta^{w}(x) \overset{w}{\underset{\beta}{\equiv}} {}_\alpha f_\beta^{w}(y)$.

2) If $\beta \in N-FN$ then the opposite implication holds, too.

3) If $\alpha \in N-FN$ and $x, y \in u_\alpha^{w}$ then $x \overset{w}{\underset{\alpha}{\equiv}} y \equiv (\forall n \in FN)({}_\alpha f_n^{w}(x) = {}_\alpha f_n^{w}(y)) \equiv (\exists \beta \in N-FN)(\beta \leq \alpha \;\&\; {}_\alpha f_\beta^{w}(x) = {}_\alpha f_\beta^{w}(y$

Proof: 1) By the induction based on the complexity of w. Only the induction step for $\mathcal{P}w$ is not obvious. Let u prove this step. Let $t \in {}_\alpha f_\beta^{\mathcal{P}w}(x)$ and let $\bar{t} \in x$ be such that $t = {}_\alpha f_\beta^{w}(\bar{t})$. There is a $\bar{v} \in y$ s.t. $\bar{v} \overset{w}{\underset{\alpha}{\equiv}} \bar{t}$. By the induction assumption we have $t \overset{\mathfrak{M}}{\underset{\beta}{=}} {}_{\alpha}f_{\beta}^{\mathfrak{M}}(\bar{v})$. If we change x, y, then we proceed analogously.

2) We again use the induction and only the step for $\mathcal{P}\mathfrak{M}$ is not obvious. Let $t \in x$. It is sufficient to find a $\bar{s} \in y$ s.t. $t \overset{\mathfrak{M}}{\underset{\alpha}{=}} \bar{s}$. Let $s \in {}_{\alpha}f_{\beta}^{\mathcal{P}\mathfrak{M}}(y)$ be s.t. ${}_{\alpha}f_{\beta}^{\mathfrak{M}}(t) \overset{\mathfrak{M}}{\underset{\beta}{=}} s$ (the existence is implied by the assumption of the implication). Let $\bar{s} \in y$ be s.t. $s = {}_{\alpha}f_{\beta}^{\mathfrak{M}}(\bar{s})$. By the induction assumption we have $\bar{s} \overset{\mathfrak{M}}{\underset{\alpha}{=}} \bar{t}$.

3) The fact that the second assertion is implied by the first one can be proved by 1) and the fact that for $n \in FN$ $\overset{\mathfrak{M}}{\underset{n}{=}}$ is the identity. The fact that the third assertion is implied by the second one follows from L1.1.2). Using 2) we prove that the first assertion is implied by the third one.

Corollary 3.5: If $FN \subseteq \beta \leq \alpha \in N \mathbin{\&} x \in u_{\alpha}^{\mathfrak{M}}$ then ${}_{\alpha}f_{\beta}^{\mathfrak{M}}(x) \overset{\mathfrak{M}}{\underset{\alpha}{=}} x$.

Proof: Put $y = {}_{\alpha}f_{\beta}^{\mathfrak{M}}(x)$. $y \in u_{\beta}^{\mathfrak{M}}$ hence ${}_{\alpha}f_{\beta}^{\mathfrak{M}}(y) = y = {}_{\alpha}f_{\beta}^{\mathfrak{M}}(x)$ (see 3.3.2)). Hence $y \overset{\mathfrak{M}}{\underset{\alpha}{=}} x$ (see 3.4.3)).

<u>Theorem 3.6</u>: Let $FN \subseteq \beta \leq \alpha \in N$. If $\sigma_{\beta/\alpha} \subseteq u_{\beta/\alpha}$ are figures in $\overset{\mathfrak{M}}{\underset{\beta/\alpha}{=}}$ then ${}_{\alpha}f_{\beta}^{\mathfrak{M}}{}''\sigma_{\alpha} = \sigma_{\alpha} \cap u_{\beta}$ and $({}_{\alpha}f_{\beta}^{\mathfrak{M}})^{-1}{}''\sigma_{\beta} = \mathrm{Fig}_{\overset{\mathfrak{M}}{\underset{\alpha}{=}}}(\sigma_{\beta})$. Hence $\sigma_{\alpha} = ({}_{\alpha}f_{\beta}^{\mathfrak{M}})^{-1}{}''({}_{\alpha}f_{\beta}^{\mathfrak{M}}{}''\sigma_{\alpha})$ and $\sigma_{\beta} = ({}_{\alpha}f_{\beta}^{\mathfrak{M}})''(({}_{\alpha}f_{\beta}^{\mathfrak{M}})^{-1}{}''\sigma_{\beta})$.

