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Commentationes Mathematicae Universitatis Carolinae, Vol. 24 (1983), No. 1, 1--15

Persistent URL: http://dml.cz/dmlcz/106200

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

24,1 (1983)

HOMOMORPHISM THEOREM FOR EQUATIONALLY PARTIAL ALGEBRAS

Horst REICHEL

__Abstract__: The paper demonstrates the main difference between total and equationally partial algebras. Whereas the homomorphic image of a homomorphism $f: \underline{A} \longrightarrow \underline{B}$ between total algebras can be rebuild from $\underline{A}$ and $f \subseteq A \times B$ by $\bar{f}(\underline{A}) \overset{\sim}{\longleftrightarrow} \underline{A}/\ker f$ without any knowledge of the algebraic structure of $\underline{B}$, in the case of equationally partial algebras the homomorphic image is in general only representable as a directed colimit of an infinite chain of iterated quotient algebras.

__Key-words__: Equational partiality, homomorphic image of partial algebras, partial quotient algebras.

Classification: 08A55

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1. __Introduction__. In the paper [1] we have introduced the notion of an equationally partial heterogeneous algebra and could see that this notion yields a proper calculus to describe for instance the behaviour of small categories. The paper [2] describes the construction of free and relatively free partial algebras and of colimits by the notion of a partial algebra defined by generators and relations. Both these papers could give the impression that there are no essential differences in the theories of total and of partial algebras.

The aim of this paper is to make visible an essential difference between the theories of total and of hierarchical equationally partial algebras. This difference is based on the well known theorem of homomorphisms. We state and prove a theorem of homomorphisms for hierarchical equationally partial algebras which encloses the theorem of homomorphisms of total algebras as a special case and which reflects the canonical factorization of functors between small categories described in [5].

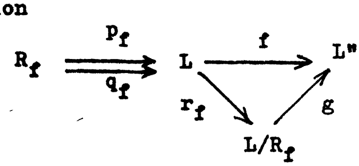
This paper represents parts of the author's thesis [6]. One of the basic concepts of the axiomatic categorical (universal) algebra of G. Richter (see [7]) is the following definition:

"The Theorem of Homomorphisms holds in a category $\mathbb{L}$, if

HOM 1: Every congruence $R \overset{p}{\underset{q}{\rightrightarrows}} L$ in $\mathbb{L}$ has a coequalizer $r: L \longrightarrow L/R$;

HOM 2: Every $\mathbb{L}$-morphism $f: L \longrightarrow L''$ has a kernel pair $p_f: R_f \longrightarrow L$, $q_f: R_f \longrightarrow L$;

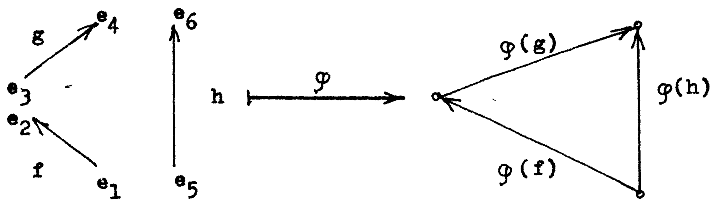
HOM 3: The homomorphism $g: L/R_f \longrightarrow L''$ in the canonical factorization

(1) 

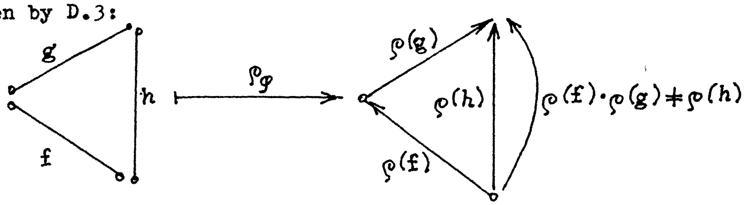
is a monomorphism for every $\mathbb{L}$-morphism $f: L \longrightarrow L''$."

