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SHIFTINGS OF THE HORIZON

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Abstract: We investigate interpretations of the alternative set theory in this theory which preserve sets and the predicate $\in$.

Key words: Alternative set theory, interpretation, finite natural numbers, endomorphic universe, standard extension, revealment.

Classification: 03E70, 03H99

In the alternative set theory (AST) we try to describe our understanding of the real world. Sets are considered as formalizations of collections we really meet, classes are formal counterparts of our idealizations and generalizations. Following this motivation, the interpretations of AST in AST which preserve sets and the predicate $\in$ are very important - they describe our different approaches to the real world; such interpretations will be called shiftings of view.

Collections converging toward the horizon of our observation ability (describing unlimited processes) are formalized in AST by countable classes. Hence countability captures in AST the notion of distance of the horizon. Among shiftings of view there are interpretations which do not preserve countability and therefore it is natural to call interpretations of this type shiftings of the horizon.

In the first section we shall see that in AST with the schema of the choice we are able to construct shiftings of the horizon. In § 2 we describe properties of shiftings of view. In particular, we describe the collection of all classes FN^* where $*$ is a shifting of the horizon which do not change properties of finite natural numbers.

We shall use notions and results of [V], [S-V 1] and [S-V 2] only.

§ 1. Let T be a theory stronger than AST. An interpretation $*$ of AST in T is called a shifting of view in T iff $T \vdash (\forall x)\, Cls^*(x) \,\&\, (\forall X^*, Y^*)(X^* \in^* Y^* \equiv X^* \in Y^*)$. A shifting of view $*$ is called a shifting of the horizon in T, if moreover $T \vdash FN^* \neq FN$ (FN being the class of all finite natural numbers).

If $*$ is a shifting of view in T then $T \vdash Set^*(X) \equiv$ $\equiv Set(X)$. In fact, Set(X) implies $(\exists y)(X \in y)$ and therefore $(\exists y)(Cls^*(y) \,\&\, X \in^* y)$; on the other hand if $Set^*(X)$ holds then we get $(\exists X)(Cls^*(Y) \,\&\, X \in^* Y)$ from which the formula $(\exists Y)\, X \in Y$ follows.

More complexly we can consider an interpretation together with an operation $\mathcal{G}$. We define that a pair $*$, $\mathcal{G}$ is a transformation of view (of the horizon respectively) in T iff $*$ is a shifting of view (of the horizon respectively) in T and $\mathcal{G}$ is an operation defined in T in such a way that for every X, $\mathcal{G}(X)$ is defined and it is a $*$-class and moreover for every (even nonnormal) formula Φ we have $T \vdash \Phi(X_1, \ldots$ $\ldots, X_n) \equiv \Phi^*(\mathcal{G}(X_1), \ldots, \mathcal{G}(X_n))$.

The following results show that we are able to construct a transformation of the horizon in AST + A 62 (with convenient fixations). The existence of a translation of the horizon in AST itself remains as an open problem.

Let AST^+ denote the theory AST with the following additional assumptions:

a) schema of choice A 62, i.e. we accept the axiom

$(\forall n \in FN)(\exists X)\Phi(n,X) \rightarrow (\exists Z)(\forall n \in FN)\Phi(n,Z''\{n\})$

for every (metamathematical) formula Φ.

b) F is an endomorphism and rng(F) = A (cf. § 2 ch. V [V])

c) Ex is a standard extension on A (cf. [S-V 1])

d) $A[d] = V$ (cf. [S-V 1])

e) $d \in (Ex(FN) - FN)$

Let us note that we require the last assumption for simplicity only since for every $\bar{d} \in V - A$ there is a countable X with $\bar{d} \in (Ex(X) - X)$.

If Φ is a formula then Φ^A is the formula resulting from Φ by restriction of all quantifiers binding set variables to elements of A and all quantifiers binding class variables to subsets of A.

Let $*$ be the interpretation determined by formulae

$Cls^*(X) \equiv (\exists Y \subseteq A)\ X = Ex(Y)''\{d\}$

$X^* \in^* Y^* \equiv X^* \in Y^*$.

The following statement is a variant of Loš's theorem.

<u>Metatheorem</u>. For every formula Φ we can prove in AST^+

$(\forall Y_1,\ldots,Y_k \subseteq A)(\Phi^*(Ex(Y_1)''\{d\},\ldots,Ex(Y_k)''\{d\}) \equiv$

$\equiv d \in Ex(\{n; \Phi^A(Y_1{}'' \{n\},\dots,Y_k{}'' \{n\})\}))$.