Proof: Let $x \in \sigma_{\alpha}$. We have $x \overset{\mathfrak{M}}{\underset{\alpha}{=}} {}_{\alpha}f_{\beta}^{\mathfrak{M}}(x) \in \sigma_{\alpha} \cap u$. The first equality is an easy consequence. The second equality is also an easy consequence of $x \overset{\mathfrak{M}}{\underset{\alpha}{=}} {}_{\alpha}f_{\beta}^{\mathfrak{M}}(x)$.

<u>Theorem 3.7</u>: The operations $-$, $\times$, $\mathcal{P}$ commutate with f in the following sense: Let $FN \subseteq \beta \leq \alpha \leq \gamma \in N$.

1) If $\sigma_{1/2} \subseteq u_{\alpha}$ are figures in $\overset{\mathfrak{M}}{\underset{\alpha}{=}}$ then ${}_{\alpha}f_{\beta}^{\mathfrak{M}}{}''\sigma_1 - {}_{\alpha}f_{\beta}^{\mathfrak{M}}{}''\sigma_2 = {}_{\alpha}f_{\beta}^{\mathfrak{M}}{}''(\sigma_1 - \sigma_2)$.

2) If $\sigma_{1/2} \subseteq u_{\alpha}^{\mathfrak{M}_{1/2}}$ are figures then $({}_{\alpha}f_{\beta}^{\mathfrak{M}_1}{}''\sigma_1) \times$

$\times({}_{\alpha}f_{\beta}^{\vartheta_2}$ " $\sigma_2) = {}_{\alpha}f_{\beta}^{(\vartheta_1\times\vartheta_2)}$ "$(\sigma_1\times\sigma_2)$.

3) If $\sigma\subseteq u_{\alpha}^{\vartheta}$ is a figure then $\mathcal{P}({}_{\alpha}f_{\beta}^{\vartheta}$ "$\sigma) = {}_{\alpha}f_{\beta}^{\mathcal{P}\vartheta}$ "$\mathcal{P}(\sigma)$.

For $({}_{\gamma}f_{\alpha}^{\vartheta})^{-1}$ hold assertions analogous to 1),2),3).