If we check the validity of the Theorem of Homomorphisms in the category **CAT** of small categories, then we see that HOM 1 and HOM 2 are evidently satisfied but HOM 3 is not true. This can be seen by the following counter-example.

Let $\varphi: C \longrightarrow C''$ be the functor given by the following diagram:

(2) 

The coequalizer $\rho_\varphi : C \longrightarrow Q$ of the kernel pair of $\varphi : C \longrightarrow C''$ is given by D.3:

(3) 

The canonical functor $\gamma_\varphi : Q_\varphi \longrightarrow C''$ with $\varphi = \rho_\varphi \cdot \gamma_\varphi$ is given by $\gamma_\varphi(\rho(g)) = \varphi(g)$, $\gamma_\varphi(\rho(f)) = \varphi(f)$, $\gamma_\varphi(\rho(h)) = \varphi(h)$, and $\gamma_\varphi(\rho(f)\cdot\rho(g)) = \varphi(h)$, hence $\gamma_\varphi : Q_\varphi \longrightarrow C''$ is not a monomorphism. But, condition HOM 3 is true for the canonical factor $\gamma_\varphi : Q_\varphi \longrightarrow C''$. In [5] it was proved that condition HOM 3 is true for every canonical factor $\gamma_\varphi : Q \longrightarrow C''$ starting with an arbitrary functor $\varphi : C \longrightarrow C''$ between small categories.
This example shows that the behaviour of small categories and functors between small categories cannot be studied in the axiomatic categorical universal algebra of G. Richter, although this approach works very well for total heterogeneous algebras.

In the case of small categories the factorization of a functor $\varphi : C \longrightarrow C''$ can be iterated and the resulting factor after the second step is then a monomorphism, and so there is no further non-trivial factorization of this monomorphism. In the sequel we show that for hierarchical equationally partial algebras this iteration does not terminate after finitely many steps. However, we can prove that the set-theoretical colimit of the chain of iterated quotients is isomorphic to the homomorphic image. The case of total algebras is characterized by the fact that the chain of iterated quotients terminates after the first step.

2. <u>The chain of iterated quotients</u>. We assume that $\Theta = (S, \alpha : \Sigma \longrightarrow S^* \times S, (\mathrm{def}\ \sigma \mid \sigma \in \Sigma))$ is any hierarchical equationally partial heterogeneous operator domain (hep-domain) and that $\mathcal{O}$ is any set of elementary implications (in the sense of [1] and [2]).

<u>Definition 2.1</u>. Let A be any $(\Theta, \mathcal{O})$-algebra and $\wp = (\wp_s \mid s \in S)$ an S-indexed family of binary relations $\wp_s \subseteq A_s \times A_s$, $s \in S$. A homomorphism $r : A \longrightarrow Q$ between $(\Theta, \mathcal{O})$-algebras is called a <u>natural homomorphism</u> to $\wp$, and Q is called a <u>quotient</u> of A to $\wp$ if

(1) For all $s \in S$, $(x,y) \in \wp_s$ holds $r_s(x) = r_s(y)$;

(2) For every homomorphism $f : A \longrightarrow B$ between $(\Theta, \mathcal{O})$-algebras with $f_s(x) = f_s(y)$ for all $s \in S$, $(x,y) \in \wp_s$ there is exactly one homomorphism $f^* : Q \longrightarrow B$ with $f = r \circ f^*$.

Different to the same notion of total algebras here the notion of a natural homomorphism and of a quotient is only unique up to isomorphisms. Since it is not possible to define on the sets of classes of congruent elements a $(\Theta, \mathcal{O})$-algebra by operating with representatives, the notion of a quotient cannot be introduced in a more restrictive way. The impossibility of defining a quotient-category by operating with representatives of congruence-classes is demonstrated by the small category C from the introduction and by the kernel-congruence of the functor $\varphi : C \longrightarrow C''$ (see Diagram 2). But, the functor $\wp_\varphi : C \longrightarrow Q_\varphi$ as given by Diagram 3 is natural to the kernel-congruence of $\varphi : C \longrightarrow C''$, and therefore Q_φ is a quotient of C to the kernel-congruence of $\varphi : C \longrightarrow C''$.

