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METAMATHEMATICS OF THE ALTERNATIVE SET THEORY III

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Abstract: In the paper we continue in the investigation of metamathematics of the Alternative Set Theory (cf. [S 1] and [S 2]). We show independence of axioms of this theory and some basic facts about models of this theory in ZF.

Key words: Alternative Set Theory, independence, interpretation, consistency, semantical model, ultrapower.

Classification: Primary 03E70, 03H99
Secondary 03H20

The alternative set theory (AST) as a formal system of axioms was introduced in [S 1] where even an introduction to the whole series can be found. We use the notions defined in [V], [S 1] and [S 2].

In the eighth section we show that each axiom of AST is independent on the others, furthermore we are going to prove that AST is not finitely axiomatizable. We introduce the axiom of elementary equivalence and show its undecidability in AST.

In § 9 we deal with models of AST in ZF. In particular we show that AST is a conservative extension of ZF_{Fin} and that FN corresponds in some sense to metamathematical natural numbers. The reduction of every model of AST to sets gives us a recursively saturated model of ZF_{Fin}. At the end of this section, undecidability of the axiom of reflection is shown.

The last section is devoted to some open problems.

§ 8. Independence of the axioms of AST. Let us start with two trivial observations concerning independence. If we want to violate the axiom of extensionality, it suffices to add a new "copy" of a class. For violation of the axiom of existence of sets it is sufficient to assume that there is only one set - a model of such a theory is obvious. The triviality of these statements points out that these theories differ essentially from AST.

To prove that the schema of existence of classes is independent even on axioms A 21 and A 22 we show that AST is not finitely axiomatizable. For this purpose we are going to use results of [M] and we interprete Montague's symbols $\langle\ \rangle$ and $\cap$ defining $\langle X \rangle = \{X\}^{\eta}$ and $\{X_1,\dots,X_n\}^{\eta} \cap \{Y_1,\dots,Y_m\}^{\eta} = \{X_1,\dots,X_n, Y_1,\dots,Y_m\}^{\eta}$. Thus in TC we can prove all Montague's axioms (for "finite", non-empty sequences of classes") from the page 54 [M] and hence if T is a theory (of the language of set theory) stronger than TC then by the third theorem of [M], for every T_0 (metamathematically) finite part of T, $ZF_{Fin} + Con(\widehat{T_0})$ is interpretable in T. According to the metatheorem of the last section we have $T \vdash Con_F(\mathcal{T}_0)$ for every recursive T, i.e. we get that T is "reflective". By Gödel's theorem T cannot be finitely axiomatizable (cf. also Theorem 4 [M]). In particular, we have demonstrated the following statement.

Metatheorem. AST is not finitely axiomatizable.

The subject of this section is to demonstrate independence of the axioms A 4, A 5 and A 7 on the other axioms of the alternative set theory. The independence of the axiom of choice was recently proved by A. Vencovská (see [Ve 3]).

Let us start at first to investigate the axiom of Gb-class. The theory $KM^- + V = L \& V = HC$ can serve as a strengthening of the theory $AST_{-4} + \neg A\ 41$. Although the first theory differs essentially from AST (admitting actually infinite sets from the Cantor's point of view), it has an interpretation in AST according to § 6 and therefore the axiom A 41 cannot be proved in AST_{-4} (of course, we assume that AST itself is consistent in the whole of this section).

To construct an interpretation of $AST_{-4} + A\ 41 + \neg A\ 4$ in AST it is sufficient to construct an interpretation of the first theory in AST + A 62 (cf. § 6). If Z is a constant denoting a nontrivial ultrafilter with $FN \eta Z$ then the formulas

$Cls^*(X) \equiv dom(X) = FN$

$X^* \in^* Y^* \equiv \{n;\ X''\{n\} \in Y''\{n\}\} \eta Z$

$X^* =^* Y^* \equiv \{n;\ X''\{n\} = Y''\{n\}\} \eta Z$

$d^*{}''\{n\} = n$

determine an interpretation of $AST + \overline{1} < d + \overline{2} < d + \dots + d \in FN$ in AST + A 62 ($\overline{k}$ being the formalization of a metamathematical natural number k). To prove this it is sufficient to show (by metamathematical induction; for the induction step concerning the existential quantifier we use A 62) the following form of Łoš's theorem: For every formula $\Phi(Z_1,\dots,Z_k)$ we have $\Phi^*(X_1,\dots,X_k) \equiv \{n;\ \Phi(X_1''\{n\},\dots,X_k''\{n\})\} \eta Z$.