Proof: We use Th. 3.6. We prove only the most complicated case and namely the case 3). Let $x\in\mathcal{P}({}_{\alpha}f_{\beta}^{\vartheta}$ "$\sigma) = \mathcal{P}(\sigma\cap u_{\beta}^{\vartheta})$. Thus $x\subseteq\sigma\ \&\ x\in u_{\beta}^{\mathcal{P}\vartheta}\Rightarrow x = {}_{\alpha}f_{\beta}^{\mathcal{P}\vartheta}(x)$ (see L.3.3.2)) $\Rightarrow x\in{}_{\alpha}f_{\beta}^{\mathcal{P}\vartheta}$ "$\mathcal{P}(\sigma)$. Let on the other hand $x = {}_{\alpha}f_{\beta}^{\mathcal{P}\vartheta}(y)\ \&\ y\subseteq\sigma$. We have to prove that $(\forall t\in x)(t\in{}_{\alpha}f_{\beta}^{\vartheta}$ "$\sigma(=\sigma\cap u_{\beta}^{\vartheta}))$. Let for an arbitrary $t\in x$ an element $s\in y$ be s.t. $t = {}_{\alpha}f_{\beta}^{\vartheta}(s)$. We have $t \overset{\vartheta}{\underset{\alpha}{\equiv}} s$ (see C.3.5), $t\in u_{\beta}^{\vartheta}$ hence $t\in\sigma\cap u_{\beta}^{\vartheta}$ as σ is a figure. We now give the proof for $({}_{\gamma}f_{\alpha}^{\vartheta})^{-1}$. Let $x\in\mathcal{P}(({}_{\gamma}f_{\alpha}^{\vartheta})^{-1}$"$\sigma) = \mathcal{P}(\mathrm{Fig}_{\overset{\vartheta}{\underset{\gamma}{\equiv}}}(\sigma))$ (i.e. $x\subseteq\mathrm{Fig}_{\overset{\vartheta}{\underset{\gamma}{\equiv}}}(\sigma) = \bar{\sigma}$). We have to prove that ${}_{\gamma}f_{\alpha}^{\mathcal{P}\vartheta}(x) = {}_{\gamma}f_{\alpha}^{\vartheta}$"$x\subseteq\sigma$. If t is an arbitrary element of x then ${}_{\gamma}f_{\alpha}^{\vartheta}(t)\in\bar{\sigma}\cap u_{\alpha}^{\vartheta} = \sigma$ (see Th. 3.6). Let on the other hand $x\in(({}_{\gamma}f_{\alpha}^{\mathcal{P}\vartheta})^{-1}$"$\mathcal{P}(\sigma))$. Hence ${}_{\gamma}f_{\alpha}^{\vartheta}$"$x\subseteq\sigma$. If t is an arbitrary element of x then ${}_{\gamma}f_{\alpha}^{\vartheta}(t)\in\sigma$. Hence $x\in\mathcal{P}(({}_{\gamma}f_{\alpha}^{\vartheta})^{-1}$"$\sigma)$.

Definition 3.8: Let $\alpha\in N-FN$, let $\sigma_{\alpha}\subseteq u_{\alpha}^{\vartheta}$ be a figure in $\overset{\vartheta}{\underset{\alpha}{\equiv}}$. We define a system $\mathcal{K}_{\sigma_{\alpha}}$ of functions $F:FN\longrightarrow V_{FN}$ in the following manner: $F\in\mathcal{K}_{\sigma_{\alpha}}\equiv(\exists x\in\sigma_{\alpha})(\forall n\in FN)(F(n) = {}_{\alpha}f_{n}^{\vartheta}(x))$.

Remarks: 1) The notation $F\in\mathcal{K}_{\sigma_{\alpha}}$ is not correct as F cannot be a set. We use this notation as it is objective. $\in$ can be understood in the external sense or in the sense of codable classes (see [V]).

2) Let us note that $\mathcal{K}_{\sigma_{\alpha}}$ is a system of parts of the standard submodel.

Theorem 3.9: Let $\alpha \in N-FN$, let $\sigma_\alpha \subseteq u_\alpha^{w}$ be a figure in $\overset{w}{\underset{\alpha}{\equiv}}$.

1) $t \in \sigma_\alpha \equiv (\exists F \in \mathcal{K}_{\sigma_\alpha})(\forall n \in FN)(F(n) = {}_\alpha f_n^{w}(t)$ & & $t \in u_\alpha^{w})$.

2) For $\beta > \alpha$ let us put $\sigma_\beta = \mathrm{Fig}_{\overset{w}{\underset{\beta}{\equiv}}}(\sigma_\alpha)$. We have $\mathcal{K}_{\sigma_\beta} = \mathcal{K}_{\sigma_\alpha}$.

Proof: 1) $\Longrightarrow$ see the definition of $\mathcal{K}_{\sigma_\alpha}$. $\Longleftarrow$ For t satisfying the righthand side let $\bar{t} \in \sigma_\alpha$ be such that $(\forall n \in FN)({}_\alpha f_n^{w}(\bar{t}) = {}_\alpha f_n^{w}(t))$ (see the definition of $\mathcal{K}_{\sigma_\alpha}$ for the existence of $\bar{t}$). We have $t \overset{w}{\underset{\alpha}{\equiv}} \bar{t}$ (see Th. 3.4.) and hence $t \in \sigma_\alpha$.