For every homomorphism $f : A \longrightarrow B$ between $(\Theta, \mathcal{O}\!\!\mathcal{L})$-algebras we denote by $\ker f = ((\ker f)_s \mid s \in S)$ the kernel-congruence of $f : A \longrightarrow B$, i.e.

$$(\ker f)_s = \{(x,y) \in A_s \times A_s \mid f_s(x) = f_s(y)\}, \; s \in S.$$

It is easy to see that ker f is a carrier of a $(\Theta, \mathcal{O}\!\!\mathcal{L})$-subalgebra of $A \times A$ and that $p_f : \ker f \longrightarrow A$, $q_f : \ker f \longrightarrow A$ with $(p_f)_s((x,y)) = x$, $(q_f)_s((x,y)) = y$ for all $s \in S$, $(x,y) \in (\ker f)_s$ are homomorphisms. In the terminology of categorical algebra (p_f, q_f) is called the kernel-pair of $f : A \longrightarrow B$.

Proposition 2.2. Let A be any $(\Theta, \mathcal{O}\!\!\mathcal{L})$-algebra and $\wp = (\wp_s \mid s \in S)$ an S-indexed family of binary relations $\wp_s \subseteq A_s \times A_s$, $s \in S$. Then there is a homomorphism $q : A \longrightarrow Q$ between $(\Theta, \mathcal{O}\!\!\mathcal{L})$-algebras natural to $\wp = (\wp_s \mid s \in S)$.

Proof: The existence of $q : A \longrightarrow Q$ is demonstrated by use of the construction $F(\mathcal{O}\!\!\mathcal{L}, G)$ of [2], i.e. by the construction of a $(\Theta, \mathcal{O}\!\!\mathcal{L})$-algebra freely generated by an appropriate set of equations (G/v), $v : X \longrightarrow S$. We define

$X_A = \{(a,s) \mid a \in A_s, s \in S\}$, $v_A : X_A \longrightarrow S$ by $v_A(a,s) = s$ and

$(G_\wp / v_A) = \{(x,s) = (y,s) \mid s \in S, (x,y) \in \wp_s\}$.

Let $q_\wp : A \longrightarrow F(\mathcal{O}\!\!\mathcal{L}, G_\wp)$ be given by $(q_\wp)_s(a) = [(a,s), G_\wp] \in F(\mathcal{O}\!\!\mathcal{L}, G_\wp)_s$ for all $s \in S$, $a \in A_s$.

Condition (1) of Definition 2.1 is evidently satisfied by $q_\wp : A \longrightarrow F(\mathcal{O}\!\!\mathcal{L}, G_\wp)$. If $f : A \longrightarrow B$ is any homomorphism between

$(\Theta, \mathcal{O}\!\!\iota)$-algebras with $f_s(x) = f_s(y)$ for all $s \in S$, $(x,y) \in \rho_s$, then $f \in B_{v_A}$ is a solution of (G_ρ / v_A) in B such that exactly one homomorphism $f^* : F(\mathcal{O}\!\!\iota, G_\rho) \longrightarrow B$ with $f^*([(a,s), G_\rho]) = f_s(a)$ for all $s \in S$, $a \in A_s$ exists, i.e. $q_\rho \circ f^* = f$.

<u>Definition 2.3</u>. A finite or countable sequence $((f_i, q_i, f_{i+1}) \mid i \in I)$, with $I = \{0,1,\ldots,n\}$ or $I = \{0,1,\ldots\}$, of homomorphisms between $(\Theta, \mathcal{O}\!\!\iota)$-algebras is called a <u>chain of iterated quotients</u> of $f_0 : A \longrightarrow B$ if

(1) $f_i = q_i \circ f_{i+1}$ for all $i \in I$;

(2) Every $q_i : Q_i \longrightarrow Q_{i+1}$, $i \in I$, is not an isomorphism;

(3) $q_i : Q_i \longrightarrow Q_{i+1}$ is natural to ker f_i for every $i \in I$.

