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MODELS OF AST WITHOUT CHOICE

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Abstract: In this paper, we present two models of the alternative set theory with the negation of the axiom of choice; in the second model even the negation of the weak axiom of choice is valid. The constructions which in several aspects remind the classical method of symmetric models, lie, however, basically on topological means of AST and the fact (also proved here) that there exists an increasing sequence of endomorphic universes with standard extension.

Key words: Alternative set theory, basic equivalence, figure, fully revealed class, endomorphic universe, standard extension, ultraproduct, model.

Classification: Primary 03E70

Secondary 03E35, 03E25

The axiom of choice (AC) is in fact in the alternative set theory (AST) equivalent with the axiom of extensional coding (see [V], ch. II, § 3). However, its independence on the other axioms of AST was, for a long time, an open question. A partial answer, not yet published, was given by the first author who constructed a model of AST in which the Gödel's scheme, the weak form of the axiom of cardinalities (i.e. every two infinite sets are equivalent) and the negation of the axiom of choice hold. A further contribution to this problem comes from A. Vencovská. Her paper (quite recently published): "Independence of the axiom of choice in AST" contains a model of the whole AST with the negation of the axiom of choice in which the weak axiom of choice (WAC) is valid. The construction uses the axiom of reflection (see [S-V3]). Some notions and results from this paper will be used later.

Here, we give two interpretations in AST.

The first one is a model of

$$AST - AC + \neg AC + WAC,$$

the second one is a model of

$$AST - AC + \neg WAC.$$

In addition, in both models the following assertion holds:

(∗) Each uncountable class of the model contains a countable class which is not a class of the model.

The following intuitive image gives a good picture of the nature of both constructed models. Let us iterate countably many times the ultraproduct construction on the universal class V (the index set is FN). The "enlargements" of classes obtained from finite iterations and other "suitable" classes (e.g. some countable classes) will represent classes of our models. Just for the description of these "suitable" classes, we shall use substantially topological techniques of AST. Into the second model, we add, moreover, a special class FR (and, of course, other classes which are obtained from FR, e.g. by Gödelian operations) such that dom (FR) = FN and for each $n \in FN$ the class FR" $\{n\}$ is the "enlargement" of FN from the n-th iteration of the ultraproduct. This class prevents the validity of WAC.

As to the validity of other axioms in our models, we shall show that:

Axioms for sets follow from the fact the the ultraproduct is an elementary superstructure of the starting structure;

the Morse's scheme will be obtained by a technique similar to symmetric models;

$\neg$AC and the axiom of cardinalities result from the fact that the cardinality of the "enlargement" of every infinite class (in classical sense, of every infinite set) is the continuum;

the axiom of prolongation is the consequence of the selection of countable classes. In our models, there are namely only the countable classes which one can obtain already on a certain step of the iteration. This circumstance implies also the validity of the assertion ($\ast$).

Up to now, we have quoted the notion of the iterated ultraproduct which is more currently used in mathematical literature. In our article we shall work, however, with another technique, specific for AST, namely with creating a system of endomorphic universes with standard extension. This method lies in the existence (proved in § 4) of an increasing sequence of endomorphic universes with standard extension. We shall understand the "smallest" member of the sequence as the universal class V and the following endomorphic universes as successive iterations of the ultraproduct construction.

Now we shall briefly recall some notions concerning our problems (see [V],[S-V1]).

A function F is an endomorphism iff dom (F) = V and for every set-formula $\varphi(z_1,\dots,z_n)$ of the language FL, the normal formula

(1) $$\psi_\varphi(F) \sim (\forall x_1,\dots,x_n \in \mathrm{dom}\ (F))(\varphi(x_1,\dots,x_n) \equiv \equiv \varphi(F(x_1),\dots,F(x_n))$$

holds.

If F is an endomorphism and rng (F) = V, we call F an automorphism.

A class A is an endomorphic universe iff there is an endomorphism F with rng (F) = A.

Let A be an endomorphic universe. An operation Ex defined for all subclasses of A is called a standard extension on A iff for an arbitrary normal formula $\varphi(Z_1,\dots,Z_n) \in FL_A$ and arbitrary $X_1,\dots,X_n \subseteq A$ we have

$$\varphi^A(X_1,\dots,X_n) \equiv \varphi(\mathrm{Ex}\ (X_1),\dots,\mathrm{Ex}\ (X_n)),$$

where φ^A denotes the formula resulting from φ by the restriction of all quantifiers binding set variables to the elements of A and all quantifiers binding class variables to the subclasses of A.

Let A, B be endomorphic universes, $A \subset B$. An operation Ex defined for all subclasses A is called a standard extension on A with respect to B x) iff for an arbitrary normal formula $\varphi(Z_1,\dots,Z_n) \in FL_A$ and arbitrary $X_1,\dots,X_n \subseteq A$ we have $\mathrm{Ex}\ (X_i) \subseteq B$ $(i = 1,\dots,n)$ and

$$\varphi^A(X_1,\dots,X_n) \equiv \varphi^B(\mathrm{Ex}\ (X_1),\dots,\mathrm{Ex}\ (X_n)).$$

If an endomorphic universe A has a standard extension (for necessary and sufficient conditions see [S-V1]), the extension is uniquely determined - we shall denote it Ex_A. Analogously, we denote by $\mathrm{Ex}_{A \to B}$ the uniquely determined standard extension on A with respect to B.

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x) The notion was introduced by A. Vencovská. Some of her (recently published) results will be used later here and denoted by [AV].

From [S-V1] let us recall several assertions:

Let $A \neq V$ be an endomorphic universe with standard extension, $X, Y \subseteq A$. Then

(A1) $X \subseteq Ex_A(X)$

(A2) $X = Ex_A(X) \cap A$

(A3) $Ex_A(A) = V$

(A4) $Ex_A(FN) \neq FN$

(A5) $Ex_A(\mathrm{dom}\ (X)) = \mathrm{dom}\ (Ex_A(X))$

(A6) $Ex_A(Y"X) = (Ex_A(Y))"\ Ex_A(X)$

(A7) $(\forall a)(a \in A \Rightarrow Ex_A(a \cap A) = a)$

(A8) Let $F: X \longleftrightarrow Y$; then $Ex_A(F): \ Ex_A(X) \longleftrightarrow Ex_A(Y)$

(A9) $X \subseteq Y \equiv Ex_A(X) \subseteq Ex_A(Y)$

(A10) If x is definable by a normal formula from $Ex_A(X)$, then x is definable, in A, by a normal formula from X and $x \in A$.

All these facts, except (A4), are immediate consequences of the definition of Ex. The assertion (A4) follows from the facts that A can be ordered by the type Ω and $A \neq V$.

§ 1. <u>Some properties of endomorphic universe</u>. In this section we shall prove several assertions concerning endomorphic universes and fully revealed classes, which we shall use later.

Up to the end of this paper let A, A_1, A_2,... denote endomorphic universes with standard extension.

<u>Lemma 1</u>. [AV] Let $(F$ be an automorphism$)^A$, $X \subseteq A$ and $F"\ X = X$. Then $Ex_A(F)$ is an automorphism and the condition

$$(Ex_A(F))"\ Ex_A(X) = Ex_A(X)$$

holds.

Proof. Let $\psi_\varphi(F)$ be formulas (1) from the definition of endomorphism. Since (F is an automorphism)^A the formulas $\psi_\varphi{}^A(F)$ are valid. From the definition of standard extension we obtain

$$\psi_\varphi{}^A(F) \equiv \psi_\varphi(Ex_A(F)),$$

which implies that $Ex_A(F)$ is a similarity. Since dom $(F) = A$, the following equality holds (see (A5),(A3))

$$\mathrm{dom}\ (Ex_A(F)) = Ex_A\ (\mathrm{dom}\ (F)) = Ex_A(A) = V.$$

Hence $Ex_A(F)$ is an automorphism. Therefore - notice that $F''\ X = X$ - we have

$$(Ex_A(F))''\ Ex_A(X) = Ex_A(X).$$

Lemma 2. Let (F be an automorphism)^A, $X \subseteq A$, $F''\ X = X$. Then we have the following:

(i) $(Ex_A(F))''\ (X) = X$

(ii) $(Ex_A(F))''\ (A) = A$

(iii) $(Ex_A(F)) \restriction Ex_A(Def)$ is an identity.

