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Jerzy Kąkol

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ON SUBSPACES OF ULTRABORNOLOGICAL SPACES

J. KĄKOL

Abstract: This paper is concerned with the inheritance of the ultrabornology by subspaces of topological vector spaces.

Key words: Ultrabornological and ultrabarrelled topological vector spaces.

Classification: 46A09

In [4] S. Dierof and P. Lurje constructed a bornological and barrelled locally convex space containing a dense subspace of countable infinite codimension which is barrelled but not bornological. On the other hand, a subspace with the property (b) in a bornological space is bornological [10]. In [5] Iyahen introduced the concepts of ultrabornological and quasi-ultrabarrelled spaces in non locally convex situations. It is known [1] that every finite codimensional subspace of an ultrabornological or quasiultrabarrelled space is a space of the same type, respectively.

In the present paper it is proved that every closed subspace G with the property (b) [resp. with a countable codimension] of an ultrabornological [resp. and ultrabarrelled] space E is of the same type, and every algebraic complement to G in E is a topological complement and carries the finest vector topology.

It is proved also that every subspace with the property (b) of an ultrabornological boundedly summing space is ultrabornological. In particular, every subspace with the property (b) of a locally convex ultrabornological space is ultrabornological. A subspace G of a topological vector space (tvs) E is said to have property (b) if for every bounded subset B of E the codimension of G in the linear span of $G \cup B$ is finite.

Following [3] a sequence (U_n) of balanced and absorbing subsets of a vector space E is called a string if $U_{n+1} + U_{n+1} \subset U_n$ for all $n \in N$. A string (U_n) in a tvs is closed, if every U_n is closed; bornivorous, if every U_n absorbs all bounded subsets of E; topological, if every U_n is a neighbourhood of zero in E.

A tvs E is ultrabornological [ultrabarrelled] if every bornivorous [closed] string in E is topological [3] (Adasch, Ernst and Keim call these spaces bornological and barrelled, respectively).

The following assertions are equivalent, [3], (2), p. 61:

(i) (E, τ) is ultrabornological.

(ii) Every bounded linear map from (E, τ) into a tvs is continuous.

(iii) Every bounded linear map from (E, τ) into a metrizable complete tvs is continuous.

(iv) Every vector topology on E having the same bounded sets as τ is coarser than τ.

Throughout we consider (Hausdorff) tvs over the field K of the real or complex scalars. A tvs E with the topology τ

is denoted by (E,τ), or simply by E, and by $(G,\tau|G)$, or G, we denote a subspace of E endowed with the induced topology. A sequence (x_n) in E is said to be a local null-sequence if there exists a sequence of scalars (a_n) such that $a_n \to \infty$ and $a_n x_n \to 0$. We say that $x_n \to x$ locally if $x_n - x \to 0$ locally. A subspace G of E is locally dense if for every $x \in E$ there exists a sequence in G which locally converges to x. A linear map from E into a tvs F is locally continuous if it maps every local null-sequence into a local null-sequence. As easily seen, a linear map from E into a tvs is locally continuous if and only if it is bounded (= bounded on bounded subsets of E), [1], p. 31. For any set M of a tvs (E,τ) we denote by $\overline{M}^{\tau}$ and $\overline{M}^{1}$ the closure of the set M, with respect to the topology τ, and the set of all local limits of sequences of M, respectively.

A tvs E is boundedly summing [3], p. 74, if for every bounded subset B of E there exists a sequence of scalars $(t_n), t_n \neq 0$, $n \in N$, such that $\sum_n t_n N := \bigcup_n \sum_{k=1}^{n} t_k B$ is bounded. Clearly, every almost convex space, locally convex space, locally pseudoconvex space, are boundedly summing.

Inheritance properties. In [6] there was proved the following result, which will be needed later.

Lemma 1. Let (E,τ) be a tvs and G its finite codimensional subspace with a co-base $(x_1, x_2, \dots x_p)$. Let (A_n) be a sequence of (balanced) subsets of G such that

(i) $G = \bigcup_n A_n$ and $A_n + A_n \subset A_{n+1}$ for all $n \in N$;

(ii) every $\tau|G$ bounded subset is contained in some A_m.