If $I = \{0,1,\ldots,n\}$ then n is called the length of the chain of iterated quotients of $f_0 : A \longrightarrow B$.

The theorem of homomorphisms of total algebras implies that every chain of iterated quotients of a homomorphism $f_0 : A \longrightarrow B$ between total algebras is of length one. In [5] it is shown that any chain of iterated quotients of a functor between small categories is at most of length two.

<u>Theorem 2.4</u>. There are chains of iterated quotients of homomorphisms between equationally partial algebras of infinite length.

Proof by construction of an infinite chain: Let us consider the following hep-domain

Θ = <u>sorts</u> N,Q

<u>oprs</u> $n : \longrightarrow N$

$s : N \longrightarrow N$

$m : Q \longrightarrow N$

$r : N$ <u>iff</u> $s(x) = n \longrightarrow Q$

and Θ

and the Θ-algebras A^o, B with

$A^o_N = \{(0,x) \mid x \in \{0,1,\dots\} = \mathbb{N}\}$,

$A^o = \emptyset$,

$n_A o = (0,0)$, $s_A o(0,x) = (0,x+1)$ for all $x \in \mathbb{N}$, and

$m_A o$, $r_A o$ are operations with empty domains,

$B_N = \mathbb{N} \times \mathbb{N}$, $B_Q = \mathbb{N}$, $n_B = (0,0)$, $s_B(x,y) = (0,0)$ for all $(x,y) \in B_N$,

$m_B(x) = (x+1,0)$ for all $x \in \mathbb{N} = B_Q$, dom $r_B = B_N$ and

$r_B(x,y) = x$ for all $(x,y) \in B_N$.

It is easy to see that A^o is an initial Θ-algebra, i.e. for every Θ-algebra B' there is exactly one homomorphism $f: A^o \longrightarrow B'$. The uniquely determined homomorphism $f: A^o \longrightarrow B$ is then defined by $f_N(0,x) = (0,0)$ for all $(0,x) \in A^o_N$, and $f_Q: \emptyset \longrightarrow B_Q$, so that $(\ker f)_N = A^o_N \times A^o_N$, $(\ker f)_Q = A^o_Q \times A^o_Q$.

According to the proof of Proposition 2.2 we construct a homomorphism $q_o: A^o \longrightarrow A^1$ by setting

$A^1_N = \{(0,0),(1,0),\dots,(1,x),\dots \mid x \in \mathbb{N}\}$, $A^1_Q = \{0\}$, $n_{A^1} = (0,0)$,

$s_{A^1}(0,0) = (0,0)$, $s_{A^1}(1,x) = (1,x+1)$ for all $x \in \mathbb{N}$,

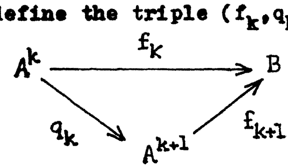
dom $r_{A^1} = \{(0,0)\}$, $r_{A^1}(0,0) = 0$, $m_{A^1}(0) = (1,0)$,

$(q_o)_N(0,x) = (0,0)$ for all $x \in \mathbb{N}$, and $(q_o)_Q: \emptyset \longrightarrow \{0\}$.

This homomorphism is natural to ker f. The uniquely determined factor $f_1: A^1 \longrightarrow B$ is given by $(f_1)_N(j,0) = (j,0)$ for $j=0,1$, $(f_1)_N(1,x) = (0,0)$ for all $x \geq 1$, and $(f_1)_Q(0) = 0$.