Proof. For (i) and (ii) notice that $Ex_A(F) \supseteq F$ (see (A1)); hence $(Ex_A(F))''\ (X) = X$. The assertion (iii) follows from the fact that $F \restriction Def$ is an identity.

Lemma 3. Let A_1, A_2 be such endomorphic universes that $A_1 \subset A_2$ and let $X \subseteq A_1$. Then the following holds:

(i) $Ex_{A_2}(Ex_{A_1 \to A_2}(X)) = Ex_{A_1}(X)$ (commutativity of Ex)

(ii) $Ex_{A_1 \to A_2}(X) = Ex_{A_1}(X) \cap A_2$

Proof. Since $Ex_{A_2}(Ex_{A_1 \to A_2}(X))$ is a standard extension on A_1, the formula (i) is valid. The assertion (ii) follows immediately from (A2).

The existence of endomorphic universes with properties mentioned in Lemma 3 will be proved in the fourth section.

For the following considerations we shall recall two notions (see [V] and [S-V1]).

A class X is revealed iff for each countable $Y \subseteq X$ there is a set u such that $Y \subseteq u \subseteq X$.

A class X is called fully revealed iff for every normal formula $\varphi(z,Z) \in FL_V$ the class $\{x; \varphi(x,X)\}$ is revealed.

<u>Remark</u>. Note that classes definable by normal formulas of the language FL_V from a fully revealed class play the role of a generalization of Sd classes. We shall often use this analogy for our proofs. Instead of giving precise argumentations, we shall only quote the corresponding assertions from [V] and leave it to the reader to replace the words "a set formula of the language FL" by "a normal formula of the language $FL_{\{X\}}$" in their proofs.

The following assertion is proved in [S-V1]:

(A11) The class $Ex_A(X)$ is fully revealed for every $X \subseteq A$.

To see this fact notice that Y is fully revealed iff FN cannot be defined by any normal formula of the language FL_V from Y.

An immediate consequence of (A11) asserts that: If $A_1 \subseteq A_2$, $X \subseteq A_1$, then $(Ex_{A_1 \to A_2}(X)$ is fully revealed$)^{A_2}$.

<u>Theorem 1</u>. Let X be a fully revealed class and let $X_1 \supset \supset X_2 \supset \ldots$ be a descending sequence of classes definable from X

by a normal formula (and thus revealed). Then $\bigcap X_i$ is a non-empty and revealed class.

Proof. See [V], ch. II, § 5.

To the construction of our models we shall need a new type of the equivalence of indiscernibility.

Definition [AV]. Let X be a class. We put $x \overset{\circ}{=}_{\{X\}} y$ iff for each normal formula $\varphi(z,Z) \in FL$ we have $\varphi(x,X) \equiv \varphi(y,X)$.

Notice that each class X is a figure in $\overset{\circ}{=}_{\{X\}}$.

Remark. It follows from Theorem 1 that for any fully revealed class X the equivalence $\overset{\circ}{=}_{\{X\}}$ is compact. In other words, the equivalence $\overset{\circ}{=}_{\{X\}}$ has, in this case (from topological point of view), as "sensible" properties as the equivalence $\overset{\circ}{=}$.

Lemma 4.

(i) Monads in $\overset{\circ}{=}_{\{X\}}$ are either trivial or they contain an infinite set.

(ii) Let $\mu_{\overset{\circ}{=}_{\{X\}}} = \{a\}$, then a is definable by a normal formula from X.

(iii) There are only countably many trivial monads in $\overset{\circ}{=}_{\{X\}}$.

Proof. See an "analogous" theorem in [V], ch. V, § 1.

Lemma 5. Let F be such an automorphism that $F''X = X$. Then

$$(\forall x)\ F(x) \overset{\circ}{=}_{\{X\}} x.$$

Proof. See [V], ch. V, § 1 and adapt the proof of the "analogous" theorem.

Lemma 5 implies that each automorphism which "preserves" the class X, "preserves" also monads and figures in $\underset{\{X\}}{\overset{\circ}{=}}$.

<u>Theorem 2</u> [AV]. Let $x \underset{\{X\}}{\overset{\circ}{=}} y$, X be fully revealed. Then there exists an automorphism F such that F(x) = y and F" X = X.

<u>Proof</u>. Use the back and forth method. In greater details - adapt the proof of the theorem on the existence of an automorphism from [V], ch. V, § 1.

<u>Lemma 6</u>. Let X be fully revealed. Then

$$x \underset{\{X\}}{\overset{\circ}{=}} y \equiv x \underset{\{X,FN\}}{\overset{\circ}{=}} y.$$

<u>Proof</u>. Suppose at first $x \underset{\{X\}}{\overset{\circ}{=}} y$. We have to prove that for every normal formula $\varphi(x,X,FN) \in FL$ the formula

(2) $$\varphi(x,X,FN) \equiv \varphi(y,X,FN)$$

holds.

From our assumption it follows (see Theorem 2) that there is an automorphism F such that F(x) = y and F" X = X. But $\varphi(x,X,FN)$ is a normal formula. Therefore (since F is an automorphism) we obtain

$$\varphi(x,X,FN) \equiv \varphi(F(x),F"X, F"FN),$$

(see [V], ch.V, § 1). We know, moreover, that F(x) = y, F" X = = X and F" FN = FN (which is the consequence of the assertion that $FN \subseteq Def$). Therefore the formula (2) is valid.

Since the relation $\underset{\{X,FN\}}{\overset{\circ}{=}}$ is finer than $\underset{\{X\}}{\overset{\circ}{=}}$, the converse implication is obvious.

<u>Remark</u>. Replacing FN by Ex (FN) in the previous lemma, we obtain an uncorrect statement: It suffices now to put X = V; we have then that Ex (FN) is a figure in $\underset{\{Ex(FN)\}}{\overset{\circ}{=}}$. But the class is not a figure in $\overset{\circ}{=}$ since Ex (FN) is not a real class (for details see [Č-V]).

__Corollary__. Let X be a fully revealed class. Then it is possible to define a from X and FN if and only if a is definable only from X.

__Proof__. Notice that $\{a\}$ is a monad in $\overset{\circ}{=}_{\{X,FN\}}$ iff $\{a\}$ is a monad in $\overset{\circ}{=}_{\{X\}}$.

Our next observations will deal with a special type of the equivalence of indiscernibility, i.e. with $\overset{\circ}{=}_{\{Ex_A(X)\}}$, which we shall use substantially in the next two sections.

__Theorem 3__. Let μ be a monad in $\overset{\circ}{=}_{\{Ex_A(X)\}}$, $X \subseteq A$. Then we have

(i) $\mu \cap A = \emptyset$ or ($\mu \cap A$ is a monad in $\overset{\circ}{=}_{\{X\}}$)^A; moreover,

$$Ex_A(\mu \cap A) \subseteq \mu .$$

(ii) If $a \in A$, then

$$a \subseteq \mu \equiv a \cap A \subseteq \mu \cap A.$$

(iii) If (X is fully revealed)^A, then $\mu \cap A \neq \emptyset$.

__Proof__. For (i) it is sufficient to prove:
Let $(Y = \{t; \varphi(t,X)\})^A$, where φ is a normal formula; then

$$(Y \cap (\mu \cap A) \neq \emptyset \Rightarrow Y \supseteq (\mu \cap A))^A.$$

Suppose therefore $Y \cap (\mu \cap A) \neq \emptyset$. Then $Ex_A(Y) \cap \mu \neq \emptyset$. From this it follows (since μ is a monad in $\overset{\circ}{=}_{\{Ex_A(X)\}}$ and $Ex_A(Y)$ is definable from $Ex_A(X)$) that $Ex_A(Y) \supseteq \mu$. Thus $Y = Ex_A(Y) \cap A \supseteq \supseteq \mu \cap A$.

Since $\mu \cap A \subseteq Y$, we have that $Ex_A(\mu \cap A) \subseteq Ex_A(Y)$ – see (A9). The validity of $Ex_A(\mu \cap A) \subseteq \mu$ follows now from the fact that $\mu = \bigcap_{i \in FN} Ex_A(Y_i)$ for suitable Y_i definable from X.