Let us proceed in $AST + \overline{1} < d + \overline{2} < d + \dots + d \in FN$. Since

ZF_{Fin} is not finitely axiomatizable (see [M] and [Mo]) there is a model $\mathfrak{A}$ satisfying all $\mathcal{ZF}_{Fin}$-axioms which are smaller than d and such that there is $\varphi \in \mathcal{ZF}_{Fin}$ so that $\mathfrak{A} \models \neg\varphi$. Then $\mathfrak{A}$ is an interpretation of AST_{-4} + A 41 + + $\neg$ A 4 in AST + $\overline{1} < d$ + $\overline{2} < d$ + ... + $d \in FN$. The composition of the interpretations mentioned above gives us an interpretation we looked for and hence we have demonstrated the following statement.

<u>Metatheorem</u>. There is an interpretation of AST_{-4} + + A 41 + $\neg$ A 4 in AST.

Now, let us deal with the prolongation axiom. By § 4 we know that AST_{-5} + $\neg$ A 52 is equivalent to KM_{Fin} and therefore it has an interpretation in AST and hence the axiom A 52 is not provable in AST_{-5}. The theory AST_{-5} + A 51 is not interpretable in AST_{-5} + A 52 according to § 7 and thence one cannot prove A 51 in AST_{-5} + A 52. It remains to show an interpretation of AST_{-5} + A 51 + $\neg$ A 5 in AST.

The symbol Def_X denotes the class of all sets definable using parameters from X (see [V 1]). If $\alpha \in N - FN$ then the models $\mathfrak{A} = \{Def_{\overline{P}(\alpha)}, E \cap Def^2_{\overline{P}(\alpha)}\}^{\eta}$ and $\{V, E\}^{\eta}$ are elementarily equivalent. Thus according to the fifth section, $\mathfrak{A}$ is an interpretation of AST_{-5} in AST. Moreover,

$$(\forall X \subseteq FN)(\exists x \in \overline{P}(\alpha))(x \cap FN = X)$$

and therefore we get A $51^{\mathfrak{A}}$. According to A 4 (in detail to formal replacement schema) for every $\varphi \in FL$ there is the greatest ξ which is definable by φ using parameters from $\overline{P}(\alpha)$. Further $Def_{\overline{P}(\alpha)} \cap N$ has no greatest element and hence there is a countable subclass X of $Def_{\overline{P}(\alpha)} \cap N$ which is cofinal (i.e. $\bigcup X = \bigcup(\mathrm{Def}_{\overline{F}(\alpha)} \cap N)$). From this observation the formula $\neg A\ 5^{a}$ follows.

To investigate the axiom of cardinalities we proceed in $AST_{-7} + \neg A\ 7$. The class $On^{\#} = \{y;\ \mathfrak{A}^{\#} \models$ "y is an ordinal"$\}$ (cf. § 6) is uncountable and $E^{\#} \cap On^{\#}$ is a well-ordering such that every its segment is countable. If $\leq$ is a well-ordering of V then every well-ordering of $On^{\#}$ is isomorphic to $\leq \restriction \{y;\ y \leq x\}$ for some x (since $On^{\#}$ and V have different cardinalities). Thus every well-ordering of $On^{\#}$ can be coded by a set. By [M-S] (cf. § 6) there is an interpretation $*$ of $KM^{-} + A\ 6 + A\ 7$ in our theory such that every $*$-class can be coded by a set and hence there is a model of $KM^{-} + A\ 6 + A\ 7$ in $AST_{-7} + \neg A\ 7$. Since there is an interpretation of AST in $KM^{-} + A\ 6 + A\ 7$, we see that $\mathcal{AST}$ has a model in $AST_{-7} +$ $+ \neg A\ 7$ and therefore by § 3 we obtain that the formula $\mathrm{Con}_{F}(\mathcal{AST})$ is provable in $AST_{-7} + \neg A\ 7$. Thus according to § 7 there is no interpretation of $AST_{-7} + \neg A\ 7$ in AST.

According to the last result one cannot prove independence of A 7 on the other axioms of the alternative set theory using an interpretation in AST. Hence it is necessary to choose a stronger theory for this purpose - e.g. ZF. Doing this, we drop for a moment our idea of the alternative set theory as the world of mathematics, nevertheless independence of A 7 will be demonstrated conclusively enough.

In ZF + V = L we can fix constants a, $\mathfrak{A} = \langle A, \widetilde{E} \rangle$ so that $\mathfrak{A} \models ZF_{Fin}\ \&\ \mathrm{card}(\{x;\ \mathfrak{A} \models x \in a\}) = \aleph_2$. Let $\mathcal{B}$ be an ultrapower of $\mathfrak{A}$ w.r.t. a nontrivial ultrafilter on ω. If d, k are elements of A^{ω} with $(\forall n \in \omega)(\mathfrak{A} \models$ "d(n) is the n-th natural number" $\&\ k(n) = a)$ then $\mathrm{card}(\{g;\ \mathcal{B} \models g \in d\})$ [illegible]

$\leq \aleph_0^{\aleph_0} = \aleph_1 < \aleph_2 \leq \mathrm{card}(x) \leq \aleph_2^{\aleph_0} = \aleph_2$ for every x with $x_{\mathscr{G}} = \{g; \mathscr{G} \models g \in k\}$. Hence by § 5, $\mathfrak{B}$ is an interpretation of $AST_{-7} + \neg A\ 71$ in ZF + V = L where $\mathscr{G}$ is fixed as described above (and ZF + V = L has an interpretation in ZF by the famous Gödel´s result; see [G]).

