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INTERIOR REGULARITY FOR THE QUASILINEAR ELLIPTIC SYSTEMS WITH NONSMOOTH COEFFICIENTS

Jiří KOTTAS

Abstract: The interior $C^{0,\alpha}$-regularity for a weak solution of the quasilinear second order elliptic system is investigated. The positive answer is obtained for systems which are "not far" from the Laplace equations. This situation is described by means of the dispersion of eigenvalues of the coefficients matrix.

Key words: Quasilinear elliptic systems, interior regularity.

Classification: 35B65, 35J60

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1. Introduction. The paper deals with $C^{0,\alpha}$-regularity of solutions of second order quasilinear elliptic systems with nonsmooth coefficients satisfying certain conditions of the dispersion of eigenvalues. This condition was firstly established by A.I. Koshelev. (See [3] for references.) Our aim is to obtain a simpler proof and to this end we use a modification of the method of J. Nečas for smooth coefficients described in [2]. We consider a slightly more general condition of ellipticity than in [3] (which does not guarantee unicity of solutions of Dirichlet problem) and we prove that every weak solution is locally Hölder-continuous.

2. Notations and definitions. We consider the quasilinear system

$$(2.1)\qquad \sum_{\alpha,\beta=1}^{n} \sum_{j=1}^{m} D_\alpha(a_{ij}^{\alpha\beta}(x,u)\, D_\beta u^j)=0 \qquad i=1,\dots,m,$$

where $u=[u^1,\dots,u^m]$ is a vector function defined on a bounded domain $\Omega\subset \mathbb{R}^n$.

The coefficients $a_{ij}^{\alpha\beta}:\Omega\times \mathbb{R}^m \longrightarrow \mathbb{R}$ are bounded Carathéodory functions, symmetric (i.e. $a_{ij}^{\alpha\beta}=a_{ji}^{\beta\alpha}$) and satisfying the following ellipticity condition:

(2.2) There are two positive numbers λ_0, λ_1 such that the inequalities

$$\lambda_0 |\xi|^2 \le \langle \mathbb{A}(x,p)\xi \, ; \xi\rangle \le \lambda_1 |\xi|^2$$

hold for all $\xi \in \mathbb{R}^{m\times n}$, $p \in \mathbb{R}^m$ and a.e. $x \in \Omega$.

$\mathbb{A}$ denotes here the matrix of coefficients $(a_{ij}^{\alpha\beta})_{i,j=1,\dots,m}^{\alpha,\beta=1,\dots,n}$.
$\langle \, ; \rangle$ is the inner product on the Euclidean space $\mathbb{R}^k$ $(k=m\times n)$,
$| \ |$ is the norm generated by this inner product.

In what follows, we shall suppose that $m \ge 2$, $n \ge 2$.

Definition 2.1. We say that the function $u \in W^1_{2,loc}(\Omega)$ (we shall write $W^1_{2,loc}(\Omega)$ instead of $[W^1_{2,loc}(\Omega)]^m$) is a weak solution of the system (2.1) if for each $\varphi \in \mathcal{D}(\Omega)$ we have

$$\int_\Omega \langle A(x,u)\, Du; D\varphi\rangle \, dx = 0.$$

Definition 2.2. The system (2.1) is said to be regular if each weak solution of (2.1) is locally Hölder-continuous on Ω.

We shall use so called Campanato spaces (denoted by $\mathcal{L}_{2,\lambda}(\Omega)$ or $L^c_{2,\lambda}(\Omega)$ - see [4]) which are for $\lambda \in]n,n+2]$ isomorphic to the spaces $C^{0,\alpha}(\bar{\Omega})$ with $\alpha = \frac{\lambda - n}{2}$

Introduce now in $\mathbb{R}^n$ the polar coordinates with the origin at the point y:

$x_1-y_1=r\cos\varphi_1$, $x_2-y_2=r\sin\varphi_1\cos\varphi_2, \dots,$
$x_{n-1}-y_{n-1}=r\sin\varphi_1\dots\sin\varphi_{n-2}\cos\varphi_{n-1}$,
$x_n-y_n=r\sin\varphi_1\,\dots\,\sin\varphi_{n-1}$

and define the symbols $\partial_1 v, \dots, \partial_n v$ as

$$\partial_1 v = \frac{\partial v}{\partial r}, \quad \partial_2 v = \frac{1}{r}\frac{\partial v}{\partial \varphi_1}, \quad \partial_3 v = \frac{1}{r\sin\varphi_1}\frac{\partial v}{\partial \varphi_2},$$

$$\partial_n v = \frac{1}{r\sin\varphi_1\dots\sin\varphi_{n-2}}\frac{\partial v}{\partial \varphi_{n-1}}.$$

Denote further $D_B v = [\partial_2 v, \dots, \partial_n v]$,

$B(y,R) = \{x \in \mathbb{R}^n; |x-y| < R\}$,

$S(y,R) = \{x \in \mathbb{R}^n; |x-y| = R\}$.