2) For $x \in \sigma_\beta$ we have ${}_\beta f_n^{w}(x) = {}_\alpha f_n^{w}({}_\beta f_\alpha^{w}(x))$ (see L.3.3.3)) and ${}_\beta f_\alpha^{w}(x) \in \sigma_\alpha$ (see Th. 3.6).

Corollary 3.10: For any normal formula $\varphi(x, \varsigma, \vec{z})$ there are a normal formula $\psi(x,y,\vec{z})$ and a system of functions $\mathcal{K} \subseteq {}^{FN}V_{FN}$ such that for any $\vec{a}$, t the following equivalence holds: $\varphi(t, FN, \vec{a}) \equiv (\exists F \in \mathcal{K})(\forall n \in FN)\ \ \psi(t, F(n), \vec{a})$.

Proof: Let us denote (1),(2) the lefthand side and the righthand side of the equivalence respectively. Using the theorem (Th. 1.2) and the corollary (C.2.4) we find an equivalent of (1) of the form $(\exists \bar{t} \in \sigma_\alpha)\,\overline{\psi}(t, \bar{t}, \vec{a})$. We know that $\sigma_\alpha \subseteq$ $\subseteq u_\alpha^{w}$ is a figure in $\overset{w}{\underset{\alpha}{\equiv}}$ for a suitable word w and an arbitrary infinitely large α. Using the theorem Th. 3.9 we obtain an equivalent (3) of the form $(\exists F \in \mathcal{K}_{\sigma_\alpha})(\forall n \in FN)$ $\overline{\overline{\psi}}(F(n), n, \alpha, t, \vec{a})$. We know that $\mathcal{K}_{\sigma_\alpha}$ is not dependent on the choice of α and that $\alpha_1 < \alpha_2 \Rightarrow (\overline{\overline{\psi}}(F(n), n, \alpha_1, t, \vec{a}) \rightarrow$ $\Rightarrow \overline{\overline{\psi}}(F(n), n, \alpha_2, t, \vec{a}))$ (Th. 3.9.2), Th. 1.3, Th. 3.6). α does not occur in the formula φ. Using the logical law

$\varphi \equiv \psi(\alpha) \vdash \varphi \equiv (\exists \alpha)\, \psi(\alpha)$ we obtain the equivalent (4) $(\exists F \in \mathcal{K}_{\sigma_\alpha})(\exists \alpha \in N\text{-}FN)(\forall n \in FN)\, \overline{\overline{\psi}}(F(n), n, \alpha, t, \vec{a})$. We prove that (4) is equivalent to (5) $(\exists F \in \mathcal{K}_{\sigma_\alpha})(\forall n \in FN)(\exists \alpha \in N, \alpha > n)\, \overline{\overline{\psi}}(F(n), n, \alpha, t, \vec{a})$. Let us fix a $F \in \mathcal{K}_{\sigma_\alpha}$. Let $\beta \in N - FN$ be an arbitrary element of N-FN, let $s \in \sigma_\beta$ be such that $(\forall n \in FN)(F(n) = {}_n f_\beta^{u}(s))$ (for the existence of s see Df. 3.8, Th. 3.9.2)). Let us define the set function g by the following description: For $\delta \leq \beta$ let $g(\delta)$ = the least $\alpha \geq \beta$ such that $\overline{\overline{\psi}}({}_\delta f_\beta^{u}(s), \delta, \alpha, t, \vec{a})$. We have $FN \subseteq dom(g)$ hence there is a $\gamma \in N\text{-}FN$ such that $\gamma \subseteq dom(g)$. Let us put $\alpha_0 = \max\{g(\delta); \delta \in \gamma\}$. $\alpha_0 \in N\text{-}FN$ and we have $(\forall n \in FN)\, \overline{\overline{\psi}}({}_n f_\beta^{u}(s), n, \alpha_0, t, \vec{a})$ (remember that $(\alpha_1 < \alpha_2 \,\&\, \overline{\overline{\psi}}(\ldots\alpha_1\ldots)) \Rightarrow \overline{\overline{\psi}}(\ldots\alpha_2\ldots)$). We have proved $(5) \Rightarrow (4)$ in view of $F(n) = {}_n f_\beta^{u}(s)$. $(4) \Rightarrow (5)$ is obvious. To finish the proof it suffices only to put $\psi(x, y, \vec{z}) \equiv (\exists x_1, x_2)\,(x = \langle x_1, x_2\rangle \,\&\, (\exists \alpha \in N, \alpha > x_2)\, \overline{\overline{\psi}}(x_1, x_2, \alpha, y, \vec{z}))$ and $\mathcal{K} = \{F; dom(F) = FN \,\&\, (\exists \overline{F} \in \mathcal{K}_{\sigma_\alpha})(\forall n \in FN)(F(n) = \langle \overline{F}(n), n\rangle)\}$.