In ZF + AC + $2^{\aleph_0} = \aleph_2$ we can fix countable $\mathfrak{A} \models ZF_{Fin}$ and let $\mathscr{G}$ be an ultrapower of $\mathfrak{A}$ w.r.t. a nontrivial ultrafilter on ω. We have $(\forall f)((\forall x)(x_{\mathscr{G}} = \{g; \mathscr{G} \models g \in f\} \rightarrow \rightarrow \mathrm{card}(x) = \aleph_2) \vee (\exists x)(\mathrm{card}(x) \in \omega \ \&\ x_{\mathscr{G}} = \{g; \mathscr{G} \models g \in f\}))$ (cf. e.g. § 3 ch. 6 [B-S]). Thus $\mathfrak{B}$ is an interpretation of $AST_{-7} + A\ 71 + \neg A\ 7$ in ZF + AC + $2^{\aleph_0} = \aleph_2$ where $\mathscr{G}$ is fixed as described above (and ZF + AC + $2^{\aleph_0} = \aleph_2$ has an interpretation in ZF according to the famous Cohen´s extension; see [C]).

Let us deal now with the axiom of regularity. We are going to show that the axiom A 8 is not provable in AST but moreover we shall see that the axioms A 81 and A 82 are not provable one from the second one. The construction (in AST) can be done e.g. as follows. Choosing $\alpha \notin FN$ we put $A_1^o = \{\langle \alpha, \alpha \rangle\}$, $A_2^o = \{\langle \alpha + n, \alpha \rangle ; n \in FN\}$, $A_i^{n+1} = P(A_i^n) - \{\{x\}; x \in A_i^o\}$, $A_i = \bigcup \{A_i^n : n \in FN\}$, $E_1 = E \cap (A_1)^2 \cup \cup \{\langle\langle \alpha, \alpha \rangle, \langle \alpha, \alpha \rangle\rangle\}$, $E_2 = E \cap (A_2)^2 \cup \{\langle\langle \alpha + n + 1, \alpha \rangle, \langle \alpha + n, \alpha \rangle\rangle ; n \in FN\}$ and $\mathfrak{A}_i = \{A_i, E_i\}^{\eta}$. Evidently $\mathfrak{A}_i \models A\ 01 \ \&\ A\ 11$ and moreover $\mathfrak{A}_i \models A\ 3 \ \&\ A\ 41$ since $\mathfrak{A}_1 \models (\forall x)(x \in \langle \alpha, \alpha \rangle \equiv x = \langle \alpha, \alpha \rangle)$ and $(\forall n \in FN)(\mathfrak{A}_2 \models (\forall x)(x \in \langle \alpha + n, \alpha \rangle \equiv x = \langle \alpha + n + 1, \alpha \rangle))$. Further $\mathfrak{A}_1 \models \langle \alpha, \alpha \rangle \in \langle \alpha, \alpha \rangle$ and thence $\mathfrak{A}_1 \models \neg A\ 81$: on the other hand for every $n > 1$ we have $\mathfrak{A}_1 \models \mathrm{Tran}(A_1^n) \ \&\ \&\ \mathrm{Set}(A_1^n)$ and as a consequence we get $\mathfrak{A}_1 \models A\ 82$. Furthermore it is $x \in A_2 \rightarrow \mathrm{Fin}(x)$ and therefore there is no x with $(\forall n \in FN)(\mathcal{A}_2 \models \langle \alpha + n, \alpha \rangle \in x)$, from which $\mathcal{A}_2 \models \neg$ A 82 follows. On the contrary A 81 is satisfied in $\mathcal{A}_2$ trivially. If $\mathcal{L}_1$ ($\mathcal{L}_2$ respectively) is a revealment of $\mathcal{A}_1$ ($\mathcal{A}_2$ respectively) then $\mathcal{B}_1$ ($\mathcal{B}_2$ respectively) is an interpretation of AST_{-8} + ¬ A 81 + A 82 (AST_{-8} + A 81 + ¬ A 82 respectively) in AST by the fifth section.

At the end of this section we are going to introduce an interesting axiom.

A 9. <u>Axiom of elementary equivalence</u>. FV is elementarily equivalent to V.

Assuming this axiom we are able to prove a great number of statements and moreover the work in AST + A 9 is much more similar to the work in the Robinson's nonstandard methods (see [Ro] or [M-H]) than in the alternative set theory without this axiom. On the other hand, the alternative set theory with the negation of the axiom A 9 seems very interesting, too. Let us note that A 8 is a consequence of A 9.