The implication $\Longrightarrow$ in (ii) is trivial. The converse assertion is an immediate consequence of (A7),(A9) and (i): Since $a \in A$ and $a \cap A \subseteq \mu \cap A$, according to our assumption, we have

$$a = Ex_A(a \cap A) \subseteq Ex_A(\mu \cap A) \subseteq \mu .$$

For proving (iii) notice that $\mu = \bigcap_{i \in FN} Ex_A(Y_i)$, where $Y_i \subseteq A$ and $Y_1 \supset Y_2 \supset \dots$ is a descending sequence of classes definable from X and hence (revealed)^A. Then, according to Theorem 1, we obtain that $\bigcap Y_i \neq \emptyset$ and hence (see (i)) also $\mu \cap A \neq \emptyset$.

This completes the proof.

<u>Lemma 7</u>. Let (X be fully revealed)^A, $a, b \in A$ and $a \subseteq \mu_{\{Ex_A^{\doteq}(X)\}}(b)$. Then $a \subseteq \mu_{\{Ex_A^{\doteq}(X), Ex_A(FN)\}}(b)$.

<u>Proof</u>. To prove our statement by contradiction, let us assume that there is $t \in a$ such that $t \notin \mu_{\{Ex_A^{\doteq}(X), Ex_A(FN)\}}(b)$. Then for a normal formula ψ both $\neg \psi(t, Ex_A(X), Ex_A(FN))$ and $\psi(b, Ex_A(X), Ex_A(FN))$ hold. Denote

$$\varphi(a, Ex_A(X), Ex_A(FN)) \sim (\exists t \in a)(\neg \psi(t, Ex_A(X), Ex_A(FN));$$

obviously φ is a normal formula.
Since

$$\varphi(a, Ex_A(X), Ex_A(FN)) \equiv \varphi^A(a, X, FN),$$

we obtain that there is $\bar{t} \in a \cap A$ such that

$$\neg \psi(\bar{t}, Ex_A(X), Ex_A(FN)).$$

We shall show that this fact is in contradiction with the assumption $a \subseteq \mu_{\{Ex_A^{\doteq}(X)\}}(b)$. To this end, notice that the existence of $\bar{t}$ implies that $(a \not\subseteq \mu_{\{X, \doteq FN\}}(b))^A$. But according to Lemma 6

$$\mu_{\{X,\overset{\circ}{=}FN\}}(b) = \mu_{\{\overset{\circ}{\overline{X}}\}}(b)$$

and thus $(a \not\subseteq \mu_{\{\overset{\circ}{\overline{X}}\}}(b))^{A}$.

For completing the proof it suffices to show that

$$a \subseteq \mu_{\{Ex_A^{\overset{\circ}{=}}(X)\}}(b) \equiv (a \subseteq \mu_{\{\overset{\circ}{\overline{X}}\}}(b))^{A}.$$

To see this, notice (use Theorem 3) that

$$(a \subseteq \mu_{\{\overset{\circ}{\overline{X}}\}}(b))^{A} \equiv a \cap A \subseteq \mu_{\{Ex_A^{\overset{\circ}{=}}(X)\}}(b) \cap A \equiv$$
$$\equiv a \subseteq \mu_{\{Ex_A^{\overset{\circ}{=}}(X)\}}(b)\ .$$

§ 2. <u>Model of AST - AC + $\neg$ AC + WAC</u>. In this part, we shall construct the first model. For creating it we suppose to have an increasing sequence $A_1 \subset A_2 \subset \dots$ of endomorphic universes with standard extension (for its construction see § 4). Let us denote

$$V^* = \bigcup \{A_n; n \in FN\}.$$

The definition of classes in this model (we shall denote them X^*, Y^*, X_1^*, ..., etc.) lies substantially on the relation $\{\overset{\circ}{\overline{X}}\}$, more precisely, on the relation $\{Ex_{A_m}^{\overset{\circ}{=}}(Z)\}$. For an easier typing we shall write further only $Ex_n(Z)$ instead of $Ex_{A_n}(Z)$ and similarly $Ex_{k\to\ell}(Z)$ will be the abbreviation for $Ex_{A_k\to A_\ell}(Z)$.

<u>Definition</u>. $C\ell s^*(X)$ iff $X = \overline{X} \cap V^*$, where $\overline{X}$ is a figure in an equivalence $\{Ex_m^{\overset{\circ}{=}}(Z)\}$, $Z \subseteq A_n$. Moreover,

$$(X^* \in^* Y^*) \equiv (X^* = x \cap V^* \,\&\, x \in Y^*)$$

and

$(X^* =^* Y^*) \equiv (X^* = Y^*)$.

For the reader's convenience we shall - when there is no danger of confusion - speak sometimes (when using the definition of $C\ell s^*(X)$) only of X instead of $\overline{X}$.

Remark. It is easy to see that, for each $x \in V^*$, $x \cap V^*$ is a class in our model: Let $x \in V^*$, then $x \in A_\ell$ for a suitable ℓ. According to (A7) we have $x = Ex_\ell(x)$. But x is a figure in $\{\overset{\circ}{\overline{X}}\}$ (see the note behind the definition of $\{\overset{\circ}{\overline{X}}\}$). Thus $C\ell s^*(x \cap V^*)$.

Furthermore, we shall denote by φ^* the formula which is obtained from the formula φ by restricting all its quantifiers to classes of our model and $\in$ to $\in^*$. If φ does not contain subformulas of the type $X \in Y$, then φ^* is obtained by the restriction of all its quantifiers binding classes of our model and sets to sets of our model.

Before proving the validity of the above mentioned axioms for our model, we shall formulate several lemmas which will make the proofs easier.

Lemma 1. Let $C\ell s^*(X_1), C\ell s^*(X_2), \dots, C\ell s^*(X_n)$. Then there is $k \in FN$ and a class $Y \subseteq A_k$ such that $(Y \text{ is fully revealed})^{A_k}$ and $X_i^* = Y_i \cap V^*$, where Y_i are figures in $\{\overset{\circ}{\overline{Ex_k(Y)}}\}$.

Proof. It follows directly from commutativity of Ex (see Lemma 3, § 1) that we can suppose that Y_i are figures in $\{\overset{\circ}{\overline{Ex_{k-1}(Z_i)}}\}$, $Z_i \subseteq A_{k-1}$ for a certain $k \in FN$. But the finite sequence of Z_i can be coded by one class - let us denote it Z. Put now $Y = Ex_{k-1 \to k}(Z)$. According to (A11) we have that (Y is fully revealed)^{A_k}, which completes the proof.

Lemma 2. Let $t,u \in A_\ell$, $\ell > k$. Let further $t \underset{\{Ex_k(Z)\}}{\overset{\circ}{=}} u$. Then there exists an automorphism F such that $F(t) = u$ and $F" Ex_k(Z) = Ex_k(Z)$. Moreover $F" V^* = V^*$ and $C\ell s^*(F \cap V^*)$.

Proof. Since $(Ex_{k\to\ell}(Z)$ is fully revealed)$^{A_\ell}$ - see (A11), and since $t \underset{\{Ex_k(Z)\}}{\overset{\circ}{=}} u$ we have (owing to commutativity of Ex) that $t \underset{\{Ex_\ell(Ex_{k\to\ell}(Z))\}}{\overset{\circ}{=}} u$. Therefore (see Theorem 3(i), § 1) we obtain that $(t \underset{\{Ex_{k\to\ell}(Z)\}}{\overset{\circ}{=}} u)^{A_\ell}$. From Theorem 2, § 1 we know that there exists (an automorphism $G)^{A_\ell}$ such that $G(t) = u$ and $G" (Ex_{k\to\ell}(Z)) = Ex_{k\to\ell}(Z)$. Put now $F = Ex_\ell(G)$ and use Lemma 2, § 1.

Lemma 3. Let F be such an automorphism that $F" V^* = V^*$ and $C\ell s^*(F \cap V^*)$. Then

$$C\ell s^*(X) \equiv C\ell s^*(F" X).$$

Proof. It suffices to prove the following statements:

(1) $(C\ell s^*(Y) \;\&\; C\ell s^*(X)) \Longrightarrow C\ell s^*(Y" X)$

(2) $C\ell s^*(X) \Longrightarrow C\ell s^*(X^{-1})$.

We shall show only the validity of (1); the proof of (2) is analogous.

Since X, Y are classes of our model, they are figures in $\{Ex_k^{\circ}(Z)\}$ (see Lemma 1).