Remarks: 1) For $\alpha \in N\text{-}FN$ the factor space $u_\alpha^{u} / \underset{\alpha}{\overset{u}{=}}$ can be endowed with a natural topology (a compact metric space is obtained if $^*\mathcal{U}$ is ω_1-saturated). $\bigcup\{u_n; n \in FN\}$ forms a dense subset. The members of $\mathcal{K}_{\sigma_\alpha}$ are sequences and their limits form a subset of the topological space corresponding to the figure σ_α. For more details see [V]. Interesting is also the connection between the obtained space and the Cantor's disconti nuum.

2) We have found an equivalent of the promised form in th set-theoretical language except of the usage of the function $G: FN \leftrightarrow V_{FN}$. Using the section 0 we can translate the found eq

valent into the arithmetical language. In the last section we give the description of $\mathcal{K}$ using only the standard submodel and the standard system of the model.

§ 4. In this section we have to solve a problem typical for the beginning of the $\epsilon - \delta$ method in the calculus. Namely: How to find new definitions of notions easily definable with the help of infinitely large (infinitely small) quantities. The new definitions may be more complicated, may be less objective but must not use infinitely large or infinitely small quantities. In our case we consider the operations $-, \times$, $\mathcal{P}$ (power class in formally finite sets).

Definition 4.1: We put $\mathcal{S} = \{x \cap V_{FN}; x \in V\}$. We call $\mathcal{S}$ the standard system (of our nonstandard model V).

Remarks: 1) Remember that we suppose the powerset axiom (the whole induction schema) hence we are in accordance with the usual definition of the standard system.

2) Note that if our model is ω_1-saturated then $\mathcal{S} = \{X; X \subseteq V_{FN}\}$.

Lemma 4.2: If $F \in \mathcal{S}$ is a function then there is a function f such that $F = f \cap V_{FN}$. Especially: If $F \in \mathcal{S}$ & $\mathrm{dom}(F) = FN$ then there is a function f such that $F = f/FN$.

Proof: Let x be such that $F = x \cap V_{FN}$. Let $\varphi(\alpha, x)$ be the formula "$x \cap V_\alpha$ is a function". φ is satisfied for every $n \in FN$ hence there is an $\alpha \in N-FN$ such that φ is satisfied (see L.1.1.2)). It is sufficient to put $f = x \cap V_\alpha$.

Theorem 4.3: Let $\alpha \in N-FN$. 1) $\mathcal{K}_{u_\alpha}(u_1 \times u_2) =$

$= \{F; \mathrm{dom}(F) = FN \,\&\, (\exists F_1 \in \mathcal{K}_{u_\alpha^{\mathcal{U}_1}}) (\exists F_2 \mathcal{K}_{u_\alpha^{\mathcal{U}_2}})(\forall n \in FN)(F(n) =$
$= \langle F_1(n), F_2(n)\rangle)\}$. We also have $\mathcal{K}_{u_\alpha^{(\mathcal{U}_1 \times \mathcal{U}_2)}} \subseteq \mathcal{S}$.

2) $\mathcal{K}_{u_\alpha}^{\mathcal{P}\mathcal{U}} = \{F; \mathrm{dom}(F) = FN \,\&\, F \in \mathcal{S} \,\&\, (\forall n \in FN)(F(n) \in u_n^{\mathcal{U}}) \,\&$
$\&\, (\forall m, n \in FN)\ (m < n \Rightarrow (F(m) = {}_n f_m^{\mathcal{P}\mathcal{U}}\ (F(n))))\} \subseteq \mathcal{S}$.