For every model $\mathcal{A}$ let $\mathrm{Th}(\mathcal{A}) = \{\varphi \in \mathrm{FL};\ \mathcal{A} \models \varphi\}$. If $\mathcal{A} \models \mathcal{Z}\mathcal{F}_{\mathrm{Fin}}$ is revealed then $\mathcal{A}$ is an interpretation of AST in AST according to the fifth section, furthermore the formula A 9 holds iff $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{F}\mathcal{V})$ (since $FV^{\mathcal{A}}$ is isomorphic to FV). There are models $\mathcal{A}$ such that $\mathrm{Th}(\mathcal{A}) \neq \mathrm{Th}(\mathcal{F}\mathcal{V})$ since otherwise there would be only one finitely consistent theory stronger than $\mathcal{Z}\mathcal{F}_{\mathrm{Fin}}$, which is absurd. By § 3 we have $\mathcal{F}\mathcal{V} \models \mathcal{Z}\mathcal{F}_{\mathrm{Fin}}$. Since every model has a revealment we see that both AST + A 9 and AST + ¬ A 9 have an interpretation in AST.

The next theorem shows some statements equivalent to the axiom A 9; to prove it we use the following lemma.

Lemma. If $\Phi(z,Z)$ is a normal formula then in AST for every class X and every set x there is a revealment Y of X such that $\Phi(x,X) \rightarrow \Phi(x,Y)$.

Proof. Let x, X with $\Phi(x,X)$ be given and let $W \times \{0\} \cup \cup \{\langle y,1\rangle\}$ be a revealment of the class $X \times \{0\} \cup \{\langle x,1\rangle\}$ (cf. § 2 [S-V 2]). Thus $(\forall \varphi \in FL)(V \models \varphi(x) \equiv V \models \varphi(y))$ by the definition of revealment and hence there is an automorphism F with $F(y) = x$ (see § 1 ch. V [V]). The class F"W is a revealment of X according to § 2 [S-V 2] and from the assumption $\Phi(y,W)$ the formula $\Phi(F(y),F"W)$ (i.e. $\Phi(x,F"W)$) follows by the second theorem of § 1 ch. V [V].

<u>Theorem</u>. Each of the following statements is equivalent to the axiom A 9:

(a) For every $\alpha \notin FN$ there is a fully revealed endomorphic universe A with $A \subseteq \overline{P}(\alpha)$;

(b) Def = FV

Proof. To prove the implication A 9 $\rightarrow$ (a) let $\alpha \notin FN$ be given. By the last lemma we can choose a revealment of FV with $A \subseteq \overline{P}(\alpha)$. According to the definition of revealment, for every set-formula $\varphi(z_1,\dots,z_n)$ of the language FL we have

$$(\forall y_1,\dots,y_n \in FV)(\varphi(y_1,\dots,y_n) \equiv \varphi^{FV}(y_1,\dots,y_n)) \rightarrow$$
$$\rightarrow (\forall a_1,\dots,a_n \in A)(\varphi(a_1,\dots,a_n) \equiv \varphi^{A}(a_1,\dots,a_n)).$$

Since the assumption of the last implication is a consequence of A 9, A is a fully revealed endomorphic universe by the eighth theorem of § 1 [S-V 1].

The implication (a) $\longrightarrow$ (b) is a consequence of the fact that Def is a subclass of each endomorphic universe (cf. § 2 ch. V [V]). The remaining implication follows from the statement $(\exists x)\varphi(x) \longrightarrow (\exists x \in \mathrm{Def})\varphi(x)$ holding for every set-formula φ of the language FL (cf. § 1 ch. V [V]).

§ 9. Models of AST. In this section we are going to investigate models of AST in ZF. If $\mathfrak{A} \models$ AST then we define $\mathfrak{V}_{\mathfrak{A}} = \langle \{x;\ \mathfrak{A} \models \mathrm{Set}(x)\},\ \{\langle x,y\rangle;\ \mathfrak{A} \models x \in y \,\&\, \mathrm{Set}(y)\}\rangle$ (reduct of $\mathfrak{A}$ to sets) and we shall write $\mathrm{FN}_{\mathfrak{A}} = \omega$ if FN in the sense of $\mathfrak{A}$ is (isomorphic to) ω.

Theorem. If T is a consistent theory (in the language of set theory) stronger than $\mathrm{ZF}_{\mathrm{Fin}}$ then there is a model $\mathfrak{A} \models$ AST such that $\mathfrak{V}_{\mathfrak{A}} \models$ T and $\mathrm{FN}_{\mathfrak{A}} = \omega$.