In general, we define the triple (f_k, q_k, f_{k+1}) for $k \geq 1$, see Diagram 4

(4)



by

$A^k_N = \{(0,0),(1,0),\ldots,(k,0),(k,1),\ldots,(k,x),\ldots \mid x \in \mathbb{N}\}$,

$A^k_Q = \{0,1,\ldots,k-1\}$,

$n_{A^k} = (0,0)$, $s_{A^k}(0,0) = s_{A^k}(1,0) = \ldots = s_{A^k}(k-1,0) = (0,0)$,

$s_{A^k}(k,x) = (k,x+1)$ for all $x \in \mathbb{N}$,

$\mathrm{dom}\ r_{A^k} = \{(0,0),(1,0),\ldots,(k-1,0)\}$, $r_{A^k}((j,0)) = j$ for $j = 0,\ldots,k-1$,

$m_{A^k}(j) = (j+1,0)$ for $j = 0,1,\ldots,k-1$,

the homomorphism $f_k:A^k \longrightarrow B$ by

$(f_k)_N(j,0) = (j,0)$ for $j = 0,1,\ldots,k$,

$(f_k)_n(k,x) = (0,0)$ for $x \neq 0$, $x \in \mathbb{N}$,

$(f_k)_Q(j) = j$ for $j = 0,1,\ldots,k-1$,

and the homomorphism $q_k:A^k \longrightarrow A^{k+1}$ by

$(q_k)_N(j,0)$ for $j = 0,1,\ldots,k$,

$(q_k)_N(k,x) = (0,0)$ for $x \neq 0$, $x \in \mathbb{N}$,

$(q_k)_Q(j) = j$ for $j = 0,1,\ldots,k-1$.

Simple calculations show that (f_k,q_k,f_{k+1}), $k = 0,1,2,\ldots$, is really a chain of iterated quotients of $f_o = f:A^o \longrightarrow B$, and evidently it is an infinite one.

3. The Theorem of Homomorphisms. In the case of total algebras for every homomorphism $f:A \longrightarrow B$ the set-theoretical image is always a carrier of a subalgebra of B, and according to the Special Theorem of Homomorphisms this homomorphic image is isomorphic to the quotient A/ker f.

For partial algebras the very notion of the homomorphic image is not so evident, since the set-theoretical image in general is not the carrier of a subalgebra. For the notion of a subalgebra of an equationally partial algebra see [1].

Let us consider the homomorphism $f: A^{o} \longrightarrow B$ as defined in the proof ot Theorem 2.4, and look for the homomorphic image. At first we will define this notion for equationally partial algebras.

<u>Definition 3.1</u>. Let Θ be a hep-domain and $f: A \longrightarrow B$ a homomorphism between Θ-algebras.

$$f(A) = ((f(A)_s \mid s \in S), (\sigma_{f(A)} \mid \sigma \in \Sigma))$$

denotes the smallest Θ-subalgebra of B with $f_s(x) \in f(A)_s$ for all $s \in S$, $x \in A_s$. $f(A)$ is said to be the <u>homomorphic image</u> of $f: A \longrightarrow B$.

Returning to the example $f: A^{o} \longrightarrow B$ we get

$$f(A^{o})_N = \{(x,0) \mid x \in \mathbb{N}\} \subseteq \mathbb{N} \times \mathbb{N} = B_N \text{ and } f(A^{o})_Q = \mathbb{N} = B_Q.$$

The homomorphic image $f(A^{o}) \subseteq B$ is infinite with respect to both sorts N and Q, although the set-theoretical image is one element only, namely $f_N(A^{o}_N) = \{(0,0)\}$, $f_Q(A^{o}_Q) = \emptyset$.

Now we are going to look for significant relations between the homomorphic image of a homomorphism $f: A \longrightarrow B$ and its maximal chain of iterated quotients.

The example of the proof of Theorem 2.4 gives us the idea. With increasing k the iterated quotient A^{k} becomes more and more similar to the homomorphic image.