Let now $u \in Y" X$, $t \in V^*$ and $u \underset{\{Ex_k(Z)\}}{\overset{\circ}{=}} t$. Then for a suitable $\ell > k$ it is true that $u,t \in A_\ell$. Let F be an automorphism from Lemma 2. This automorphism "keeps" obviously also figures in $\{Ex_k \stackrel{o}{=} (Z)\} = \{Ex_\ell (\stackrel{o}{=}\!\!Ex_{k\to\ell}(Z)\}$ and therefore $t \in Y'' X$. Thus $Y'' X$ is a figure in $\{Ex_k \stackrel{o}{=} (Z)\}$. As $F'' V^* = V^*$ we have hence $C\ell s^*(Y'' X)$.

<u>(Morse's scheme)*</u>. For every formula $\varphi(x, X_1, \ldots, X_n) \in FL$ and for every $X_1^*, \ldots, X_n^*$ there exists a class Y such that $C\ell s^*(Y)$ and

$$(\forall x \in V^*)(x \in Y \equiv \varphi^*(x, X_1^*, \ldots, X_n^*)).$$

<u>Proof.</u> We can suppose (see Lemma 1) that $X_1, \ldots, X_n$ are figures in an equivalence $\{Ex_k \stackrel{o}{=} (Z)\}$ where $Z \subseteq A_k$ and $(Z \text{ is fully revealed})^{A_k}$.

Define

$$Y = \{x \in V^*; \varphi^*(x, X_1^*, \ldots, X_n^*)\}.$$

We shall prove $C\ell s^*(Y)$. To this end, it suffices to show that Y is a figure in $\{Ex_k \stackrel{o}{=} (Z)\}$, i.e. that for every $u \in Y$ and $t \in V^*$ such that $t \; \{Ex_k \stackrel{o}{=} (Z)\} \; u$ we have $t \in Y$.

Let $\ell > k$ be such a number that $t, u \in A_\ell$. Let further F be an automorphism from Lemma 2. Since $u \in Y$, the formula $\varphi^*(u, X_1^*, \ldots, X_n^*)$ holds.

We show the validity of the formula

$$(3) \qquad \varphi^*(u, X_1^*, \ldots, X_n^*) \equiv \varphi^*(F(u), X_1^*, \ldots, X_n^*).$$

Notice that $(\exists X^*)\psi$ means $(\exists X)(C\ell s^*(X) \;\&\; \psi)$. Since (see Lemma 3) $C\ell s^*(X) \equiv C\ell s^*(F'' X)$ and $F'' V^* = V^*$, according to Lemma 2, § 1, we can replace $(\exists X) C\ell s^*(X)$ by $(\exists F'' X) C\ell s^*(F'' X)$ and $(\exists x)\, x \in V^*$ by $(\exists x) F(x) \in V^*$. But then

$$\varphi^*(u, X_1^*, \ldots, X_n^*) \equiv \varphi^*(F(u), F''^* X_1, \ldots, F''^* X_n)$$

– see [V], ch. V, § 1. Formula (3) follows now immediately from the fact that $F^{"*}X_i = X_i^*$ $(i = 1,\dots,n)$. This completes the proof.

Further we shall investigate countable classes in our model.

Lemma 4. $FN^* = FN$.

Proof. Since $FN \subseteq Def$ (see [V], ch. V, § 1), the class FN is a figure in each equivalence $\overset{\circ}{=}_{\{X\}}$. Moreover, $FN \subseteq A_\ell$ for every ℓ; this follows from the fact that Def is a subclass of each endomorphic universe (see [S-V1]). Therefore $FN \subseteq V^*$. Hence $C\ell s^*(FN)$. For proving $FN^* = FN$ notice that $FN^* \not\subset FN$, since in our model there is a smaller amount of classes than in AST.

Theorem 1. Let X^* be a countable class of V^*. Then there exists an endomorphic universe A_k such that $X^* \subseteq A_k$.

Proof. Since $C\ell s^*(X)$, the class X is a figure in $\overset{\circ}{=}_{\{Ex_{k}(Z)\}}$ for $Z \subseteq A_k$ and $(Z \text{ fully revealed})^{A_k}$. Moreover, since X^* is a countable class, all monads in $\overset{\circ}{=}_{\{Ex_{k}(Z)\}}$ there are trivial - see Lemma 4, § 1. Suppose now $t \in X^*$. Then $\{t\} = \mu$ is a monad in $\overset{\circ}{=}_{\{Ex_{k}(Z)\}}$. From Theorem 3, § 1 it follows that $(\mu \cap A_k$ is a monad in $\overset{\circ}{=}_{\{Z\}}\ )^{A_k}$. Hence $t \in A_k$.

Corollary. The property "to be countable" is absolute for the classes of our model; i.e.

$$\mathrm{Count}^*(X^*) \equiv \mathrm{Count}\,(X^*).$$

Proof. From Lemma 4 we know that $FN^* = FN$. Suppose at first $\mathrm{Count}^*(X^*)$. Then there exists $F^*: FN \leftrightarrow X^*$. But F^* is a one-one mapping in AST, too.

If we assume $\mathrm{Count}\,(X^*)$ we obtain - in accordance with Theorem 3, § 1 - that $X^* \subseteq A_k$ for a suitable k. Therefore there is such a mapping F that $F: FN \leftrightarrow X^*$ and moreover, $F \subseteq A_k$. From the axiom of prolongation in AST it follows that $F = f \upharpoonright FN$ for a certain $f \in A_k$. But $f \in V^*$, which completes the proof.

<u>(Axiom of prolongation)*</u> . Let (F^* be a countable function)* , then there is a function f^* such that $F^* \subseteq f^*$.

<u>Proof.</u> From the Corollary of Theorem 1 it follows that F^* is a countable function. Now proceed similarly as in the second part of the proof of the Corollary.

Before proving the axiom of cardinalities, we shall formulate a useful assertion.

<u>Theorem 2.</u> For each uncountable class X^* there is a set $a \in A_k$, for a suitable $k \in FN$, such that $a \subseteq X^*$ and a is an infinite set.

<u>Proof.</u> The class X^* is a figure in $\{\overset{\circ}{=}_{Ex_k(Z)}\}$, where $Z \subseteq A_k$, (Z fully revealed)^{A_k}. Since X^* is an uncountable class and since there is only a countable amount of trivial monads in $\{\overset{\circ}{=}_{Ex_k(Z)}\}$ (see Lemma 4, § 1), the class X^* has to contain a non-trivial monad. Such a monad contains, however, an infinite set - this follows from Lemma 4, § 1 and Theorem 3, § 1.

<u>(Axiom of cardinalities)*</u> . Each uncountable class X^* can be mapped by a one-one function onto V^*.

<u>Proof.</u> Owing to Theorem 2 and Cantor-Bernstein's theorem it is sufficient to prove: If $a \in V^*$ and a is an infinite set, then there exists $F^* : a \longleftrightarrow V^*$.

Let $a \in A_\ell$. Then there is $(G: a \longleftrightarrow A_\ell)^{A_\ell}$. Put now $F^* =$ $= Ex_\ell\,(G) \cap V^*$.

(Negation of the axiom of choice)*. (There is no class X such that $\in \restriction X$ is an ordering of the type Ω.)*

Proof. Such a class X would have to be uncountable and could not contain any infinite set, at the same time (see [V], ch. II, § 3 and Theorem 2).

(Weak axiom of choice)*. Let R^* be a relation, $\mathrm{dom}(R^*) = FN$. Then there is a function $F^* \subseteq R^*$ such that $\mathrm{dom}(F^*) = FN$.

Proof. R^* is a figure in $\{Ex_{\mathcal{R}}^{\doteq}(Z)\}$ for $Z \subseteq A_k$, (Z fully revealed)^{A_k}. We claim that $\mathrm{dom}(R^* \cap A_k) = FN$. For this, it suffices to realize that for each $n \in FN$ the class $R'' \{n\}$ is a figure and moreover (see Theorem 3, (iii), § 1) $R^{*}{}'' \{n\} \cap A_k \neq \emptyset$.

Since the axiom of choice holds in the endomorphic universe A_k (and, obviously, the weak axiom of choice, too), there exists a function $g \in A_k$ such that

$$g \restriction FN \subseteq R^* \cap A_k \subseteq R^* .$$

Put now $F^* = g \restriction FN$.

Theorem 3. Each uncountable class X^* contains a countable class Y such that $\neg\, C\ell s^*(Y)$.

Proof. Let $T = \{a_1, a_2, \dots\}$ where $a_1 \in A_1$, $a_m \in A_m - A_{m-1}$ for $m = 2, 3, \dots$. Obviously $T \subseteq V^*$.