Proof. Without loss of generality we can suppose the continuum hypothesis (working in the inner model L(T); cf. [H]). Let $\mathfrak{M} \models$ T and let $\mathfrak{M}' = \langle M', E'\rangle$ be the ultrapower of $\mathfrak{M}$ with respect to a nontrivial ultrafilter on ω. Let $\mathfrak{A}$ be the model expanding $\mathfrak{M}'$ by all its subsets, i.e. we put $q = \{x \subseteq M';\ \neg(\exists z \in M')\ x = \{y;\ \mathfrak{M}' \models y \in z\}\}$ and $\mathfrak{A} = \langle M' \cup q,\ E' \cup (E \restriction q)\rangle$ (without loss of generality we can suppose that $q \cap M' = 0$). Thus $\mathfrak{A} \models$ AST according to § 5 ($\mathfrak{A} \models$ A 7 follows from $\mathrm{card}(M') = \aleph_1$) and $\mathfrak{V}_{\mathfrak{A}} = \mathfrak{M}' \models$ T by the Łoš's theorem.

Theorem. AST is a conservative extension of $\mathrm{ZF}_{\mathrm{Fin}}$, i.e. for every set-formula Φ we have

AST $\vdash \Phi$ iff $\mathrm{ZF}_{\mathrm{Fin}} \vdash \Phi$.

Proof. By § 1 ch. I [V] we know that AST is stronger than ZF_{Fin} and therefore the implication from right to left is evident. Assuming that a set-formula Φ is not provable in ZF_{Fin}, i.e. that $ZF_{Fin} + \neg\Phi$ is consistent, we obtain according to the previous result that AST + $\neg\Phi$ is consistent, too.

The above theorem can be expressed in the way that the axioms of AST which are not set-formulas do not change the provability of set-formulas. On the other hand, the axiom A 9 which is neither set-formula has not the same property. In fact, by § 3, $Con_F(\mathcal{ZF}_{Fin})$ is provable in AST and therefore $Con(\overline{ZF}_{Fin})$ is provable in AST + A 9; contrariwise we cannot, of course, prove $Con(\overline{ZF}_{Fin})$ in ZF_{Fin}. Further let us note that according to the last theorem, in AST we can prove less set-formulas than in KM_{Fin} though KM_{Fin} is strictly weaker than AST in the sense of interpretability (cf. § 7).

There are models of AST such that their FN is not isomorphic to ω, e.g. by the Gödel's theorem there is a model $\mathfrak{A}$ with $\mathfrak{A} \models AST + \neg Con_F(\mathcal{AST})$ (and, of course, there is no element of which is (code of) a proof of inconsistency of AST). Nevertheless, the following result shows that in some sense members of FN give a true picture of ω (metamathematical natural numbers from our standpoint), namely we are able to describe precisely enough only those elements of FN which correspond to elements of ω (cf. also the usual definition of ω-consistency).

<u>Theorem</u>. Let $\Phi(z)$ be a set-formula. If AST $\vdash (\exists n \in FN)\, \Phi(n)$ then there is $m \in \omega$ such that $ZF_{Fin} \vdash (\exists n < \overline{m})\, \Phi(n)$.

Proof. Let us suppose that $ZF_{Fin} \nvdash (\exists n<\overline{m})\ \Phi(n)$ for every $m \in \omega$. Thus the theory $T = ZF_{Fin} \cup \{\neg \Phi(\overline{m});\ m \in \omega\}$ is consistent and therefore by the first theorem there is $\mathfrak{A} \models$ AST such that $FN_{\mathfrak{A}} = \omega$ and such that $\mathfrak{A} \models \neg \Phi(\overline{m})$ for every $m \in \omega$. Hence AST $\nvdash (\exists n \in FN)\ \Phi(n)$.

Remark. We have proved that for every set-formula $\Phi(z)$ from the assumption AST $\vdash (\exists n \in FN)\ \Phi(n)$ there follows the existence of $m \in \omega$ with AST $\vdash (\exists n<\overline{m})\ \Phi(n)$. In this result, the class FN plays an important role - the analogical statement without this constant does not hold because the assumption AST $\vdash (\exists \alpha \in N)\ \Phi(\alpha)$ (i.e. $ZF_{Fin} \vdash (\exists \alpha)(\Phi(\alpha)\ \&$ & "α is a natural number")) does not imply the existence of $m \in \omega$ with AST $\vdash (\exists n<\overline{m})\ \Phi(n)$ (i.e. $ZF_{Fin} \vdash (\exists n<\overline{m})\ \Phi(n)$). Following the Hájek's idea we define $\Phi(\alpha)$ as the property $(Con(\overline{ZF}_{Fin}) \longrightarrow \alpha = 0)\ \&\ (\neg Con(\overline{ZF}_{Fin}) \longrightarrow$ "α is the smallest proof of inconsistency of $\overline{ZF}_{Fin}$") (proofs being conveniently coded). In fact, AST $\vdash (\exists \alpha \in N)\ \Phi(\alpha)$ and for every $m \in \omega$ we have AST $\nvdash (\exists n \in \overline{m})\ \Phi(n)$ since AST $\vdash Con_F(\mathcal{ZF}_{Fin})$ and AST $\nvdash Con(\overline{ZF}_{Fin})$.