To prove this conjecture in general we study at first the construction of a colimit of a directed diagram of $(\Theta, \mathcal{A})$-algebras, where $\mathcal{A}$ is any set of elementary implications. Let $(J, \preceq)$ be a directed partially ordered set and

$\Phi : (J, \leq) \longrightarrow \underline{\underline{ALG(\Theta, \mathcal{A})}}$ a directed diagram in the category of $(\Theta, \mathcal{A})$-algebras, i.e., for every $j \in J$ $\Phi(j)$ is a $(\Theta, \mathcal{A})$-algebra, for every pair (i,j) with $i \leq j$ $\Phi(i,j) : \Phi(i) \longrightarrow \Phi(j)$ is a homomorphism, and for every pair $i_1, i_2 \in J$ there is at least one $k \in J$ with $i_1 \leq k$ and $i_2 \leq k$.

At first we build up the set-theoretical colimit from the given diagram by forming the S-indexed family $(L_s^* \mid s \in S)$ with $L_s^* = \{(x,j) \mid j \in J,\ x \in \Phi(j)_s\}$ for every $s \in S$, by defining an S-indexed family of equivalence relations $(\equiv_s \mid s \in S)$ with $(x, j_1) \equiv_s (y, j_2)$ if and only if there is a $k \in J$ with $j_1 \leq k$,

$$j_2 \leq k \text{ and } \Phi(j_1,k)(x) = \Phi(j_2,k)(y),$$

and by setting $L_s = L_s^* / \equiv_s$ for every $s \in S$. By $[x,j] \in L_s$ we denote an arbitrary equivalence-class.

As next we extend this S-indexed family of sets to a $(\Theta, \mathcal{A})$-algebra $L = ((L_s \mid s \in S), (\sigma_L \mid \sigma \in \Sigma))$. Let $\sigma : s_1 \ldots s_n \longrightarrow s$ be any operator. The domain of σ_L is defined by

$([x_1, j_1], \ldots, [x_n, j_n]) \in \operatorname{dom} \sigma_L$ iff there is a $k \in J$ and there are

$$y_1, \ldots, y_n \text{ with } j_1 \leq k, \ldots, j_n \leq k,\ [x_1, j_1] = [y_1, k], \ldots, [x_n, j_n] = [y_n, k], \text{ and } (y_1, \ldots, y_n) \in \operatorname{dom} \sigma_{\Phi(k)}.$$

In this case we set

$$\sigma_L([x_1, j_1], \ldots, [x_n \cdot j_n]) = [\sigma_{\Phi(k)}(y_1, \ldots, y_n), k].$$

Since $\Phi : (J, \leq) \longrightarrow \underline{\underline{ALG(\Theta, \mathcal{A})}}$ is a directed diagram one can easily prove that the domain of σ_L and that the value $\sigma_L([x_1, j_1], \ldots, [x_n, j_n])$ is defined independently from the choice of representatives, and that L is really a Θ-algebra.

Using the finiteness of the premise of any elementary implication $(H \longrightarrow t = t'/v) \in \mathcal{A}$ and again the fact that Φ is a directed diagram one can prove without any problems that every axiom out of $\mathcal{A}$ is satisfied !

Finally, we remark that for every $j \in J$ the S-indexed family of mappings $g(j) = (g(j)_s : \Phi(j)_s \longrightarrow L_s \mid s \in S)$, with $g(j)_s(x) = [x,j]$ for every $s \in S$, $x \in \Phi(j)_s$, is a homomorphism $g(j) : \Phi(j) \longrightarrow L$. The cone $(g(j) : \Phi(j) \longrightarrow L \mid j \in J)$ is a colimit of the diagram $\Phi : (J, \leq) \longrightarrow \underline{\underline{ALG(\Theta, \mathcal{O}\!\mathcal{l})}}$. This statement follows from the fact that the forgetful functor from $\underline{\underline{ALG(\Theta, \mathcal{O}\!\mathcal{l})}}$ to the underlying set-category, whose objects are S-indexed families of sets, preserves directed colimits (see [4],[3]).