Demonstration. According to § 2 [S-V 1] we have $Ex(Y_1)'' \{d\} \in^* Ex(Y_2)'' \{d\} \equiv d \in \{\alpha; Ex(Y_1)'' \{\alpha\} \in$ $\in Ex(Y_2)'' \{\alpha\}\} \equiv d \in Ex(\{\alpha \in A; Y_1{}'' \{\alpha\} \in Y_2{}'' \{\alpha\}\})$. Moreover $d \in Ex(FN)$ and thus $Ex(Y_1)'' \{d\} \in^* Ex(Y_2)'' \{d\} \equiv d \in$ $\in Ex(FN) \cap Ex(\{\alpha \in A; Y_1{}'' \{\alpha\} \in Y_2{}'' \{\alpha\}\}) \equiv d \in Ex(\{n;$ $Y_1{}'' \{n\} \in Y_2{}'' \{n\}\})$. The induction step for & and $\neg$ is trivial because of $d \in (Ex(X) \cap Ex(Y)) \equiv d \in Ex(X \cap Y)$ and $d \in Ex(X) \equiv$ $\equiv d \notin Ex(FN - X)$ for every $X, Y \subseteq FN$. If $((\exists X)\, \Phi\, (X, Ex(Y_1)'' \{d\},\dots,Ex(Y_k)'' \{d\}))^*$ then there is $Y \subseteq A$ such that $\Phi^*(Ex(Y)'' \{d\}, Ex(Y_1)'' \{d\},\dots,Ex(Y_k)'' \{d\})$ and using the induction hypothesis we get $d \in Ex(\{n; \Phi^A(Y'' \{n\},$ $Y_1{}'' \{n\},\dots,Y_k{}'' \{n\})\}) \subseteq Ex(\{n; ((\exists X)\, \Phi\, (X, Y_1{}'' \{n\},\dots$ $\dots, Y_k{}'' \{n\}))^A\})$. On the other hand let us suppose that $d \in Ex(\{n; ((\exists X)\, \Phi\, (X, Y_1{}'' \{n\},\dots,Y_k{}'' \{n\}))^A\}) = Ex(\{n;$ $(\exists X \subseteq A)\, \Phi^A(X, Y_1{}'' \{n\},\dots,Y_k{}'' \{n\})\})$. Thus, by A 62 (FN being a subclass of A) there is $Y \subseteq A$ with $\{n; (\exists X \subseteq A)\, \Phi^A(X, Y_1{}'' \{d\},\dots$ $\dots, Y_k{}'' \{n\})\} = \{n; \Phi^A(Y'' \{n\}, Y_1{}'' \{n\},\dots,Y_k{}'' \{n\})\}$ and at the end we obtain using the induction hypothesis $\Phi^*(Ex(Y)'' \{d\}, Ex(Y_1)'' \{d\},\dots,Ex(Y_k)'' \{d\})$, from which $((\exists X)\, \Phi\, (X, Ex(Y_1)'' \{d\},\dots,Ex(Y_k)'' \{d\}))^*$ follows.

Let $\mathcal{G}$ denote the operation defined by $\mathcal{G}(X) = Ex(F''X)$.

<u>**Metatheorem**</u> **.** The pair $*$, $\mathcal{G}$ is a shifting of the horizon in AST$^+$ and moreover in AST$^+$ we can prove $FN^* =$ $= Ex(FN)$.

Demonstration. Proceeding in AST$^+$ we have at first to prove $(\forall x)\, Cls^*(x)$. If x is given then according to the assumption $A[d] = V$, we can choose $f \in A$ with $f(d) = x$ and define $Y = \{\langle y,n\rangle ;\ n \in FN \,\&\, y \in A \cap f(n)\}$. We have $(\forall y \in A)(\forall n \in FN)(\langle y,n\rangle \in Y \equiv y \in f(n))$ and hence by the definition of standard extension we obtain $(\forall y)(\forall \alpha \in Ex(FN))(\langle y,\alpha\rangle \in Ex(Y) \equiv y \in f(\alpha))$ which implies $Ex(Y)''\{d\} = f(d)$. We have proved $(\exists Y \subseteq A)(x = Ex(Y)''\{d\})$ and therefore x is a $*$-class.

By the second theorem of § 1 ch. V [V] we have $\Phi(X_1,\dots,X_k) \equiv \Phi^A(F''X_1,\dots,F''X_k)$. Defining $Y_1 = (F''X_1) \times \times FN, \dots, Y_k = (F''X_k) \times FN$, we obtain the following equivalences according to the last metatheorem: $\Phi^*(Ex(F''X_1),\dots \dots, Ex(F''X_k)) \equiv \Phi^*(Ex(Y_1)''\{d\},\dots,Ex(Y_k)''\{d\}) \equiv \equiv d \in Ex(\{n;\ \Phi^A(F''X_1,\dots,F''X_k)\}) \equiv \Phi^A(F''X_1,\dots,F''X_k) \equiv \equiv \Phi(X_1,\dots,X_k)$.