We shall prove at first that $\neg\, C\ell s^*(T)$. The class T is, evidently, countable. Suppose $C\ell s^*(T)$. Then – according to Theorem 1 – there exists A_k such that $T \subseteq A_k$. From the construction of T it follows, however, that $a_{k+1} \in A_{k+1} - A_k$, i.e. $a_{k+1} \notin A_k$, and simultaneously $a_{k+1} \in T$, which is a contradiction.

Since X^* is an uncountable class, there is $F^* : V^* \longleftrightarrow X^*$. Put now $Y = F^{*}{}'' T$.

Remark. The previous theorem implies that there exists a countable system of classes in our model which cannot be coded. This circumstance raises hopes that it could be possible to create a model in which even the weak axiom of choice does not hold. Such a model is described in the following section.

§ 3. Model of AST − AC + ¬ WAC. This model will contain all the classes from the first model. In addition, we join here a special class (and therefore many other classes that we can obtain from it, e.g., by Gödelian operations) which prevents the validity of WAC. The class will be denoted FR (in fact, FR is a relation created from standard extensions of FN) and defined as follows:

Definition. FR is such a class that dom (FR) = FN and

$$(\forall n \in FN)\ FR''\{n\} = Ex_n(FN).$$

Note that the larger the endomorphic universe A_n is, the smaller is the extension $Ex_n(FN)$.

Lemma 1. For each $n \in FN$

$$FR \restriction n = Ex_n(Z),$$

where $(Z \text{ is a fully revealed class})^{A_n}$.

Proof. From the definition of FR, (A2), Lemma 3, § 1 and (A11) it follows that

$$FR \restriction n = Ex_n\ (Ex_{n-1\to n}(FR \restriction n \cap A_{n-1}));$$

this completes the proof.

Now we shall introduce new relations of indiscernibility in which the class FR will be a figure.

Let us denote $\overset{o}{=}_{\{Y,\, FR\restriction n\}}$ by ${}^{n}\overset{o}{=}_{\{Y\}}$.

Corollary. $(\forall n \in FN) \overset{n}{\underset{\{Y\}}{\stackrel{o}{=}}} = \underset{\{\overline{Y}\}}{\stackrel{o}{=}}$
for a suitable $\overline{Y}$.

Proof. It follows directly from Lemma 1.

Definition. Let us put

$$\overset{\omega}{\underset{\{Y\}}{\stackrel{o}{=}}} = \bigcap_{n \in FN} \overset{n}{\underset{\{Y\}}{\stackrel{o}{=}}}$$

The relation $\overset{\omega}{\underset{\{Y\}}{\stackrel{o}{=}}}$ is obviously a refinement of all relations $\overset{n}{\underset{\{Y\}}{\stackrel{o}{=}}}$, $n \in FN$.

Lemma 2. The class FR is, for each Y, a figure in $\overset{\omega}{\underset{\{Y\}}{\stackrel{o}{=}}}$.

Proof. It is sufficient to realize (see the definition of $\overset{n}{\underset{\{Y\}}{\stackrel{o}{=}}}$) that for each $n \in FN$ the class $FR \upharpoonright n + 1$ is a figure in $\overset{n}{\underset{\{Y\}}{\stackrel{o}{=}}}$.

The next assertion that will further be used substantially, is a generalization of Lemma 7, § 1.

Theorem 1. Let $(X \text{ be fully revealed})^{A_n}$, $a,b \in A_n$, $Y = Ex_n(X)$ and $a \subseteq \mu_{\overset{n}{\underset{\{Y\}}{\stackrel{o}{=}}}}(b)$. Then $a \subseteq \mu_{\overset{\omega}{\underset{\{Y\}}{\stackrel{o}{=}}}}(b)$.

Proof. Obviously it suffices to prove that for each $k \in FN$, $k > n$, the inclusion $a \subseteq \mu_{\overset{k}{\underset{\{Y\}}{\stackrel{o}{=}}}}(b)$ holds. This fact follows - using induction - from Lemma 7, § 1 and the equality (see the definition of FR):

$FR \upharpoonright k + 1 = Ex_k((FR \upharpoonright k) \cap A_k) \cup Ex_k(FN \times \{k\})$.

We shall create now the second model. The definitions of classes, relations $=^*$ and $\in^*$ are similar to those ones in the first model. We have only to substitute there $\{Ex_n^{\stackrel{o}{=}}(Z)\}$ by

$\overset{\omega_{\underline{\varrho}}}{\{Ex_n(Z)\}}$. We leave the detailed reformulation to the reader.

Notice that X^*, Y^*,... will mean now classes in the second model. To prevent any misunderstanding when further speaking about classes of the first model, we then shall express this explicitly.

Remark. Note that the definition of classes in this model really ensures that each class in the first model is also a class in the second one (the converse assertion is not, of course, true owing to FR). This fact will help us to verify here the individual axioms (and auxiliary statements, too). If it is possible, we shall not give further detailed argumentations but only modify procedures of the analogous assertions from § 2.

Lemma 3. Let $t, u \in A_\ell$, $\ell > k$. Then

$$t \overset{\omega_{\underline{\varrho}}}{\{Ex_k(Z)\}} u \equiv t \overset{\ell_{\underline{\varrho}}}{\{Ex_k(Z)\}} u.$$

Proof. The assertion is an obvious consequence of Theorem 1. Put there e.g. $a = \{t\}$ and $b = u$.

Lemma 4. Let $(Z \text{ be fully revealed})^{A_k}$. Then

(i) If $x \overset{\omega_{\underline{\varrho}}}{\{Ex_k(Z)\}} y$, $x \neq y$ and if $x, y \in A_\ell$, where $\ell \geq k$, then there is $a \notin Fin$, $a \in A_\ell$ such that $a \subseteq \mu_{\overset{\omega_{\underline{\varrho}}}{\{Ex_k(Z)\}}}(x)$.

(ii) If $\mu_{\overset{\omega_{\underline{\varrho}}}{\{Ex_k(Z)\}}}(x) = \{x\}$, then $x \in A_k$.

Proof. For (i), at first, notice (see Lemma 3) that $x \overset{\omega_{\underline{\varrho}}}{\{Ex_k(Z)\}} y \equiv x \overset{\ell_{\underline{\varrho}}}{\{Ex_k(Z)\}} y$. We claim that $\mu_{\overset{\ell_{\underline{\varrho}}}{\{Ex_k(Z)\}}}(x)$ is a non-trivial monad which contains an infinite set from A_ℓ . This assertion follows (see Lemma 1 and Corollary) from the fact that $\overset{\ell}{\underset{\{Ex_k(Z)\}}{\stackrel{\circ}{=}}} = \underset{\{Ex_k(\bar{Z})\}}{\stackrel{\circ}{=}}$ for a suitable $\bar{Z}$, ($\bar{Z}$ fully revealed)^{A_k}, and from Lemma 4, § 1 and Theorem 3, § 1.

For proving (ii) let us assume that ℓ, ($\ell \geq k$), is the smallest number for which $x \in A_\ell$. We show, by contradiction, that $\ell = k$. Suppose therefore $\ell > k$. Then $\mu\, \overset{\ell}{\underset{\{Ex_k(Z)\}}{\stackrel{\circ}{=}}}(x) \cap A_\ell =$ $= \{x\}$ since for $t, u \in A_\ell$ we have - in accordance with Lemma 3 - that

$$t \overset{\ell}{\underset{\{Ex_k(Z)\}}{\stackrel{\circ}{=}}} u \equiv t \overset{\omega}{\underset{\{Ex_k(Z)\}}{\stackrel{\circ}{=}}} u.$$

Hence x is definable in A_ℓ from $Ex_{k\to\ell}(Z)$ and $Ex_{\ell-1\to\ell}(FR \restriction \ell) \cap$ $\cap A_{\ell-1}$. Thus, using commutativity of Ex and (A10), we obtain that x is definable in $A_{\ell-1}$ from $Ex_{k\to\ell-1}(Z)$ and $(FR \restriction \ell \cap A_{\ell-1})$, which contradicts the choice of ℓ.

<u>Lemma 5</u>. Let $t, u \in A_\ell$, $\ell > k$. Let further $t \overset{\omega}{\underset{\{Ex_k(Z)\}}{\stackrel{\circ}{=}}} u$. Then there is an automorphism F such that $F(t) = u$ and $F'' Ex_k(Z) =$ $= Ex_k(Z)$. Moreover, $F'' V^* = V^*$ and $C\ell s^*(F \cap V^*)$.