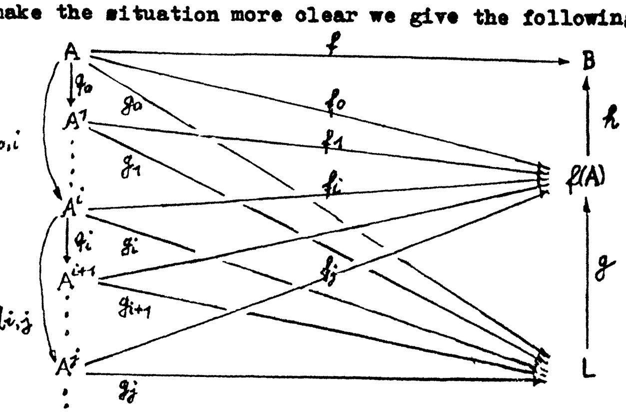
3.2. Theorem of Homomorphisms: Let Θ be any hep-domain and $\mathcal{O}\!\mathcal{l}$ any set of elementary implications. For every homomorphism $f : A \longrightarrow B$ between $(\Theta, \mathcal{O}\!\mathcal{l})$-algebras the homomorphic image $f(A)$ is isomorphic to the colimit of the maximal chain (f_i, q_i, f_{i+1}), $i = 0,1,\ldots,$ of iterated quotients of $f_0 : A \longrightarrow f(A)$, where $f = f_0 \circ h$, and where $h : f(A) \longrightarrow B$ is the inclusion of the homomorphic image.

Proof: Above all we introduce the following abbreviation: For all $i,j \in \mathbb{N}$ with $i \leq j$ and $i \neq j$ let

$$q_{i,j} = q_i \circ q_{i+1} \circ \cdots \circ q_{i+(j-i)} : A^i \longrightarrow A^j.$$

To make the situation more clear we give the following diagram

(5)



In this diagram denotes $L \in ALG(\Theta, \mathcal{O}\!\mathcal{l})$ together with the family $(g_j : A^j \longrightarrow L \mid j \in \mathbb{N})$ of canonical injections the colimit of the chain of iterated quotients of $f_0 : A \longrightarrow f(A)$. Since $(f_j : A^j \longrightarrow f(A) \mid j \in \mathbb{N})$ is also a cone for the chain of iterated quotients, there exists exactly one homomorphism $g : L \longrightarrow$ $\longrightarrow f(A)$ with $g_1 \cdot g = f_j$ for all $j \in \mathbb{N}$.

We prove the Theorem by showing that this homomorphism $g : L \longrightarrow$ $\longrightarrow f(A)$ is an isomorphism. Because of the hierarchy of the hep-domain Θ it is sufficient to show that $g : L \longrightarrow f(A)$ is bijective (see Theorem 2.4 of [11]).

We start with the injectivity of $g : L \longrightarrow f(A)$. Let be $s \in S$, $x, y \in L_s$ with $g_s(x) = g_s(y)$. Since L is a colimit of a directed diagram we can use the preceding construction of such a colimit. Hence there are $i \in \mathbb{N}$, $x', y' \in A^i_s$ with $(g_i)_s(x') = x$ and $(g_i)_s(y') = y$. Because of $f_i = g_i \circ g$ it follows

$$(f_i)_s(x') = (g_i \circ g)_s(x') = g_s((g_i)_s(x')) = g_s(x) = g_s(y) = (f_i)_s(y').$$

Since q_i is natural to ker f_i, this equality implies $(q_i)_s(x') = (q_i)_s(y')$. Due to this we see $x = (g_i)_s(x') =$ $= (q_i \circ g_{i+1})_s(x') = (g_{i+1})_s((q_i)_s(x')) = (g_{i+1})_s((q_i)_s(y')) =$ $= (q_i \circ g_{i+1})_s(y') = (g_i)_s(y') = y$, i.e. $g : L \longrightarrow f(A)$ is an injective homomorphism.