In the third section we introduced notions related to notions of finite formula and formula in AST. There was also emphasized that more important role in the alternative set theory play notions with the attribute "finite", therefore theories are for us subclasses of FL and not arbitrary subclasses of L. We did not deal with statements concerning simultaneously notions with the adjective "finite" and without it, but it is not excluded that such statements can be used for the development of mathematics in AST. The following result dealing with consistency of theories can serve as example of this approach.

In [Sh 1] (cf. also [N]) it is shown that the theories GB_{Fin} and ZF_{Fin} are equiconsistent and therefore dealing with two kinds of formalization of metamathematics in AST we get, of course, AST $\vdash Con(\overline{ZF}_{Fin}) \equiv Con(\overline{GB}_{Fin})$ and AST $\vdash Con_F(\mathcal{ZF}_{Fin}) \equiv Con_F(\mathcal{GB}_{Fin})$. Let us note that GB_{Fin} has finitely many axioms only and hence AST $\vdash \overline{GB}_{Fin} = \mathcal{GB}_{Fin}$; let ZF^n_{Fin} denote the first n axioms of ZF_{Fin}.

<u>Theorem</u>. The theory AST + $Con(\mathcal{ZF}_{Fin}) + \neg Con(\mathcal{GB}_{Fin})$ is consistent.

Proof. We have to prove that in AST the statement $\neg Con(\mathcal{ZF}_{Fin}) \vee Con(\mathcal{GB}_{Fin})$ is not provable. Proceeding in AST we have $\neg Con(\mathcal{ZF}_{Fin}) \equiv (\exists n \in FN) \neg Con(\overline{ZF}^n_{Fin})$ since $x \subseteq \mathcal{ZF}_{Fin} \rightarrow (\exists n \in FN)\ x \subseteq \overline{ZF}^n_{Fin}$ and all proofs are sets (and therefore for each proof even the class of all nonlogical axioms occurring in it is a set). To obtain a contradiction let us suppose AST $\vdash (\exists n \in FN) \neg Con(\overline{ZF}^n_{Fin}) \vee Con(\overline{GB}_{Fin})$. Thus, according to the last theorem there is $m \in \omega$ such that $ZF_{Fin} \vdash (\exists n < \overline{m}) \neg$ $\neg Con(\overline{ZF}^n_{Fin}) \vee Con(\overline{GB}_{Fin})$. Since ZF_{Fin} is reflexive (see [M] and [Mo]) we get $ZF_{Fin} \vdash (\forall n < \overline{m}) Con(\overline{ZF}^n_{Fin})$ and hence we obtain $ZF_{Fin} \vdash Con(\overline{GB}_{Fin})$ and since GB_{Fin} is equiconsistent to ZF_{Fin} we get at the end $ZF_{Fin} \vdash Con(\overline{ZF}_{Fin})$ which contradicts the Gödel's theorem.

<u>Remark</u>. The previous theorems of this section can be proved in theories much weaker than ZF (e.g. KM^- + "HC is a set" is strong enough). The crucial point of the previous considerations was that the ultrapower construction was available,

i.e. that subclasses of ultrapower were sets. Contrariwise in AST (supposing its consistency) we are not able to prove analogical statements e.g. in AST one cannot prove neither the formula

$$(\forall \varphi \in \mathrm{FL})(\text{"}\varphi \text{ is a set-formula"} \longrightarrow ((\mathcal{AST} \vdash_F \varphi) \equiv$$
$$\equiv (\mathcal{ZF}_{Fin} \vdash_F \varphi)))$$

(since AST $\nvdash \mathrm{Con}_F(\mathcal{AST})$ and AST $\vdash \mathrm{Con}_F(\mathcal{ZF}_{Fin})$) nor the formula

$$(\forall \varphi \in \mathrm{FL})\ (\text{"}\varphi \text{ is a set-formula"} \longrightarrow ((\overline{\mathrm{AST}} \vdash \varphi) \equiv$$
$$\equiv (\overline{\mathrm{ZF}}_{Fin} \vdash \varphi)))$$

($\mathrm{Con}(\overline{\mathrm{ZF}}_{Fin})$ is provable in AST + A 9, according to the last section there is an interpretation of AST + A 9 + + $\neg \mathrm{Con}_F(\mathcal{AST})$ in AST + $\neg \mathrm{Con}_F(\mathcal{AST})$ and the lastly mentioned theory is consistent by the Gödel's theorem).

As a consequence of the first theorem of this section we see that for every $\mathfrak{M} \models \mathrm{ZF}_{Fin}$ there is a model $\mathfrak{A} \models$ AST such that $\mathcal{V}_{\mathfrak{A}}$ is elementarily equivalent to $\mathfrak{M}$. The following theorem shows that this statement cannot be strengthened; there are models of ZF_{Fin} which cannot be expanded to models of AST. For the definition of recursively saturated models see e.g. [B-S]; the following result for $\mathrm{FN}_{\mathfrak{A}} = \omega$ was independently proved by M. Raškovič cf. [Ra].

