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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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TOPOLOGICAL CATEGORIES WITH BOTH EPIREFLECTIVE AND COREFLECTIVE PROPER SUBCATEGORIES

Eraldo GIULI

Abstract: It is known that some familiar topological categories - for example the category of all topological spaces and the category of all uniform spaces - have no proper subcategories which are both epireflective and coreflective.

In this paper we produce a class of topological categories which contain the category Rere of all reflexive relations as proper, both epireflective and coreflective subcategory.

All these categories are cartesian closed and have other nice properties.

Key words: Topological category, epireflective subcategory, coreflective subcategory.

Classification: 18A40

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1. It is shown in ([6]) that the topological category Top of all topological spaces has no proper subcategory which is both epireflective and coreflective (i.e. closed under the formation of products, subspaces, coproducts and quotient spaces). The same statement is true for the categories Mer of all merotopic spaces ([7]), Unif od all uniform spaces ([5]) and Born of all bornological spaces ([1]).

The category Simp of all abstract simplicial complexes ([9]) is an example of a topological category with a proper subcategory which is both epireflective and coreflective. In fact the subcategory $\underline{\text{Simp}}_o$ of all o-dimensional simplicial complexes (which coincides with the subcategory Discrete(Simp) of all discrete Simp-objects) is obviously closed under the formation of products, subspaces, coproducts and quotient spaces. Moreover it is shown in [1] that $\underline{\text{Simp}}_0$ is the unique subcategory of Simp with the previous properties.

Examples of topological categories (in the sense of Herrlich [2]) with a proper, non trivial (i.e. $\neq$ Discrete) both epireflective and coreflective subcategory were firstly given by Hušek in [5] and [10].

Example 1 ([5] Example 2). Let $\underline{X}$ be the full subcategory of the category Unif whose objects are all uniform spaces $(X,\mathcal{U})$, $\mathcal{U}$ uniform neighborhoods of diagonal, with the property that the intersection of equivalences from $\mathcal{U}$ belongs to $\mathcal{U}$. If $\underline{Y}$ is the full subcategory of $\underline{X}$ generated by all $(X,\mathcal{U})$ such that $\bigcap \mathcal{U} \in \mathcal{U}$ then $\underline{Y}$ is non trivial, and both coreflective and bireflective in $\underline{X}$.

Example 2 ([10] Example 6). Let $\underline{X}$ be the full subcategory of the category Top whose objects are all locally connected spaces. Then $\underline{X}$ is a topological category because it is a coreflective subcategory of the topological category Top. The full subcategory $\underline{Y}$ of $\underline{X}$ composed of all spaces the collection of open and closed sets of which coincides is a non trivial, both bireflective and coreflective subcategory of $\underline{X}$.

The aim of this paper is to describe a class of topological categories which have proper, non trivial, both epireflective and coreflective subcategories. In particular each of these categories contains as full subcategory with the previous properties the category Rere of all reflexive relations described in [3]. It is also shown that all these categories are strongly topological (i.e. they are cartesian closed and final epi-sinks are hereditary) and no universal topological categories in the sense of Marny ([8]).

2. Here topological category means concrete category $(\underline{X},U)$ over the category Set of all sets which is initially complete and well-fibred (i.e. it is small-fibred and every constant map $UX \longrightarrow UY$ underlies some $\underline{X}$-morphism).

For definitions and results on cartesian closed topological categories see [2]. All undefined terminology is that of [4].

2.1. Definition. Let A be a non empty set. $\underline{A}$ denotes the category whose objects are couples (X,F) where X is a set and F is a subset of the set $\underline{Set}(A,X)$ of all maps of A into X containing all constant maps, and whose morphisms of (X,F) into (Y,G) are the maps $h:X \longrightarrow Y$ such that $h \circ f \in G$ for each $f \in F$.

2.2. Proposition. $\underline{A}$ is a topological category.

Proof. The class of all $\underline{A}$-structures in a set X is a set and every constant map is an $\underline{A}$-morphism. Thus $\underline{A}$ is well-fibred. Let X be a set, $(Y_i,F_i)_{i\in I}$ a family of $\underline{A}$-objects and $(h_i:X \longrightarrow Y_i)_{i\in I}$ a family of maps. Then the set $F = \{f:A \longrightarrow X : (h_i \circ f) \in F_i$ for each $i \in I\}$ is the initial A-structure in X with respect to $(X,h_i,(Y_i,F_i)_{i\in I})$ uniquely determined by its defining properties. Thus $\underline{A}$ is initially complete.

2.3 (1) An epi-sink $(h_i:(X_i,F_i) \longrightarrow (Y,F))_{i\in I}$ is final iff $f \in F$ holds whenever there exist $i \in I$ and $f_i \in F_i$ such that $h_i \circ f_i = f$.