To prove the surjectivity of $g : L \longrightarrow f(A)$ we recall that for every $s \in S$ an element $b \in B_s$ is an element of $f(A)_s$ iff there is a $w : \{1, 2, \ldots, n\} \longrightarrow S$, a term $t \in T(\Omega, w)_s$ and an assignment $a \in B_w$ with $b = t_B(a)$ and $a(j) \in f_{w(j)}(A_{w(j)})$ for every $j \in \{1, 2, \ldots, n\}$, i.e. every component $a(j)$ is an element of the set-theoretical image $(f_s(A_s) \mid s \in S)$. Hence there are $i \in \mathbb{N}$, $\bar{a}_1 \in A^i_{w(1)}, \ldots, \bar{a}_n \in A^i_{w(n)}$ with $(f_i)_{w(j)}(\bar{a}_j) = a(j)$

for every $j \in \{1,2,\dots,n\}$ such that $t_{A^i}(\bar{a}_1,\dots,\bar{a}_n)$ exists. Therefore, for the element $\bar{a} = t_{A^i}(\bar{a}_1,\dots,\bar{a}_n)$ holds $(f_i)_s(\bar{a}) =$
$= (f_i)_s(t_{A^i}(\bar{a}_1,\dots,\bar{a}_n)) = t_B((f_i)_w(\bar{a}_1,\dots,\bar{a}_n)) = t_B(a(1),\dots$
$\dots,a(n)) = t_B(a) = b$, $b = (f_i)_s(\bar{a}) = (g_i \circ g)_s(\bar{a}) =$
$= g_s((g_i)_s(\bar{a}))$, i.e. $g: L \longrightarrow f(A)$ is surjective.

An important consequence of this theorem is the fact that it is not possible to reconstruct in a finite manner the global behaviour of the homomorphic image $f(A)$ of a homomorphism $f: A \longrightarrow B$ between equoids from the knowledge of the equoid A and of the set-theoretical mappings $(f_s : A_s \longrightarrow B_s \mid s \in S)$.
However, the local computation with finitely many arguments and operations in $f(A)$ can be described in a finite manner by the equoid A and the S-indexed family of mappings $(f_s : A_s \longrightarrow$
$\longrightarrow B_s \mid s \in S)$. But, this description may be arbitrarily complicated, depending from the index $i \in \mathbb{N}$ of the iterated equoid A^i to which the given finitary situation in $f(A)$ may be reduced.

It is easy to see that any two maximal chains $(f_i, q_i, f_{i+1})_{i \in I}$, $(\bar{f}_j, \bar{q}_j, \bar{f}_{j+1})_{j \in J}$ of iterated quotients of one and the same homomorphism, i.e. $f_0 = \bar{f}_0$, have the same length, i.e. $I = J$, and that they are isomorphic, i.e. there are isomorphisms h_i, $i \in I$ with $h_i \cdot q_i = \bar{q}_i \circ h_i$ for every $i \in I$, and therefore $h_i \circ f_i = \bar{f}_i$ for every $i \in I$.

Due to this we introduce the so-called <u>homomorphic number</u>, $\hom(f)$ of a homomorphism $f: A \longrightarrow B$ as the length of a maximal chain of iterated quotients of $f: A \longrightarrow B$. If this chain is infinite we set $\hom(f) = \infty$. For a hep-domain Θ and a set $\mathcal{A}$

of elementary implications, i.e. for a hep-theory $(\Theta, \mathcal{A})$, we define hom$(\Theta, \mathcal{A})$ to be the supremum of all hom(f) of homomorphisms between $(\Theta, \mathcal{A})$-algebras.

Hep-theories with a finite homomorphic number are of some interest, because of the finitary representability of the homomorphic image f(A) with respect to the equoid A and the S-indexed family $(f_s : A_s \longrightarrow B_s \mid s \in S)$ of mappings. We guess that it is recursively undecidable whether hom(T) is finite or not for any hep-theory T. It seems to be very interesting to look for conditions being necessary or sufficient for the finiteness of hom(T). Up to now we do not know any such condition.

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(Oblatum 8.7. 1982)