Furthermore putting $Y = FN \times FN$, $Y''\{n\}$ is the smallest complete proper subsemiset of N for every $n \in FN$. Thus $FN^* = Ex(Y)''\{d\} = Ex(FN)$. It remains to realize that $Ex(FN) \neq FN$ for every standard extension Ex (cf. § 2 [S-V 1]).

If T is stronger than AST + A 62 and B, C are constants such that in T it is provable that B is a revealment of C (cf. [S-V 2]) then we can fix constants F, A, d and a standard extension Ex in such a way that all properties (a) - (e) are provable and $Ex(F''C) = B$ (cf. § 2 [S-V 2]). Hence we are able to construct a transformation of the horizon $*$, $\mathcal{G}$ in T (with definitions in question) so that the equality $\mathcal{G}(C) = B$ is provable.

By the second theorem of § 1 ch. V [V] we get that if F is a constant denoting an automorphism in T (stronger than AST) then the pair of the identical interpretation and of the operation $\mathcal{G}(X) = F''X$ is a transformation of view which is no transformation of the horizon.

§ 2. In the previous section we have constructed some shiftings of view, now we are going to show some results restricting the existence of shiftings of view, in particular we shall see that there are no other types of transformations of view than were mentioned above.

<u>Metatheorem</u>. If $*$ is a shifting of the horizon in T then in T it is provable that all $*$-classes are fully revealed.

Demonstration. If a $*$-class X is not revealed then there is a countable class $Y \subseteq X$ such that there is no set u with $Y \subseteq u \subseteq X$. Let us suppose that $*$ is a shifting of view. By the prolongation axiom there is f with $\mathrm{dom}(f) \in N - FN$ & & $(\forall \alpha \in \mathrm{dom}(f))(f(\alpha) \in X \equiv \alpha \in FN)$ and therefore $FN =$ $= \{\alpha ; f(\alpha) \in X\}$ is a $*$-class. Thus $FN^* = FN$ and $*$ is no shifting of the horizon. If all $*$-classes are revealed then they are also fully revealed, since for every $*$-class X and every normal formula $\varphi(z,Z)$ of the language FL, the class $\{z; \varphi(z,X)\}$ is a $*$-class, too.

<u>Metatheorem</u>. If a pair $*$, $\mathcal{G}$ is a transformation of view in T then in T we can fix an endomorphism F so that either F is an automorphism and $\mathcal{G}(X) = F''X$ for every X (and $*$ is no shifting of the horizon in this case) or there is a standard extension Ex on $\mathrm{rng}(F) \neq V$ so that $\mathcal{G}(X) = \mathrm{Ex}(F''X)$ for every X.

Demonstration. At first let us realize that describing the satisfaction relation in question we get a (metamathematical, may be nonnormal) formula $\Theta(z_1,z_2,Z)$ such that in AST for every normal formula φ of the language FL_V we have

$$\Theta(\varphi,\langle x_1,\dots,x_n\rangle,\langle X_1,\dots,X_k\rangle) \equiv \varphi(x_1,\dots,x_n,X_1,\dots,X_k)$$

(where $\langle X_1,\dots,X_k\rangle$ denote the k-tuple of classes $X_1,\dots,X_k$; cf. [V] or formally [S 1]). Let us proceed in T. Since all sets are $*$-sets, we have $E = E^* = \mathcal{G}(E)$ and moreover for every Gödel's operation $\mathcal{F}$ we have $\mathcal{F}^*(X,Y) = \mathcal{F}(X,Y)$ (X and Y being arbitrary $*$-classes). From this, by induction we get $\mathcal{G}(n) = n$ for every $n \in FN$ and moreover $\Theta^*(\varphi,x,X) \equiv \Theta(\varphi,x,X)$ for every normal formula φ of the language FL, every set x and every $*$-class X. In particular, for every set-formula φ of the language FL we have

$$\varphi(x_1,\dots,x_n) \equiv \Theta(\varphi,\langle x_1,\dots,x_n\rangle,0) \equiv$$
$$\equiv \Theta^*(\mathcal{G}(\varphi),\mathcal{G}(\langle x_1,\dots,x_n\rangle),\mathcal{G}(0)) \equiv \Theta(\varphi,\mathcal{G}(\langle x_1,\dots,x_n\rangle),0) \equiv$$
$$\equiv \varphi(\mathcal{G}(x_k),\dots,\mathcal{G}(x_n))$$

and thence the class $F = \{\langle \mathcal{G}(x),x\rangle;\ x \in V\}$ is an endomorphism.

If F is an automorphism (i.e. $rng(F) = V$) then $x \in X \equiv \mathcal{G}(x) \in \mathcal{G}(X) \equiv F(x) \in \mathcal{G}(X)$ and therefore $\mathcal{G}(X) = F''X$.