<u>Proof</u>. From the definition of $\overset{n}{\underset{\{Y\}}{\stackrel{\circ}{=}}}$ and Lemma 1 it follows that

$$\overset{\ell}{\underset{\{Ex_k(Z)\}}{\stackrel{\circ}{=}}} = \underset{\{Ex_k(Z), FR\restriction \ell\}}{\stackrel{\circ}{=}} = \underset{\{Ex_k(\bar{\bar{Z}}), Ex_\ell(\bar{Z})\}}{\stackrel{\circ}{=}},$$

where ($\bar{Z}$ is fully revealed)$^{A_\ell}$. Moreover, commutativity of Ex implies that

$$\underset{\{Ex_k(\bar{\bar{Z}}), Ex_\ell(\bar{Z})\}}{\stackrel{\circ}{=}} = \underset{\{Ex_\ell(Ex_{k\to\ell}(Z)), Ex_\ell(\bar{Z})\}}{\stackrel{\circ}{=}}$$

Since $Ex_{k\to\ell}(Z)$ and $\bar{Z}$ are both standard extensions $Ex_{\ell-1\to\ell}(Z_1)$ for suitable Z_1, the same is valid for their couple. This couple is therefore (a fully revealed class)$^{A_\ell}$.

Now put in mind Lemma 3 and proceed analogously to Lemma

2, § 2. Let F be that automorphism. Then $F(t) = u$ and also $F'' Ex_k(Z) = Ex_k(Z)$ since $Ex_k(Z)$ is the first component of the couple which is "preserved" by F. As V^* is the same in both models, we have that $F'' V^* = V^*$. The assertion $C\ell s^*(F \cap V^*)$ follows from the fact that $F \cap V^*$ is even a class of the first model.

Lemma 6. Let F be such an automorphism that $F'' V^* = V^*$ and $C\ell s^*(F \cap V^*)$. Then

$$C\ell s^*(X) \equiv C\ell s^*(F'' X).$$

Proof. Modify the proof of Lemma 3, § 2 in such a way: replace $\overset{\circ}{\underset{\{Ex_k(Z)\}}{=}}$ by $\overset{\omega\circ}{\underset{\{Ex_k(Z)\}}{=}}$ and note that (there is, of course, A_ℓ, $\ell > k$, such that $t, u \in A_\ell$)

$$t \overset{\omega\circ}{\underset{\{Ex_k(Z)\}}{=}} u \quad \text{iff} \quad t \overset{\ell\circ}{\underset{\{Ex_k(Z)\}}{=}} u$$

Hence $t \overset{\circ}{\underset{\{Ex_k(\overline{Z})\}}{=}} u$ (see Corollary of Lemma 1).

(Morse s scheme)*. For every formula $\varphi(x, X_1, \dots, X_n) \in FL$ and for every $X_1^*, \dots, X_n^*$ there exists a class Y such that $C\ell s^*(Y)$ and

$$(\forall x \in V^*)(x \in Y \equiv \varphi^*(x, X_1^*, \dots, X_n^*)).$$

Proof. It is enough to modify the proof of the Morse´s scheme in the first model. Substitute there $\overset{\circ}{\underset{\{Ex_k(Z)\}}{=}}$ by $\overset{\omega\circ}{\underset{\{Ex_k(Z)\}}{=}}$ and instead of Lemmas 2, 3 of § 2, consider now Lemmas 5, 6.

Lemma 7. $FN^* = FN$.

Proof. Since FN is the class of the first model (see Lemma 4, § 1), we have here $C\ell s^*(FN)$, too. The assertion $FN^* = FN$

follows now from the same equality in the first model and from the fact that the second model contains a greater amount of classes.

Theorem 2. Let X^* be a countable class of V^*. Then there is an endomorphic universe A_k such that $X^* \subseteq A_k$.

Proof. $C\ell s^*(X)$ implies that X is a figure in $\overset{\omega_0}{\underset{\{Ex_{k}(Z)\}}{\doteq}}$ for $Z \subseteq A_k$, (Z fully revealed)^{A_k}. But X^* is a countable class. Therefore (see Lemma 4 (i)) all monads of X^* are trivial. For proving the fact that $X^* \subseteq A_k$, apply the second assertion of Lemma 4.

Corollary. Count* $(X^*) \equiv$ Count (X^*).

Proof. Modify, using Lemma 7 and the previous theorem, the proof of the analogous assertion from the first model.

Since sets and countable classes are the same in both models, we obtain immediately that the following statement holds:

(Axiom of prolongation)* . Let (F^* be a countable function)* , then there exists a function f^* such that $F^* \subseteq f^*$.

(Axiom of cardinalities)* . Each uncountable class X^* can be mapped by a one-one function onto V^*.

Proof. Lemma 4 (i) implies that each uncountable class of our model contains an infinite set; let us denote it a. Since, in the first model, there exists a function F such that $F: a \longleftrightarrow V^*$, this function is also a class in the second model. Now see the proof of the axiom of cardinalities in the first model.

(Negation of weak axiom of choice)* . There is such a relation R^* with dom $(R^*) = FN$ that for any function F^* with dom $(F^*) = FN$, the condition $F^* \subseteq R^*$ does not hold.

Proof. Put $R^* = FR - (FN \times FN)$ and suppose that F^* is such a function that dom $(F^*) = FN$ and $F^* \subseteq R^*$. Let us prolong F^* and denote the new function by g^*. Then $F^* = g^* \upharpoonright FN$. Since $g^* \in A_n$ for a suitable n, we have $g^*(n) \in A_n$ (notice that $n \in A_n$). Therefore $(Ex_n(FN) - FN) \cap A_n \neq \emptyset$ (according to (A2) we know that $Ex_n(FN) \cap A_n = FN$), which is a contradiction.

Theorem 3. Each uncountable class X^* contains a countable class Y such that $\neg\, C\ell s^*(Y)$.

Proof. As both models have the same countable classes, Theorem 3 follows directly from the validity of the analogous assertion in the first model and from the axiom of cardinalities.

§ 4. The construction of an increasing sequence of endomorphic universes with standard extension.

The construction of both the models mentioned above lies substantially on the existence of an increasing sequence of endomorphic universes with standard extension. The last section of our paper will be devoted just to proving that such a sequence exists. If the following text will remind someone of the construction of the iterated ultraproduct, we stress that the similarity is quite accidental and that its content is but a pure fiction.

At first we shall recall several notions and results from [S-V1], we shall further need.

For an arbitrary class A and arbitrary set d we put

$$A[d] = \{f(d);\ f \in A\}.$$

Theorem (A). Let A be an endomorphic universe and let $d \in \bigcup A$. Then $A[d]$ is the smallest endomorphic universe, the subclass of which is the class $A \cup \{d\}$.

From the definition of $A[d]$ it follows now:

Lemma 1. Let A be an endomorphic universe. Then for each function $f \in A$ and each $d \in \bigcup A$ the condition

$$A[f(d)] \subseteq A[d]$$

holds.

Theorem (B). Let A be an endomorphic universe and let $c, d \in \bigcup A$. Then $A[c] = A[d]$ iff there is a one-one mapping $f \in A$ with $c = f(d)$.

If A is an endomorphic universe, then we put for each $X \subseteq A$

$$E_A(X) = \bigcap \{u \in A;\ X \subseteq u\}.$$

Theorem (C). An endomorphic universe A has a standard extension iff

$$V = \bigcup \{E_A(X),\ X \subseteq A \ \&\ X \preccurlyeq FN\}.$$

Now we shall introduce some notions which make our next considerations easier.

Definition. An ultrafilter $\mathcal{F}$ is called an ultrafilter on FN iff

$$(\forall X \in \mathcal{F})\ FN \cap X \neq \emptyset.$$

Since we shall be further interested only in ultrafilters on semisets (namely on the countable ones), we shall restrict ourselves only on sets; ultrafilters are nowfully determined by their sets.

For ultrafilters on FN we shall define an ordering (in fact, it is Rudin-Keesler's ordering on ultrafilters; cf. [C-H]).