Proof: 1) Let $F_{1/2} \in \mathcal{K}_{u_\alpha^{\mathcal{U}_{1/2}}}$. If $x_{1/2} \in u_\alpha^{\mathcal{U}_{1/2}}$ are such that $(\forall n \in FN)\ (F_{1/2}(n) = {}_\alpha f_n^{\mathcal{U}_{1/2}}\ (x_{1/2}))$ then $(\forall n \in FN)(F(n) =$ $= {}_n f_\alpha^{(\mathcal{U}_1 \times \mathcal{U}_2)}\ (\langle x_1, x_2\rangle) = \langle F_1(n), F_2(n)\rangle)$. We also have $F =$ $= \{\langle t, \beta\rangle; \beta \leq \alpha \,\&\, t = {}_\alpha f_\beta^{(\mathcal{U}_1 \times \mathcal{U}_2)}\ (\langle x_1, x_2\rangle)\} \cap V_{FN}$. On the other hand let $F \in \mathcal{K}_{u_\alpha^{(\mathcal{U}_1 \times \mathcal{U}_2)}}$. If $\langle x_1, x_2\rangle \in u_\alpha^{(\mathcal{U}_1 \times \mathcal{U}_2)}$ corresponds to F then $F_{1/2}$ corresponding to $x_{1/2}$ are members of $\mathcal{K}_{u_\alpha^{\mathcal{U}_{1/2}}}$.

2) $\subseteq$ is obvious. We prove $\supseteq$. Let F be a member of the righthand side of the considered equality. Let g be a function prolonging F. Let $\varphi(\alpha, g)$ be the formula $g(\alpha) \subseteq u_\alpha^{\mathcal{P}\mathcal{U}}$ $\&\, (\forall \beta;\ \beta < \alpha)(g(\beta) = {}_\alpha f_\beta^{\mathcal{P}\mathcal{U}}(g(\alpha)))$. This formula is satisfied for every $\alpha \in FN$ and hence there is a $\beta \in N\text{-}FN$ such that $\varphi(\beta, g)$. Hence $F \in \mathcal{K}_{u_\beta}^{\mathcal{P}\mathcal{U}}$.

Definition 4.4: 1) $\mathcal{K}_1 \otimes \mathcal{K}_2 = \{F; \mathrm{dom}(F) = FN \,\&$ $\&(\exists F_1 \in \mathcal{K}_1)(\exists F_2 \in \mathcal{K}_2)\ (\forall n \in FN)(F(n) = \langle F_1(n), F_2(n)\rangle)$.

2) For $F \in \mathcal{K}_{u_\alpha}^{\mathcal{U}}$ and $H \in \mathcal{K}_u^{\mathcal{P}\mathcal{U}}$ let us define $F \circledcirc H \equiv$ $\equiv (\forall n \in FN)(F(n) \in H(n))$.

3) For $\mathcal{K} \subseteq \mathcal{K}_{u_\alpha^{\mathcal{U}}}$ let us define $\mathcal{K}^{\textcircled{\mathcal{P}\mathcal{U}}} = \{H \in \mathcal{K}_{u_\alpha^{\mathcal{P}\mathcal{U}}};$ $(\forall F \circledcirc H)(F \in \mathcal{K})\}$.

<u>Theorem 4.5</u>: Let $\alpha \in N\text{-}FN$. 1) If $\sigma_{1/2} \subseteq u_\alpha^{\mathcal{U}}$ are figures in $\overset{\mathcal{U}}{\underset{\alpha}{=}}$ then $\mathcal{K}_{\sigma_1 - \sigma_2} = \mathcal{K}_{\sigma_1} - \mathcal{K}_{\sigma_2}$.

2) If $\sigma_{1/2} \subseteq u_\alpha^{\mathcal{U}_{1/2}}$ are figures in $\overset{\mathcal{U}_{1/2}}{\underset{\alpha}{=}}$ then

$\mathcal{K}_{\sigma_1 \times \sigma_2} = \mathcal{K}_{\sigma_1} \otimes \mathcal{K}_{\sigma_2}$.