Let us suppose that $rng(F) \neq V$, in this case we have to prove that the operation $Ex(X) = \mathcal{G}(F^{-1}''X)$ (defined for each $X \subseteq rng(F)$) is a standard extension. If φ is a normal formula of the language FL then

$$\varphi^A(X_1,\dots,X_k) \equiv \varphi(F^{-1}''X_1,\dots,F^{-1}''X_k) \equiv$$
$$\equiv \Theta(\varphi,0,\langle F^{-1}''X_1,\dots,F^{-1}''X_k\rangle) \equiv \Theta^*(\mathcal{G}(\varphi),\mathcal{G}(0),\mathcal{G}(\langle F^{-1}''X_1,$$
$$\dots,F^{-1}''X_k\rangle)) \equiv \Theta(\varphi,0,\mathcal{G}(\langle F^{-1}''X_1,\dots,F^{-1}''X_k\rangle)) \equiv$$
$$\equiv \varphi(\mathcal{G}(F^{-1}''X_1),\dots,\mathcal{G}(F^{-1}''X_k))$$

by the second theorem of § 1 ch. V [V] and by the previous results. The equivalence $\varphi^A(X_1,\dots,X_k) \equiv \varphi(\mathcal{G}(F^{-1}''X_1,\dots,\mathcal{G}(F^{-1}''X_k))$ expresses that $\mathcal{G}(F^{-1}''X)$ is a standard extension on rng(F) and we are done.

According to the first lemma of § 2 [S-V 2] we get the following result.

<u>Corollary</u>. Let a pair $*$, $\mathcal{G}$ be a transformation of the horizon in T. Then in T it is provable that $\mathcal{G}(X)$ is a revealment of X.

According to the last metatheorem there are only two types of transformation of view; considering a shifting of view $*$ only, there are much more possibilities. On the other hand the absoluteness of some formulae implies some restriction in this case, too.

<u>Metatheorem</u>. Let $*$ be a shifting of the horizon in T such that for every formula Φ we have

$$T \vdash (\forall n \in FN)(\Phi(n) \equiv \Phi^*(n)).$$

If C is a constant definable in T, then in T we can prove that C^* is a revealment of C.

Demonstration. Let us note that under our assumptions, $*$ is an interpretation of T in T and hence our statement is meaningful because even the constant C^* is definable.

According to the last but one metatheorem and to the definition of revealment we have to show in T that for every normal formula $\varphi(Z)$ of the language FL it is $\varphi(C^*) \equiv \varphi(C)$. Let $\Psi(Z)$ be a formula defining the constant C in T (we have $T \vdash (\exists ! Z) \Psi(Z) \mathbin{\&} \Psi(C))$ and let $\Theta(z_1, z_2, Z)$ be the formula investigated in the last demonstration. In T, we have

$$(\exists Z)(Cls^*(Z) \mathbin{\&} \Psi^*(Z) \mathbin{\&} \Theta^*(\varphi, 0, \langle Z\rangle) \equiv$$
$$\equiv (\exists Z)(\Psi(Z) \mathbin{\&} \Theta(\varphi, 0, \langle Z\rangle))$$

by our assumption and hence $\Theta^*(\varphi,0,\langle C^*\rangle)\equiv\Theta(\varphi,0,\langle C\rangle)$. Furthermore $\Theta^*(\varphi,0,\langle C^*\rangle)\equiv\Theta(\varphi,0,\langle C^*\rangle)$ and therefore $\varphi(C^*)\equiv\varphi(C)$ for every normal formula φ of the language FL, which finishes the demonstration.

Corollary. Let R be a constant in T (stronger than AST + A 62). Then in T it is provable that R is a revealment of FN iff there is a shifting of the horizon $*$ in T such that $FN^* = R$ and such that for every formula Φ, the statement $(\forall n\in FN)(\Phi(n)\equiv\Phi^*(n))$ is provable in T.

The last corollary describes initial segments which can serve as shifted horizons if we consider shiftings of the horizon of the above described type. A description of initial segments which can serve as shifted horizons remains as an open problem. Let us note that if T is stronger than AST and if is a shifting of the horizon in T, then FN^* is fully revealed, but there can be even fully revealed initial segments such that the horizon cannot be shifted to them. The theory AST + $\neg$ Con(ZF_{Fin}) is consistent (relatively to ZF, say); let us fix α so that there is a proof of inconsistency of ZF_{Fin} the length of which is α. If R is an initial segment with $\alpha\in R$ then we cannot construct a shifting of the horizon with $FN^* = R$ since AST $\vdash Con_F(\mathcal{Z}\mathcal{F}_{Fin})$ (cf. [S 1]).

In this paper we dealt with interpretations of AST in AST; similar questions appear if we investigate (semantical) models of AST in ZF, some results concerning this topic can be found in [P-S].

R e f e r e n c e s

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