(2) $k:(X,F) \longrightarrow (Y,G)$ is an embedding (and (X,F) is a subspace of (Y,G)) iff f is injective and $F = \{f:A \longrightarrow X:\ k \circ f \in G\}$.

(3) If $((X_i,F_i))_{i\in I}$ is a family of $\underline{A}$-objects then the family of maps $f:A \longrightarrow \prod_{i\in I} X_i$ satisfying the condition that each component $p_i \circ f$ belongs to F_i is the product structure $\prod_{i\in I} F_i$, and the family $\{k_i \circ f: i \in I \text{ and } f \in F_i\}$, where $k_i:X_i \longrightarrow \coprod_{j\in I} X_j$ are the canonical injections, is the coproduct structure $\coprod_{i\in I} X_i$.

(4) $q:(X,F) \longrightarrow (Y,G)$ is a quotient morphism (and (Y,G) a quotient space of (X,F)) iff $G = \{q \circ f\}_{f\in F}$.

(5) (X,F) is discrete iff $F = \underline{Set}(A,X)$.

(6) (X,F) is indiscrete iff $F = \{$all constant maps$\}$.

2.4. Theorem. $\underline{A}$ is a strongly topological category, i.e.

(a) $\underline{A}$ is cartesian closed.

(b) Final epi-sinks in $\underline{A}$ are hereditary.

Proof. (a): It is to be shown ([2] Theorem 2 (1)$\Leftrightarrow$(6)) that, for each final epi-sink $(h_i:(X_i,F_i) \longrightarrow (X,F))_{i\in I}$ and $\underline{A}$-object (Y,G), the epi-sink $(h_i \times 1_Y:(X_i \times Y,F_i \times G) \longrightarrow$
$\longrightarrow (X \times Y,F \times G))_{i\in I}$ is final.

Let m be an element of $F \times G$. Since the component m_X belongs to F (2.3.(3)), by 2.3.(1) there exist $i \in I$ and $f_i \in F_i$ with $m_X = h_i \circ f_i$. Since the component m_Y belongs to G (2.3.(3)) then the map $\langle f_i,m_Y\rangle:A \longrightarrow X_i \times Y$ belongs to $F_i \times G$ and $h_i \circ (\langle f_i,m_Y\rangle) =$
$= g$.

(b): Let $(h_i:(X_i,F_i) \longrightarrow (X,F))_{i\in I}$ be a final epi-sink in $\underline{A}$ and X' a subset of X with $k:X' \longrightarrow X$ the inclusion map. Further let F' be the subspace structure in X', and, for each $i \in I$, let F_i' be the subspace structure in $h_i^{-1}(X') \subset X_i$. It follows from 2.3.(2) and 2.3.(1) that the epi-sink $(h_i/h_i^{-1}(X'):$
$:(h_i^{-1}(X'),F_i') \longrightarrow (X',F'))_{i\in I}$ is final.

2.5. Remark. If A is a two-point set then the category $\underline{A}$ coincides with the category Rere of all reflexive relations described in [3].

The subcategory SRere of all symmetric reflexive relations properly contains the trivial subcategory Discrete(Rere) of all discrete Rere-objects. Moreover the functors T_1, T_2: Rere $\longrightarrow$ SRere which respectively send each reflexive relation F into the smallest symmetric relation containing F and into the largest symmetric relation contained in F, are left and right adjoint of the inclusion functor J: SRere $\longrightarrow$ Rere.

Thus SRere is a proper non trivial subcategory of Rere which is both bireflective and coreflective.

For each surjective map $q:A \longrightarrow B$ let $\underline{A}_q$ denote the full subcategory of $\underline{A}$ whose objects are all (X,F) satisfying the following condition: for every $f \in F$ there exists $g:B \longrightarrow X$ such that $g \circ q = f$.

2.6. Theorem. For each surjective map $q:A \longrightarrow B$ the category $\underline{A}_q$ is an epireflective and coreflective subcategory of $\underline{A}$. Moreover, if $|A| > 2$ and q is not injective, then $\underline{A}_q$ is a proper and non trivial subcategory of $\underline{A}$.