It is clear that for $x \in S(y,R)$, $(D_B v)(x)$ is the vector of derivatives of v in tangent directions to the sphere S(y,R). Put

$u_{y,R} = \frac{1}{\mu(B(y,R))}\int_{B(y,R)} u\,dx$ (μ is the Lebesgue measure on $\mathbb{R}^n$) and finally

$$K(n)=\frac{\sqrt{1+\frac{(n-2)^2}{n-1}}-1}{\sqrt{1+\frac{(n-2)^2}{n-1}}+1}.$$

3. Soft theorem. We present here Theorem 3.1 which is weaker than Theorem 4.1, because it can be proved in a transparent way.

Theorem 3.1. Let $\frac{\lambda_0}{\lambda_1} > \frac{n-2}{n}$. Then the system (2.1) is regular

Remark 3.2. Each solution of the quasilinear system (2.1) is also the solution of the linear system with bounded measurable coefficients ($b_{ij}^{\alpha\beta}(x)=a_{ij}^{\alpha\beta}(x,u(x))$), which satisfies the conditions (2.2) with the same constants λ_0, λ_1, hence it is sufficient to prove the theorem only for linear systems.

Proof: Let u be a weak solution of the system (2.1) and let $\Omega_1 \subset\subset \Omega$. In order to prove that $u \in C^{0,\alpha}(\overline{\Omega}_1)$ we have to show that for some $\beta > n$ the function $g(x_0,R)=R^{-\beta}\int_{B(x_0,R)}|u-u_{x_0,R}|^2dx$ is bounded on the set $M=\Omega_1\times]0;d[$, where $d=\frac{1}{2}\operatorname{dist}(\Omega_1,\partial\Omega)$. Using Poincaré inequality

$$\int_{B(x_0,R)}|u-u_{x_0,R}|^2\,dx \le c\,R^2\int_{B(x_0,R)}|Du|^2\,dx$$

we can see that it suffices for some $\gamma > n-2$ to show the boundedness of the function

$$f(x_0,R)=R^{-\gamma}\int_{B(x_0,R)}|Du|^2\,dx \text{ on } M.$$

The function f is bounded on the set $\Omega_1\times\{d\}$, hence it suffices to prove that $\frac{\partial f}{\partial R}(x_0,R)\ge 0$ for all $x_0\in\Omega_1$ and a.e. $R\in]0,d[$. The derivative $\frac{\partial f}{\partial R}$ exists for all $x_0\in\Omega_1$ and a.e. $R\in]0,d[$ and

$$\frac{\partial f}{\partial R}=-\gamma R^{-\gamma-1}\int_{B(x_0,R)}|Du|^2\,dx+R^{-\gamma}\int_{\partial B(x_0,R)}|Du|^2\,dS.$$

For $(x_0,R)\in M$ we denote $v=v(x,x_0,R)$ the vector function which is a weak solution in $W_2^1(B(x_0,R))$ of the system

$$\Delta v^j=0 \qquad j=1,\dots,m$$

and satisfies the stable boundary condition $u-v\in \mathring{W}_2^1(B(x_0,R))$.

Now we shall prove two lemmas to finish the proof of the theorem.

Lemma 3.3. For all $(x_0,R)\in M$ the inequality

$$(3.1)\qquad \int_{B(x_0,R)} |Du|^2\, dx \le \frac{\lambda_1+\lambda_0}{2\lambda_0} \int_{B(x_0,R)} |Dv|^2\, dx$$

holds.

Lemma 3.4. Let for some $a \in]1, \frac{n-1}{n-2}[$ and for all $(x_0,R) \in M$

$$\int_{B(x_0,R)} |Du|^2\, dx \le a \int_{B(x_0,R)} |Dv|^2\, dx .$$

Then

$u \in C^{0,\alpha}(\bar{\Omega}_1)$ with $\alpha = \frac{1}{2}(\frac{n-1}{a} - n+2)$.