Definition. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be ultrafilters on FN. We shall say that $\mathcal{F}_2$ is stronger than $\mathcal{F}_1$ with respect to a function f (denotation $\mathcal{F}_1 \overset{f}{\preceq} \mathcal{F}_2$) iff dom (f) $\supseteq$ FN, f"FN $\subseteq$ FN and, for each $x \in \mathcal{F}_2$, f"$x \in \mathcal{F}_1$. We say, moreover, that $\mathcal{F}_2$ is stronger than $\mathcal{F}_1$ (denotation $\mathcal{F}_1 \preceq \mathcal{F}_2$) iff there exists a function f such that $\mathcal{F}_1 \overset{f}{\preceq} \mathcal{F}_2$.

Let further A denote, similarly to previous paragraphs, an endomorphic universe with standard extension.

Definition. Let $x \in Ex_A(FN)$. The class

$$\{ y;\ x \in Ex_A(FN \cap y) \}$$

will be called a filter determined by x and denoted by $\mathcal{F}il\,(x)$.

Obviously, for each $x \in Ex_A(FN)$, the class $\mathcal{F}il\,(x)$ is an ultrafilter on FN.

Lemma 2. Let $f \in A$ be a function. Then

$(\forall d \in \mathrm{dom}\,(f))\ \mathcal{F}il\,(f(d)) \overset{f}{\preceq} \mathcal{F}il\,(d)$.

Proof is evident.

Definition. Let $\mathcal{F}$ be an ultrafilter on FN. Then the class

$$\bigcap \{ Ex_A(y \cap FN);\ y \in \mathcal{F} \}$$

is called a monad of ultrafilter $\mathcal{F}$ and denoted by $\mu\,(\mathcal{F})$.

Let us note that there is an ultrafilter $\mathcal{F}$ on FN such that $\mu\,(\mathcal{F}) = \emptyset$.

From the definitions of ordering on ultrafilters and monads of ultrafilters, the next two assertions follow immediately.

Theorem 1. (i) Let $x \in Ex_A(FN)$. Then $x \in \mu\,(\mathcal{F}il\,(x))$.

(ii) Let $\mathcal{F}$ be an ultrafilter on FN. Then

$$(\forall x \in \mu(\mathcal{F})) \quad \mathcal{F} = \mathcal{F}il\,(x).$$

<u>Theorem 2.</u> Let $\mathcal{F}$ be an ultrafilter on FN, $x \in Ex_A(FN)$. Then

$$\mathcal{F} \preceq \mathcal{F}il\,(x) \equiv (\exists f \in A)(\mathrm{dom}(f) \supseteq FN \,\&\, f''FN \subseteq FN \,\&\, \mathcal{F} = \mathcal{F}il\,(f(x))).$$

<u>Theorem 3.</u> For each ultrafilter $\mathcal{F}$ on FN there exists an endomorphic universe A (with standard extension) and $x \in Ex_A(FN)$ such that

$$V = A[x] \;\&\; \mathcal{F} = \mathcal{F}il\,(x).$$

<u>Proof.</u> See [S-V1], § 3.

<u>Definition.</u> We say that c_1 is <u>much smaller than</u> c_2 (denotation $c_1 << c_2$) iff

$$(c_1 \in Ex_A(FN)) \,\&\, (\forall f \in A)\,[\,c_2 \in Ex_A(f^{-1}{}''FN)) \Rightarrow$$
$$\Rightarrow (f(c_2) \in FN \vee f(c_2) \sim c_1)\,].$$

<u>Definition.</u> Let $\alpha \in Ex_A(FN)$ and let $f \in A$ be a function with dom $(f) \supset FN$. We say that $\beta \in Ex_A(FN)$ is <u>the second component of</u> α <u>with respect to</u> f iff α is the β-th element of $f^{-1}{}''\{f(\alpha)\}$.

Let $x \in Ex_A(\sigma)$, where σ is a countable subclass of A. Let $f \in A$ be a function with dom $(f) \supset \sigma$. We call $\beta \in Ex_A(\sigma)$ <u>the second component of</u> x <u>with respect to</u> f <u>and</u> σ iff x is the β-th element of $f^{-1}{}''\{f(x)\}$ in a fixed chosen ordering of σ by the type ω.

<u>Remark.</u> Notice that all the above mentioned definitions and assertions concerning ultrafilters on FN can be, in an obvious manner, reformulated for ultrafilters on countable subclasses of A. We shall further suppose to have such modifica-

tions.

__Lemma 3.__ Let $x \in Ex_A(\sigma)$, σ a countable subclass of A. Le $f \in A$ be a function with dom $(f) \supseteq \sigma$. Then $f(x) \in E_A(f''(\sigma))$.

__Proof__ is easy and can be left to the reader.

__Theorem 4.__ Let $A[d] = V$, where $d \in Ex_A(\sigma)$ and $\sigma \subseteq A$ is countable. Let $f \in A$ be a function with dom $(f) \supseteq \sigma$ and let $\beta \in$ $\in Ex_A(\sigma)$ be the second component of d with respect to f and σ. If $\beta << f(d)$, then $A[f(d)]$ is an endomorphic universe with standard extension.

__Proof.__ $A[f(d)]$ is evidently an endomorphic universe; therefore it remains to prove that $A[f(d)]$ can be standardly extended. Without loss of generality, we can suppose that σ = FN and $f''FN \subseteq FN$. Then $d \in Ex_A(FN)$. Put $c = f(d)$. We show that $\beta \in$ $\in E_{A[c]}$ (FN). To this end it is necessary and sufficient to prove

$(x \in A[c] \;\&\; x \supset FN) \Longrightarrow \beta \in x.$

Put $\gamma = \max\{\delta ; \delta \subset x\}$. Then $\delta \in A[c]$ and thus for a suitable function $g \in A$, we have $\delta = g(c)$. Since $\beta << c$, we obtain $\beta \in \delta$ $(\delta \supset FN)$. But $\delta \subset x$; hence $\beta \in x$.

Now we show that for suitable σ_1, where σ_1 is a countable subclass of $A[c]$, it is true that $d \in E_{A[c]}(\sigma_1)$. In accordance with Lemma 3 it suffices to prove that there is such a function g that $g \in A[c]$ and $d = g(\beta)$.

Let the function $\bar{g}$ be defined as follows: $\bar{g}(t,\alpha)$ is the α-th element of $f^{-1}{}''\{t\}$. Obviously $\bar{g} \in A$. Put now $g(\alpha) =$ $= \bar{g}(c,\alpha)$.

For completing the proof it is now enough to realize that for every $x \in V$ we have $x = h(d)$, for suitable $h \in A \subset A[c]$, and apply once more Lemma 3.

Remark. Let us stress the fact that if $f(d) << \beta$, then $A[f(d)]$ has no standard extension. This result is not quite obvious.

For the construction of an increasing sequence of endomorphic universes with standard extension it suffices now to find a suitable endomorphic universe A with standard extension, a suitable element $d \in V$ and such a sequence of functions $f_1, f_2, \ldots$ from A for which the second component β_i $(i \in FN)$ of d with respect to f_i and σ (σ is a countable subclass of A such that $d \in E_A(\sigma)$) is much smaller than $f_i(d)$ and $f_i(d) \overset{A}{\prec} f_{i+1}(d)$.

We define the symbol $\overset{A}{\prec}$ as follows:

$x \overset{A}{\preccurlyeq} y \equiv (\exists f \in A)\ x = f(y)$;

$x \overset{A}{\prec} y \equiv x \overset{A}{\preccurlyeq} y$ and there is no function $g \in A$ such that g is a one-one mapping and $x = g(y)$.

If we put now $A_i = A[f_i(d)]$, we obtain a sequence of endomorphic universes with standard extension for which $A_1 \subsetneqq A_2 \subsetneqq \ldots$. The ideas, just described, will be now precised.

Firstly, we give a definition.

Definition. Let $\mathcal{F}_i$ be ultrafilters on σ_i, $\mathcal{F}$ be an ultrafilter on σ , where σ , σ_i are countable classes ($i \in \sigma$). Then the ultrafilter $\overline{\mathcal{F}} = \mathcal{F} - \sum \mathcal{F}_i$ is called an $\mathcal{F}$ - sum of ultrafilters $\mathcal{F}_i$ and defined in such a way:

$\overline{\mathcal{F}}$ is an ultrafilter on $\underset{i \in \sigma}{\times} \sigma_i = \{\langle x,i\rangle;\ x \in \sigma_i \,\&\, i \in FN\}$ and

$$(\forall u)(u \in \overline{\mathcal{F}}) \equiv (\forall t)(t \supset \{i; u''\{i\} \in \mathcal{F}_i\} \Rightarrow t \in \mathcal{F}).$$

If $\mathcal{F}_i$ are equal we write instead of $\overline{\mathcal{F}} = \mathcal{F} - \sum \mathcal{F}_i$ only $\overline{\mathcal{F}} = \mathcal{F} \times \mathcal{F}$.