Then every τ bounded subset of E is contained in some $\overline{A}_m^{\tau} + 2^m \{\sum_{i=1}^{p} a_i x_i : |a_i| \leq 1\}$, $a_i \in K$.

Let B_E be a family of all bounded closed and balanced subsets of a tvs E.

<u>Lemma 2</u>. Let (E,τ) be a tvs and G its closed subspace with the property (b). Let F be an algebraic complement of G in E. Then for every $B\in B_E$ there exist $G\in B_E$ and a finite dimensional bounded subset A of F such that $B\subset G\cap Q + A$.

<u>Proof</u>. Let $B\in B_E$. Then $G\cap E_B$ is a finite codimensional subspace of E_B, where $E_B = \bigcup_n B_n$ and $B_n = \sum_1^{2^{n+1}} B$, $n\in N$. Let τ_B be the finest vector topology on E_B for which all B_n are bounded. A string (V_j) in E_B is topological if every V_j absorbs all B_n. Clearly $\tau|E_B \leq \tau_B$. In view of [2], p. 15, we obtain that $(\overline{B}_n^{\tau_B})$ forms a fundamental sequence of τ_B bounded sets. By Lemma 1 there exist $n\in N$ and a finite dimensional bounded subset T such that $B\subset \overline{G\cap \overline{B}_n^{\tau_B}}^{\tau_B} + T$. Since both projections of T onto G and onto F are bounded, there exist $Q\in B_E$ and a finite dimensional bounded subset A of F such that $B\subset G\cap Q + A$.

<u>Proposition 1</u>. Let (E,τ) be an ultrabornological tvs and G its closed subspace with the property (b). Let F be an algebraic complement of G in E. Then G is ultrabornological and F is a topological complement and carries the finest vector topology.

<u>Proof</u>. Clearly, (E,τ) is the inductive limit space of the family $(E_B, \tau_B : B\in B_E)$ of ultrabornological spaces. For every $n\in N$ let $H_n(B) := \sum_1^{2^{n+1}} B\cap G$ and $B\in B_E$. Let $\tau_{B\cap G}$ be the finest vector topology on $E_{B\cap G} := \bigcup_n H_n(B)$ for which all $H_n(B)$ are bounded. Clearly, $\tau_B|E_{B\cap G} \leq \tau_{B\cap G}$. If (G,ϑ) denotes the inductive

limit space of the family of ultrabornological spaces $(E_{B\cap G}, \tau_{B\cap G}: B\in B_E)$, then (G,ϑ) is ultrabornological, [3], 4, p. 62. Since F endowed with the finest vector topology Θ is ultrabornological ([8], Example 1, [3], (4), p. 62), the topological direct sum $(E,\alpha):=(G,\vartheta)\oplus(F,\Theta)$ is ultrabornological. Clearly $\tau\leq\alpha$. By Lemma 2 the topologies α and τ have the same bounded sets. Since (E,α) and (E,τ) are ultrabornological, it follows that $\alpha = \tau$. This completes the proof.

Corollary 1. Let E be an ultrabornological and ultrabarrelled tvs and G its closed subspace of countable codimension. Then G is ultrabornological and ultrabarrelled and every algebraic complement of G in E is a topological complement and carries the finest vector topology.

Proof. Observe that G has the property (b). Indeed, let $(x_n$ be a co-base of G in E. Put $G_n:=G+\text{lin}\{x_1,x_2,\dots x_n\}$ for all $n\in N$. Let $B\in B_E$. Since E is the strict inductive limit space of closed subspaces G_n, [1], p.29, then $B\subset G_n$ for some $n\in N$, [3], p. 28. Hence G has the property (b). In view of [3], p.90, G is ultrabarrelled. Applying Proposition 1 we obtain that G is ultrabornological.

Corollary 2. Let E be an ultrabornological tvs and G its closed subspace with the property (b). Then any linear extension to E of a continuous linear functional on G is continuous.

We shall need the following

Lemma 3. Let (E,τ) be a boundedly summing tvs and G its subspace with the property (b). Let F be an algebraic complement of G in E. Then for every $B\in B_E$ there exist $Q\in B_E$ and a finite dimensional bounded subset A of E such that $B\subset \overline{G\cap Q}^1 + A$.