<u>Theorem</u>. For every $\mathfrak{A} \models$ AST the model $\mathcal{V}_{\mathfrak{A}}$ is recursively saturated.

Proof. Let $a_1,\dots,a_n \in A$ with $\mathfrak{A} \models \mathrm{Set}(a_1) \& \dots \mathrm{Set}(a_n)$ be given and let Γ be a recursive nonempty set of formulae (of the language of set theory) with one free variable and with constants $a_1,\dots,a_n$ only. For every set-formula Φ there is $X_\Phi \in A$ so that for every x, the formula $\mathcal{O}\!\mathcal{l} \models \Phi(x) \equiv$ $\equiv \mathcal{O}\!\mathcal{l} \models x \in X_\Phi$ holds. Evidently every X_Φ is set-theoretically definable in the sense of $\mathcal{O}\!\mathcal{l}$. Supposing that Γ is finitely satisfiable (i.e. that for every $\Phi_1, \dots, \Phi_k \in \Gamma$ we have $\mathcal{O}\!\mathcal{l} \models X_{\Phi_1} \cap \dots \cap X_{\Phi_k} \neq 0$) we have to show that there is $a \in A$ with $(\forall \Phi \in \Gamma)\, \mathcal{O}\!\mathcal{l} \models a \in X_\Phi$. Since Γ is recursive, there is $X \in A$ such that for every set-formula Φ we have

$$\Phi \in \Gamma \equiv (\exists m \in \omega)\, \mathcal{O}\!\mathcal{l} \models X''\{\bar{m}\} = X_\Phi$$

(and such that for every $m \in \omega$ there is a set-formula Φ with $\mathcal{O}\!\mathcal{l} \models X''\{\bar{m}\} = X_\Phi$).

If $FN_{\mathcal{O}\!\mathcal{l}} = \omega$ then the system $\{X_\Phi ; \Phi \in \Gamma\}$ is a countable system in $\mathcal{O}\!\mathcal{l}$ and therefore it is sufficient to use results of § 5 ch. II [V].

Supposing $FN_{\mathcal{O}\!\mathcal{l}} \neq \omega$ we can assume moreover $\mathcal{O}\!\mathcal{l} \models \mathrm{dom}(X) \in$ $\in FN$. Thus we can choose $\ell \in A$ with

$$\mathcal{O}\!\mathcal{l} \models \bigcap\{X''\{k\};\ k < \ell\} \neq 0\ \&\ (\ell = \mathrm{dom}(X) \vee \bigcap\{X''\{k\};\ k \leq \ell\} = 0)$$

because in AST every $X \subseteq FN$ has the first element. We have $(\forall m \in \omega)\, \mathcal{O}\!\mathcal{l} \models \bar{m} < \ell$ because $\mathcal{O}\!\mathcal{l} \models \bigcap\{X''\{k\};\ k \leq \bar{m}\} \neq 0$ and we are done.

At the end let us deal with the axiom of reflection (cf. [S-V 3]). Every codable reflecting system determines in AST a model of $\mathcal{AST}$ and thence we get $\mathrm{Con}_F(\mathcal{AST})$. Therefore by the Gödel's theorem AST with the negation of the axiom of reflection is consistent.

On the other hand, let assume V = L and let $\mathcal{M} \models ZF_{Fin}$ be countable and let $\mathcal{O}\!\mathcal{l} = \langle A, \tilde{E} \rangle$ be the model of AST expanding the ultrapower of $\mathcal{M}$ by all its subsets (cf. the first proof of this section). Thus we are able to choose $B \subseteq A$ so that B is closed under all Skolem functions, B contains all

$\mathfrak{A}$-sets and so that card(B) = $\aleph_1$. B is a reflecting system in the sense of $\mathfrak{A}$ and it is $\mathfrak{A}$-codable since card(B) = = card($\{x;\ \mathfrak{A} \models \mathrm{Set}(x)\}$) and since all subsets of the ultrapower in question are classes in the sense of $\mathfrak{A}$. We have proved that even AST with the axiom of reflection is consistent (relatively to ZF).

Let us note that according to § 7 there is no interpretation of AST with the axiom of reflection in AST.

§ 10. <u>Remarks and problems</u>. In this section we are going to mention some open problems concerning metamathematics of the alternative set theory.

The following question was motivated in § 6:

Open problem. Is there an interpretation of TC + A 51 + + A 61 in TC + A 51 ?

Let us remind that ZF + "there is an infinite set without countable subset" has an interpretation in ZF and hence the axiom A 61 is not provable in TC + A 51.