__Theorem 5.__ Let $\mathfrak{S} \subset A$ be a countable class. Let $\mathcal{F}_1$, $\mathcal{F}_2$, respectively, be non-trivial ultrafilters on $\mathfrak{S}$, FN resp. and $\mathcal{F} = \mathcal{F}_2 \times \mathcal{F}_1$. Let further $d \in Ex_A(\mathfrak{S} \times FN)$ and $\mathcal{F} = \mathcal{F}il(d)$. Then (Pr denotes the projection function)

(i) $Pr_2(d) << Pr_1(d)$

(ii) $Pr_1(d) \overset{A}{\not\prec} d$

(iii) $Pr_2(d)$ is the second component of d with respect to Pr_1 and $\mathfrak{S} \times FN$.

__Proof.__ At first we shall prove an auxiliary assertion: Under the assumptions of Theorem 5 it is true that

$$\mathcal{F}_i = \mathcal{F}il(Pr_i(d)) \qquad (i = 1,2).$$

We have to show that

$$(\forall u)\ u \in \mathcal{F}_i \equiv u \ni Pr_i(d) \qquad (i = 1,2).$$

Let $i = 1$. Then

$u \times FN \ni \mathcal{F} \equiv u \times FN \ni d \equiv u \in Pr_1(d)$.

For $i = 2$, substitute FN by $\mathfrak{S}$ and proceed analogously.

To prove (i) suppose that $f \in A$ is such a function that $f(Pr_1(d)) < Pr_2(d)$. Then the same is valid for a set of the ultrafilter $\mathcal{F}$. Thus, for a certain component j, we have (see the definition of $\mathcal{F} = \mathcal{F}_2 \times \mathcal{F}_1$) $u''\{j\} \in \mathcal{F}_1$. Hence $u''\{j\} \ni Pr_1(d)$ and therefore $f(Pr_1(d)) = f(Pr_1(\langle Pr_1(d), j\rangle)) < j$. Since $j \in FN$, the validity of (i) is demonstrated.

We prove the assertion (ii) by contradiction. Let $g \in A$ be a one-one mapping for which $Pr_1(d) = g(d)$. Then $Pr_2(d) = Pr_2(g^{-1}(Pr_1(d)))$ which contradicts $Pr_2(d) << Pr_1(d)$ - see (i).

The statement (iii) is obvious.

It follows from [V], ch. II, § 4 that there is a non-trivial ultrafilter $\mathcal{F}$ on FN.

Let us put

$$FN^i = \underbrace{FN \times FN \times \ldots \times FN}_{i\text{-times}}$$

and define ultrafilters $\mathcal{F}_i$ on FN^i in such a way:

$$\mathcal{F}_1 = \mathcal{F}\ , \quad \mathcal{F}_{i+1} = \mathcal{F} \times \mathcal{F}_i .$$

Further put $\overline{\mathcal{F}} = \mathcal{F} - \sum \mathcal{F}_i$ and denote $\wp = \underset{i \in FN}{\times} FN^i$. The class $\wp$ is, evidently, countable. From Theorem 3 we know that for $\overline{\mathcal{F}}$ on $\wp$ there is an endomorphic universe A (with standard extension) and $d \in Ex_A(\wp)$ such that $V = A[d]$ and $\overline{\mathcal{F}} = \mathcal{F}il(d)$.

On $\wp$, we shall define functions f_i: If $x \in \wp$ are such elements that $Pr_{ls}(x) > i$, (Pr_{ls} denotes the last projection), then $f_i(x) = \langle Pr_1(x), \ldots, Pr_i(x) \rangle$.

Denote $Pr_i(d) = d_i$ and put $c_i = \langle d_1, \ldots, d_i \rangle$. We would like to show that, for every i, the class $A[c_i]$ is an endomorphic universe with standard extension.

Put further $\overline{d} = \langle\langle d_1, \ldots, d_i \rangle, \langle d_{i+1}, \ldots \rangle\rangle$. Then $A[d] = A[\overline{d}]$ since there exists a one-one mapping $g \in A$ such that $d = g(\overline{d})$. If we denote $\beta = \langle d_{i+1}, \ldots \rangle$, we obtain that $\overline{d} = \langle c_i, \beta \rangle$.

Under the above stated denotations we prove

<u>Lemma 4</u>. $\mathcal{F}il(\overline{d}) = \mathcal{F}il(\beta) \times \mathcal{F}il(c_i)$.

<u>Proof</u>. Let, at first, $u \in \mathcal{F}il(\overline{d})$. Then $u \ni \overline{d}$. Let $m \in A$ be such that

$$m \supseteq \{x \in \mathrm{dom}(\overline{\wp});\ \ u''\{x\} \ni c_i\}$$

($\overline{\wp}$ is obtained from $\wp$ by an obvious manner).

We prove that $m \in \mathcal{F}il(\beta)$; i.e. that $m \ni \beta$. Since $\langle c_i, \beta \rangle = \overline{d}$, we have $u''\{\beta\} \ni c_i$ and hence $\beta \in m$. Thus $u \in \mathcal{F}il(\beta) \times \mathcal{F}il(c_i)$.

For proving the statement:

$$u \in \mathfrak{Fil}(\beta) \times \mathfrak{Fil}(c_i) \Rightarrow u \in \mathfrak{Fil}(\bar{d}),$$

follow the proof of the first part going "from bottom to top".

Theorem 6. $A[c_i]$ is, for each i, an endomorphic universe with standard extension (c_i are defined above).

Proof. Owing to Lemma 4 and Theorem 5 (iii), we know that β is the second component of c_i with respect to Pr_i. Due to Theorem 5 (i), we have further that $\beta << c_i$. Hence (see Theorem 4) $A[c_i]$ is an endomorphic universe with standard extension. Moreover $A[c_i] \subsetneqq A[d] = V$ - since, in accordance with Theorem 5 (ii) - we have $c_i \overset{A}{\prec} d$.

Theorem 7. $(\forall i \in FN)\ A[c_i] \subsetneqq A[c_{i+1}]$.

Proof. The inclusion $A[c_i] \subseteq A[c_{i+1}]$ follows from the facts that $c_i = \langle Pr_1(\langle c_{i+1}, i+1\rangle), \dots\ Pr_i(\langle c_{i+1}, i+1\rangle)\rangle$ and projections are functions from A. For proving $A[c_i] \neq A[c_{i+1}]$ it suffices to realize that

$$\mathfrak{Fil}(c_{i+1}) = \mathfrak{Fil}(d_{i+1}) \times \mathfrak{Fil}(c_i);$$

it is namely $c_{i+1} = \langle c_i, d_{i+1}\rangle$ and (see Theorem 5 (ii)) $c_i \overset{A}{\prec} c_{i+1}$.

Remark. In [AV] there is constructed a model similar to our first one. Its construction lies there on an increasing sequence $\{A_\alpha ; \alpha \in \Omega\}$ of endomorphic universes with standard extension. The existence of such a sequence is not, however, shown there explicitly. If one supposes the second order choice, i.e.

$$(\forall x)(\exists Y)\ \varphi(x,Y) \Rightarrow (\exists \bar{Y})(\forall x)\ \varphi(x, \bar{Y}''\{x\}),$$

it is possible to prove the existence of $\{A_\alpha ; \alpha \in \Omega\}$ in such a way: Starting from a fixed non-trivial ultrafilter on FN we can create in AST the structure $\mathfrak{A}$ which is Ω-times iterated ultraproduct. This structure is saturated, elementarily equivalent to V and has cardinality Ω. But V is, owing to the axiom of prolongation, also a saturated structure. Therefore there is an isomorphism $F: \mathfrak{A} \longleftrightarrow V$. Now we obtain A_α as images of α-th steps of the iteration process.

We have preferred in our paper, § 4, to avoid the second order choice and, in addition, we have used the methods being more fit for AST.

<u>Problem</u>. Thanks to WAC, in the first model, we know that each countable union of countable classes is a countable class. This assertion is also valid in the second model. A question arises: Is there such a model of AST - AC in which V is the union of countably many countable classes ? Or, in a weaker form, is it possible for V to be a union of countably many semisets there ? The answers are unknown to us.

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