Proof. Let $B \in B_E$. We construct a metrizable vector topology ϑ_B on E_B, coarser than τ_B, and such that $\tau | E_B \leq \vartheta_B$. Indeed, since (E, τ) is boundedly summing, then there exists a sequence of scalars (a_n) with $a_n > 0$ and $a_{n+1} \leq a_n$ for all $n \in N$ such that $\sum_i a_i B$ is bounded in E. If we put $V_n = \sum_i a_{2^{n-1} i} B$, then for every $n \in N$ we have $V_{n+1} + V_{n+1} \subset V_n$. Clearly, every V_n absorbs all B_n, and hence (V_n) is a string in E_B, which generates a metrizable vector topology ϑ_B on E_B such that $\tau | E_B \leq \vartheta_B$. Since τ_B is the finest vector topology on E_B for which all B_n are bounded, then $\vartheta_B \leq \tau_B$. Let $(x_1, x_2, \dots x_p)$ be a co-base of $G \cap E_B$ in E_B. In view of Lemma 1 there exists $m \in N$ such that

$$B \subset \overline{G \cap \overline{B}_m^{\tau_B} B}^{\tau_B} + 2^m \{ \sum_{i=1}^{p} a_i x_i : |a_i| \leq 1 \}.$$

Let $P := \overline{B}_m^{\tau} B$ and $Q := \overline{B}_m^{\tau}$. Clearly $\overline{G \cap P}^{\tau_B} \subset \overline{G \cap P}^{\vartheta_B}$. Since (E_B, ϑ_B) is metrizable and $\tau | E_B \leq \vartheta_B$, so we have $\overline{G \cap P}^{\vartheta_B} \subset \overline{G \cap P}^1 \subset \overline{G \cap Q}^1$. This completes the proof.

Lemma 4. Let (E, τ) be an ultrabornological tvs and G its dense subspace.

(i) If G is of finite codimension in E, then G is locally dense.

(ii) If E is boundedly summing and G has the property (b), then G is locally dense.

Proof. (i) Evidently, it suffices to carry over the proof to the case when G is of codimension one. Suppose G is not locally dense. Then G must be locally closed. Let f be a linear functional on E such that G = ker f. We prove that f is locally continuous. By [1], p. 31, f is locally continuous if and only if it is bounded on local null-sequences. Suppose f fails that property. Then $f \neq 0$, so that $f(x_0) = 1$ for some $x_0 \in E$ and there exist sequences $a_n \to \infty$ and $x_n = z_n + b_n x_0$, $z_n \in G$, $b_n \in K$, such that $a_n x_n \to 0$ and $f(x_n) = b_n \to \infty$. Since $a_n b_n (b^{-1} z_n + x_0) \to 0$, then $b_n^{-1} z_n \to -x_0$ locally. Since G is locally closed, it follows $x_0 \in G$, a contradiction. Hence f is locally continuous. Since E is ultrabornological, then f is continuous. Thus G is closed, a contradiction. We proved that G must be locally dense in E.

(ii) Let $F = \bigcup (\bigcup_n \overline{B_n \cap G}^{l} : B \in B_E)$. To conclude the proof it is enough to show that F = E. Suppose $F \neq E$ and let X be an algebraic complement of F in E. For every $B \in B_E$ let $F_B = \bigcup_n \overline{B_n \cap G}^{l}$. Let γ_B be the finest vector topology on F_B for which all $\overline{B_n \cap G}^{l}$ are bounded. Clearly, $\tau | F_B \leq \gamma_B$ and (F_B, γ_B) is ultrabornological. Let (F, ϑ) be the inductive limit space of the family $(F_B, \gamma_B : B \in B_E)$. Then the topological direct sum $(E, \alpha) := (F, \vartheta) \oplus (X, \theta)$ is ultrabornological, provided θ is the finest vector topology on X. Clearly $\tau \leq \alpha$. By Lemma 3 there exist $Q \in B_E$ and a finite dimensional bounded subset A such that $B \subset \overline{G \cap Q}^{l} + A$. Since both projections of A onto F and onto X are bounded, there exist $S \in B_E$ and a finite dimensional bounded subset R of X such that $B \subset \overline{G \cap S}^{l} + R$. Hence the topologies α and τ have the same bounded sets, and thus $\alpha = \tau$. The last is a contradiction, because F is closed in (E, α) and dense in (E, τ). Hence F = E.