Proof of Lemma 3.3. It is easy to see that

$$\int_{B(x_0,R)} \langle \mathbb{A}\, Du; D(v-u)\rangle\, dx=0$$

and

$$\int_{B(x_0,R)} \langle Dv; D(v-u)\rangle\, dx=0; \text{ hence}$$

$$\int_{B(x_0,R)} |Dv|^2\, dx= \int_{B(x_0,R)} \langle Dv;Du\rangle\, dx.$$

Now using the condition (2.2) and the symmetry of $\mathbb{A}$ we obtain

$$\int_{B(x_0,R)} |Du|^2\, dx \le \frac{1}{\lambda_0} \int_{B(x_0,R)} \langle \mathbb{A}\, Du, Du\rangle\, dx =$$

$$= \frac{1}{\lambda_0} \int_{B(x_0,R)} \langle \mathbb{A}\, Dv;Dv\rangle\, dx - \frac{1}{\lambda_0} \int_{B(x_0,R)} \langle \mathbb{A}\, D(v-u),D(v-u)\rangle\, dx \le$$

$$\le \frac{\lambda_1}{\lambda_0} \int_{B(x_0,R)} |Dv|^2\, dx - \int_{B(x_0,R)} |D(v-u)|^2\, dx =$$

$$=\frac{\lambda_1}{\lambda_0} \int_{B(x_0,R)} |Dv|^2\, dx - \int_{B(x_0,R)} |Dv|^2\, dx+2 \int_{B(x_0,R)} \langle Dv;Du\rangle\, dx -$$

$$- \int_{B(x_0,R)} |Du|^2\, dx =(1+ \frac{\lambda_1}{\lambda_0}) \int_{B(x_0,R)} |Dv|^2\, dx - \int_{B(x_0,R)} |Du|^2\, dx.$$

An easy calculation gives (3.1).

Proof of Lemma 3.4. For a weak solution $w \in W_2^1(B(x_0,R))$ of the system $\Delta w^j=0 \quad j=1,\dots,m$ the estimate

$$\int_{B(x_0,R)} |Dw|^2\, dx \le \frac{R}{n-1} \int_{S(x_0,R)} |D_B w|^2\, dS$$

holds. See [2].

As $D_B u=D_B v$ on $S(x_0,R)$ and $|D_B u|^2 \le |Du|^2$, we get from here

$$\int_{B(x_0,R)} |Du|^2\, dx \le a \int_{B(x_0,R)} |Dv|^2\, dx \le \frac{aR}{n-1} \int_{S(x_0,R)} |D_B v|^2\, dS =$$

$$= \frac{aR}{n-1} \int_{S(x_0,R)} |D_B u|^2\, dS \le \frac{aR}{n-1} \int_{S(x_0,R)} |Du|^2\, dS.$$

It easily follows that

$$R \int_{S(x_0,R)} |Du|^2\, dS - \frac{n-1}{a} \int_{B(x_0,R)} |Du|^2\, dx \ge 0.$$

Put $\gamma = \frac{n-1}{a}$. Then

$$\frac{\partial\gamma}{\partial R}(x_0,R)=R^{-\gamma}\int_{S(x_0,R)}|Du|^2\,dS-\gamma R^{-\gamma-1}\int_{B(x_0,R)}|Du|^2\,dx\geq 0 .$$

Q.E.D.

4. Hard theorem

Theorem 4.1. Let $\frac{\lambda_0}{\lambda_1}>K(n)$. Then the system (2.1) is regular.

Proof: Let us introduce the function space (see [3])

$$H_{2,\lambda}(\Omega)=\{u\in W_2^1(\Omega);\ \sup_{x_0\in\bar{\Omega}}\int_\Omega|Du|^2\,|x-x_0|^{-\lambda}\,dx<\infty\}$$

equipped with the norm

$$|u|_{H_{2,\lambda}}(\Omega)=(\int_\Omega|u|^2\,dx+\sup_{x_0\in\bar{\Omega}}\int_\Omega|Du|^2\,|x-x_0|^{-\lambda}\,dx)^{\frac{1}{2}}.$$

This space is for $\lambda>n-2$ imbedded into the space $C^{0,\alpha}(\bar{\Omega})$ with $\alpha=\frac{1}{2}(\lambda-n+2)$.