Proof. Let (X,F) be an $\underline{A}$-object. If $cF = \{f \in F:$ there exists $g:B \longrightarrow X$ with $g \circ q = f\}$, then the $\underline{A}$-morphism $c:(X,cF) \longrightarrow (X,F)$, which is the identity map on the underlying set X, is the coreflection of (X,F) in $\underline{A}_q$. Let rX be the quotient of X given by the equivalence relation generated by $x \mathcal{R} y$ iff there exist $a,b \in A$ and $f \in F$ such that $fa = x$, $fb = y$ and $qa = qb$ (i.e. rX is the largest quotient set of X such that, for each $f \in F$ there exists $g:B \longrightarrow rX$ with $g \circ q = r \circ f$, where

$r: X \longrightarrow rX$ is the quotient map). Then $r:(X,F) \longrightarrow (rX,\ r\circ F)$ is the epireflection of (X,F) in $\underline{A}_q$.

2.7. Remarks. (1) For each surjective map $q: A \longrightarrow B$ the functor $Q:\underline{B} \longrightarrow \underline{A}$ defined by $Q(X,F) = (X, F\circ q)$ is a full embedding and $Q\underline{B} = \underline{A}_q$. Then the theorem above says that the functors $P_1, P_2: \underline{A} \longrightarrow \underline{B}$, defined by $P_1(Y,G) = (Y, \{h:B \longrightarrow Y: h\circ q \in G\})$ and $P_2(Y,G) = (rY, \{h:B \longrightarrow Y:$ there exists $f \in F$ with $h\circ q = r\circ f\})$, are respectively left and right adjoint of the functor Q.

In particular if B is a two-point set (that is $\underline{B} = \underline{\text{Rere}}$) then each surjective map $q: A \longrightarrow B$ induces a full embedding $Q: \underline{\text{Rere}} \longrightarrow \underline{A}$ as well as an epireflective and coreflective subcategory.

(2) If $|A| \geq 3$ and $q: A \longrightarrow B$ is not injective then $\underline{A}_q$ is not bireflective in $\underline{A}$. It follows from (1) and Remark 2.5 that every $\underline{A}$, with $|A| \geq 2$ has a proper and non trivial, both bireflective and coreflective subcategory (namely $\underline{\text{SRere}}$).

(3) T_0-objects in $\underline{\text{Rere}}$ (i.e. objects (X,F) such that each $\underline{\text{Rere}}$-morphism of a two-point indiscrete object into (X,F) is constant) are the reflexive antisymmetric relations. Furthermore it follows from [2] Prop. 5 that a reflexive relation belongs to the bireflective hull $\underline{IT_0\text{Rere}}$ of the subcategory $\underline{T_0\text{Rere}}$ of all T_0-objects iff the largest symmetric relation which is contained in it is an equivalence relation. Since it is very easy to find relations without the property above then $\underline{\text{Rere}}$ is not universal in the sense of Marny ([8]).

(4) For each set A with $|A| \geq 3$ the topological category $\underline{A}$ is not universal. In fact if $\underline{A}$ would be a universal topologi-

cal category, then using a characterization by Marny ([8]) an $\underline{A}$-object (X,F) would be indiscrete iff $R_X = \{(x,y) \in X \times X:$ the subspace $\{x,y\}$ is indiscrete$\} = X \times X$. But if we choose (X,F) with $|X| \geq 3$ and $F = \{f: A \longrightarrow X: |\mathrm{Im} f| \leq 2\}$, then (X,F) is not indiscrete, and $R_X = X \times X$.

(5) The previous characterization of the class $\underline{\text{Ind}}$ of all indiscrete objects still holds for the category $\underline{\text{Rere}}$ even if Rere is not universal.

R e f e r e n c e s

[1] GIULI E.: Two classes of cartesian closed topological categories, preprint.

[2] HERRLICH H.: Cartesian closed topological categories, Math. Coll. Univ. Cape Town (9)(1974), 1-16.

[3] HERRLICH H., SALICRUP G. and VÁZQUEZ R.: Dispersed factorization structures, Can. J. Math. XXXI (5) (1979), 1059-1071.

[4] HERRLICH H. and STRECKER G.E.: Category Theory (Heldermann-Verlag, Berlin, 1979).

[5] HUŠEK M.: Reflective and coreflective subcategories of Unif (and Top), Sem. Unif. Sp. Prague 1973-1974 113-126.

[6] KANNAN V.: Reflexive cum coreflexive subcategories in topology, Math. Annal. 195(1972), 168-174.

[7] KATĚTOV M.: On continuity structures and spaces of mappings, Comment. Math. Univ. Carolinae 6(1965), 257-278.

[8] MARNY Th.: On epireflective subcategories of topological categories, General Topology Appl. 10(1979), 175-181.

[9] PREUSS G.: Some categorical aspects of simplicial complexes, preprint.

[10] HUŠEK M: Lattices of reflections and coreflections in continuous structures, Proc. First Categorial Topology Symposium, Lecture Notes in Math. 540 (Springer, Berlin 1976), 404-424.

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