In the last section we dealt with models of AST in ZF. Some other results and problems concerning this topic can be found in [P-S], let us mention the following question only:

Open problem. What are necessary and sufficient conditions for a model $\mathfrak{M} \models ZF_{Fin}$ to be expandable to a model of AST?

Let us note that for every model $\models$ AST, the model $\mathfrak{N}_{\mathfrak{A}}$ is recursively saturated and that if $\mathfrak{M} \models ZF_{Fin}$ is resplendent then there is a model $\mathfrak{A} \models$ AST with $\mathfrak{N}_{\mathfrak{A}} = \mathfrak{M}$ but the expandability of a model of ZF_{Fin} to a model of AST is equivalent neither to recursive saturation nor to resplendency.

Let us deal with different forms of the axiom of choice. In § 6 ch. I [V] the axiom of choice was proved in AST_{-6} using the axiom of extensional coding. In this proof the axiom of cardinalities was essentially used.

Open problem. Is A 6 provable in TC + A 4 + A 5 + the axiom of extensional coding?

It is natural to investigate even the following forms of the axiom of choice which correspond to forms of AC used in classical set theories and arithmetic

A 63 (Strong schema of choice). For every formula $\Phi(z,Z)$ we accept the axiom $(\forall x)(\exists X)\ \Phi(x,X) \rightarrow (\exists Y)(\forall x)\Phi(x,Y''\{x\})$.

A 64 (Schema of dependent choices). For every formula $\Phi(Z_1,Z_2)$ we accept the axiom $(\forall X)(\exists Y)\ \Phi(X,Y) \rightarrow$ $\rightarrow (\forall X)(\exists Z)(\mathrm{dom}(Z) = FN\ \&\ (\forall n \in FN)\ \Phi(Z''\{n\}, Z''\{n+1\})\ \&$ $\&\ Z''\{0\} = X)$.

Both axioms A 63, A 64 are consequences of the axiom of reflection. Evidently in AST it is provable:

(a) A 64 $\rightarrow$ A 62 (consider the formula $\Psi(X,Y) \equiv$ $\equiv (\forall n \in FN)(\mathrm{dom}(X) = n \rightarrow (\mathrm{dom}(Y) = n+1\ \&\ \Phi(n,Y''\{n\})))$; supposing $(\forall n \in FN)(\exists X)\ \Phi(n,X)$ we have $(\forall X)(\exists Y)\ \Psi(X,Y)$, if $Z''\{0\} = 0$ and $(\forall n \in FN)\ \Psi(Z''\{n\}, Z''\{n+1\})$ then $(\forall n \in FN)\Phi(n, (Z''\{n+1\})''\{n\}))$.

(b) A 63 $\rightarrow$ A 62

(c) A 62 $\rightarrow$ A 61 (suppose dom(X) = FN and consider the formula $\Phi(n,f) \equiv (\mathrm{dom}(f) = n\ \&\ f \subseteq X)$; if $(\forall n \in FN)\ \Phi(n,Z''\{n\})$ then put $F(n) = Z''\{n+1\}(n))$.

None of the implication A 64 $\rightarrow$ A 6, A 62 $\rightarrow$ A 6 and A 61 $\rightarrow$ A 6 is provable in AST_{-6}. To prove this, considering the Vencovská's interpretation $*$ (cf. [Ve 3]) it is sufficient to show that if $\{X_n;\ n\in FN\}$ is a collection of $*$-classes then even the class $X = \bigcup\{X_n\times\{n\};n\in FN\}$ is a $*$-class. Let X_n be a figure in the equivalence $\overset{\circ}{\underset{\{\widetilde{L}_n\}}{=}}$ and let $L_n = Ex_{\alpha_m}(\widetilde{L}_n)$. Then there is γ so that $(\forall n\in FN)\ \alpha_n < \gamma$ and we put $\widetilde{L} =$ $= \bigcup\{Ex_{\alpha_m\to\gamma}(\widetilde{L}_n)\times\{n\};n\in FN\}$ and $L = Ex_\gamma(\widetilde{L})$. Evidently $\widetilde{L}\subseteq A_\gamma$, we have to show that for every automorphism F with $F''L = L$ the equality $F''X = X$ holds. Since $F(n) = n$ for every $n\in FN$, this is the same as $F''X_n = X_n$ and for the proof of this it is sufficient to show that $F''L_n = L_n$, but this is trivial because $L''\{n\} = Ex_\gamma(\widetilde{L})''\{n\} = Ex_\gamma(\widetilde{L}''\{n\}) = Ex_\gamma(Ex_{\alpha_m\to\gamma}(\widetilde{L}_n)) =$ $= Ex_{\alpha_m}(\widetilde{L}_n) = L_n$.

Of course, there are several open problems concerning connections among these axioms in particular the following:

Open problem. Is the axiom A 61 provable in AST_{-6}?

Open problem. Are the axioms A 62, A 63 and A 64 provable in AST?

R e f e r e n c e s

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