Let h be a non-zero element of $\mathcal{D}(\mathbb{R}_n)$, supp $h\subset B(0,1)$, $h\geq 0$.

Denote $h_k(x)=c_k h(kx)$, $k\in\mathbb{N}$, where c_k are constants such that

$$\int_{\mathbb{R}^n}h_k(x)\,dx=1$$

let $\Omega_2\subset\subset\Omega_1\subset\subset\Omega$, $\partial\Omega_1$ is sufficiently smooth.

Put $R=\frac{1}{4}\operatorname{dist}(\Omega_2,\partial\Omega_1)$, $a_{ij}^{\alpha\beta}\equiv 0$ on $\mathbb{R}^n\setminus\Omega$ and ${}^k a_{ij}^{\alpha\beta}=h_k * a_{ij}^{\alpha\beta}$.

Then $\lim_{k\to\infty}{}^k a_{ij}^{\alpha\beta}(x)=a_{ij}^{\alpha\beta}(x)$ a.e. on Ω and matrices ${}^k\mathbb{A}$ satisfy (for $k\geq k_0$) on Ω_1 the condition (2.2) with the same constants λ_0, λ_1.

The boundary value problem

$$\int_{\Omega_1}\langle {}^k\mathbb{A}\,Du_k,D\varphi\rangle=0\qquad\forall\varphi\in\overset{\circ}{W}{}_2^1(\Omega_1),\ u_k-u\in\overset{\circ}{W}{}_2^1(\Omega_1)$$

has a uniquely determined solution $u_k\in W_2^1(\Omega_1)$ for each $k>k_0$.

Obviously

$$|u_k|_{W_2^1(\Omega)}\leq c(\lambda_0,\lambda_1)\ |u|_{W_2^{1/2}(\partial\Omega_1)}.$$

The space $W_2^1(\Omega_1)$ is reflexive and so we can suppose that u_k is weakly convergent to some $v\in W_2^1(\Omega_1)$.

The set $V=\{w\in W_2^1(\Omega_1),\ w-u\in\overset{\circ}{W}{}_2^1\}$ is convex and closed, hence it is weakly closed and $v-u\in\overset{\circ}{W}{}_2^1(\Omega_1)$. Now we can apply the well known convergence lemma (see [1], chapt. 4) to see that v is a weak solution of the system (2.1) and hence - because of the uniqueness - v=u.

The function u is the weak limit of the sequence $\{u_k\}_{k>k_0}$

in $W_2^1(\Omega_1)$ and hence it is the strong limit in $L_2(\Omega_1)$ so we can suppose that

$$\lim_{k\to\infty} u_k(x)=u(x) \text{ a.e. on } \Omega.$$

All functions u_k are of the class $C^\infty_{loc}(\Omega_1)$. Choose $x_0\in\Omega_2$, $\eta\in\mathcal{D}(B(x_0,2R))$, $\eta=1$ on $B(x_0,R)$, $|D\eta|<\frac{c}{R}$, and $\psi=[\psi_1,\dots,\psi_m]\in\mathcal{D}(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n}\langle {}^k\mathbb{A}\, D(u_k\eta),D\psi\rangle\, dx =$$

$$= \int_{\mathbb{R}^n}\langle {}^k\mathbb{A}\, u_k D\eta, D\psi\rangle\, dx - \int_{\mathbb{R}^n}\langle {}^k\mathbb{A}\, Du_k, D\eta\,\psi\rangle\, dx.$$

Putting $\gamma=\frac{2}{\lambda_0+\lambda_1}$, we can rewrite the last equality as

$$(4.1)\quad \int_{\mathbb{R}^n}\langle D(u_k\eta);D\psi\rangle\, dx=\int_{\mathbb{R}^n}\langle (I-\gamma\,{}^k\mathbb{A})(D(u_k\eta)+\gamma\,{}^k\mathbb{A}\,u_k D\eta\,;D\psi\rangle\, dx -$$
$$-\gamma\int_{\mathbb{R}^n}\langle {}^k\mathbb{A}\, Du_k, D\eta\,\psi\rangle\, dx.$$