Lemma 5. Let (E, τ) be a tvs and G its locally dense subspace with the property (b). Let f be a locally continuous map from G into a metrizable and complete tvs F. Then there exists a locally continuous extension $\tilde{f}$ of f to the whole space.

Proof. Let $B \in \mathcal{B}_E$. Then G is a locally dense finite codimensional subspace of $(G + E_B, \tau | G + E_B)$. According to [1], p. 32, for every $B \in \mathcal{B}_E$ there exists a locally continuous extension f_B of f to the space $G + E_B$. If $\tilde{f}(x) := f_B(x)$ for $x \in G + E_B$ we obtain a linear extension $\tilde{f}$ of f to the space E. Let $x_n \longrightarrow$ $\longrightarrow 0$ locally in E. There exist a scalar sequence $a_n \longrightarrow \infty$ and a bounded set $B := \{ t a_n x_n : |t| \leq 1, n \in N \}$ such that $a_n x_n \longrightarrow 0$ in $G + E_B$. Since $f_B(x_n) \longrightarrow 0$, so $\tilde{f}$ is locally continuous.

Corollary 3. Let E be an ultrabornological tvs and G its locally dense subspace with the property (b). Then G is ultrabornological.

Remark. In [7], Proposition 13.1, we proved that every tvs which admits a locally dense ultrabornological subspace must be ultrabornological. In view of [3], p. 112, we deduce that "locally dense" cannot be replaced by "dense".

Corollary 4 ([1], p. 33). Let E be an ultrabornological tvs and G its subspace of finite codimension. Then G is ultrabornological.

Proof. It suffices to carry over the proof to the case when G is of codimension one. Two cases are possible: G is closed. Then G is ultrabornological by Proposition 1. G is dense. Then G is locally dense by Lemma 4. Corollary 3 completes the proof.

Let E be a tvs. By E^* and E' we denote its algebraic and topological dual, respectively. Let τ and ϑ be two vector topologies on E. By $\sup(\tau, \vartheta)$ we mean the weakest vector topology on E finer than τ and ϑ.

<u>Corollary 5</u>. Let (E,τ) be an ultrabornological tvs with $E^* \neq E'$. Then there exists on E a vector topology ϑ different from τ such that (E,τ) and (E,ϑ) are linearly homeomorphic and such that $(E, \sup(\tau,\vartheta))$ is ultrabornological.

<u>Proof</u>. Let $f \in E^* \setminus E'$ and let $S_f = \ker f$. Choose x_0 with $f(x_0) = 2$. Define a linear map T of E into E by $Tx = x - f(x)x_0$ for every $x \in E$. Clearly $T^2 = id_E$. Let ϑ be a vector topology on E defined as the image of τ by T. In view of [9], the proof of Theorem 3.4, f is continuous for $\sup(\tau,\vartheta)$. As easily seen $\vartheta | S_f = \tau | S_f$. Hence $\sup(\tau,\vartheta)|S_f = \tau | S_f$. By Corollary 4, $(S_f, \tau | S_f)$ is ultrabornological, and hence we have $(E,\sup(\tau,\vartheta)) = (S_f, \tau | S_f) \oplus K$ is also ultrabornological.

<u>Proposition 2</u>. Let E be a boundedly summing ultrabornological tvs and G its subspace with the property (b). Then G is ultrabornological.

<u>Proof</u>. If G is closed, we apply Proposition 1. If G is dense, then by Lemma 4 (ii) it is locally dense. Applying Corollary 3 we obtain that G is ultrabornological. If G is neither closed nor dense, we take its closure and apply the previous arguments.

Since every locally convex tvs is boundedly summing, Proposition 2 can be applied to obtain the following

<u>Corollary 6</u>. Let E be a locally convex ultrabornological tvs and G its subspace with the property (b). Then G is ultrabornological.

<u>Problem</u>. Must $(E,\sup(\tau,\vartheta))$ be ultrabornological if τ

and ϑ are non comparable ultrabornological topologies for a vector space E ?

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Institute of Mathematics, A. Mickiewicz University, ul. Matejki 48/49 Poznań, Poland

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