3) If $\sigma \subseteq u_\alpha^{\mathfrak{N}}$ is a figure in $\overset{\mathfrak{N}}{\underset{\alpha}{\equiv}}$ then $\mathcal{K}_{\mathfrak{P}_\sigma} = \mathcal{K}_\sigma^{(\mathfrak{P}_\mathfrak{N})}$.

Proof: Only the case 3) is not obvious and hence we prove only this case. $\subseteq$ - let $H \in \mathcal{K}_{\mathfrak{P}_\sigma}$, let $y \in u_\alpha^{\mathfrak{P}_\mathfrak{N}}$ be an element corresponding to H $((\forall n \in FN)(H(n) = {}_\alpha f_n^{\mathfrak{P}_\mathfrak{N}}(y)))$, hence $y \subseteq \sigma$. Let $F \circledast H$ and let g be a prolongation of F. We know that for every $n \in FN$, $g(n) \in {}_\alpha f_n^{\mathfrak{P}_\mathfrak{N}}(y) \& (\forall \beta < n)(g(\beta) = {}_n f_\beta(g(n)))$, hence this formula is satisfied also for an infinite $\gamma \leq \alpha$ (see L.1.1.2)). Hence $g(\gamma) \in {}_\alpha f_\gamma^{\mathfrak{P}_\mathfrak{N}}(y) \subseteq \sigma$ and $F \in \mathcal{K}_\sigma$. $\supseteq$ - let $H \in \mathcal{K}_\sigma^{(\mathfrak{P}_\mathfrak{N})}$ and let $y \in u_\alpha^{\mathfrak{P}_\mathfrak{N}}$ be an element corresponding to H. We have to prove $y \subseteq \sigma$. Let x be an arbitrary element of y. Let $F \in \mathcal{K}_{u_\alpha^{\mathfrak{N}}}$ be a function corresponding to x. For any $n \in FN$ we have $F(n) \in H(n)$ as $F(n) = {}_\alpha f_n^{\mathfrak{N}}(x) \in {}_\alpha f_n^{\mathfrak{P}_\mathfrak{N}}(y) = H(n)$. Hence $F \circledast H$ and $F \in \mathcal{K}_\sigma$. Hence $x \in \sigma$ (see Th. 3.9).

Remarks: 1) The elimination of the predicate "to be infinitely large" (IL( )) is commonly used in the case of one quantification $(\exists \alpha, IL(\alpha))\ \varphi$ (Robinson's overspread lemma). The author has got to know the elimination method for two quantifiers $(\forall \alpha, IL(\alpha))(\exists \beta, IL(\beta))\ \varphi$ from P. Vopěnka [see Č 1]. It is apparent that the Cauchy's $\varepsilon - \delta$ expression of the notion of a limit is an implicit form of such an elimination. The equivalent for three quantifiers $(\exists \alpha, IL(\alpha))(\forall \beta, IL(\beta))(\exists \gamma, IL(\gamma))\ \varphi$ was found by A. Vencovská in the case of ω_1-saturated models. A help variable for real numbers (or for parts of natural numbers) appears in this equivalent.

2) An example, proving that help variables for natural numbers do not suffice, was found by P. Vopěnka in the case of ω_1-saturated models. Let us note here that if the predicate

"x is a member of the satisfactory relation on the standard submodel" (cf. § 0 for the possibility of the usage of the set-theoretical language) is a member of the standard system of the model, then it can be expressed in the form

$(\exists \alpha, \mathrm{IL}(\alpha))(\forall \beta, \mathrm{IL}(\beta))(\exists \gamma, \mathrm{IL}(\gamma))\varphi(\alpha, \beta, \gamma, x)$,

where φ is a normal formula. If we suppose that this predicate is equivalent to a formula having the prefix bounded to the standard submodel followed by a normal formula, then it is equivalent to a normal formula in the sense of the standard submodel in the case of elementary equivalence of the model and its standard submodel. An easy diagonal consideration proves that this is not possible.

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