Now we can apply

<u>Lemma 4.3</u>. Let v,g,f be from $\mathcal{D}(\mathbb{R}_n)$, $x_0\in\mathbb{R}^n$, $n\geq 3$, and let for all $\psi\in\mathcal{D}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n}\langle Dv,D\psi\rangle dx=\int_{\mathbb{R}^n}\langle f,D\psi\rangle\, dx+\int_{\mathbb{R}^n} g\,\psi\, dx.$$

Then for $\lambda\in(n-2,n)$ and $\varepsilon>0$ exist $k=k(\varepsilon,\lambda)>0$ and $a=a(\lambda)>0$ such that

$$\int_{\mathbb{R}^n}|Dv|^2|x-x_0|^{-\lambda}\, dx\leq(1+\varepsilon)a(\lambda)\Big(1+\frac{(n-2)^2}{n-1}\Big)\int_{\mathbb{R}^n}|f|^2|x-x_0|^{-\lambda}dx+$$
$$+k\int_{\mathbb{R}^n}|g|^2|x-x_0|^{-\lambda+2}\, dx$$

and $\lim_{\lambda\to(n-2)_+} a(\lambda)=1.$

We omit the proof of this lemma. It can be found in a slightly modified form in [2].

Note that we are to prove this theorem only for $n\geq 3$. In the case $n=2$, every system (2.1) is regular. (It follows e.g. from Theorem 3.1.)

From (4.1) and from the conclusion of Lemma 4.3 we obtain for ε and σ positive

$$\int_{\mathbb{R}^n}|D(u_k\eta)|^2|x-x_0|^{-\lambda}\, dx\leq(1+\varepsilon)a(\lambda)\Big(1+\frac{(n-2)^2}{n-1}\Big).$$
$$\cdot\int_{\mathbb{R}^n}|(I-\gamma\,{}^k\mathbb{A})D(u_k\eta)+\gamma\,{}^k\mathbb{A}\,u_k D\eta|^2|x-x_0|^{-\lambda}\, dx+$$
$$+K_1\int_{\mathbb{R}^n}|{}^k\mathbb{A}\,Du_k D\eta|^2|x-x_0|^{-\lambda+2}\, dx\leq$$

$$\leq (1+\varepsilon)(1+\sigma)a(\lambda)\left(1+\frac{(n-2)^2}{n-1}\right)\int_{\mathbb{R}^n}|(I-\gamma\,{}^k\mathbb{A})D(u_k\eta)|^2|x-x_0|^{-\lambda}\,dx+$$

$$+K_2\left[\int_{\mathbb{R}^n}|{}^k\mathbb{A}\,u_k D\eta|^2|x-x_0|^{-\lambda}\,dx+\int_{\mathbb{R}^n}|{}^k\mathbb{A}\,Du_k D\eta|^2|x-x_0|^{-\lambda+2}\,dx\right]\leq$$

$$\leq (1+\varepsilon)(1+\sigma)a(\lambda)\left(1+\frac{(n-2)^2}{n-1}\right)\left(\frac{\lambda_1-\lambda_0}{\lambda_1+\lambda_0}\right)^2\int_{\mathbb{R}^n}|D(u_k\eta)|^2|x-x_0|^{-\lambda}+$$

$$+K_3\int_{\mathbb{R}^n}(|u_k|^2+|Du_k|^2)\,dx$$

since supp $|D\eta|\subset P=B(x_0,2R)\setminus B(x_0,R)$ and the function $|D\eta||x-x_0|^{-\lambda}$ is bounded on P.

Now we have $\frac{\lambda_0}{\lambda_1}>K(n)$ and so we can choose positive ε,σ and $\lambda>n-2$ such that

$$(1+\varepsilon)(1+\sigma)a(\lambda)\left(\frac{\lambda_1-\lambda_0}{\lambda_1+\lambda_0}\right)^2\left(1+\frac{(n-2)^2}{n-1}\right)<1$$

and hence

$$\int_{B(x_0,R)}|Du_n|^2|x-x_0|^{-\lambda}\leq K_4|u_k|_{W_2^1(\Omega_1)}\leq c=$$

$$=c\left(\Omega_1,\Omega_2,\frac{\lambda_1}{\lambda_0},\lambda_1|u|_{W_2^{1/2}(\partial\Omega_1)}\right).$$

If we take into account the definition of the space $H_{2,\lambda}(\Omega_2)$ and its imbedding into $C^{0,\alpha}(\overline{\Omega}_2)$ we have for $x,y\in\Omega_2$

$$\left|\frac{u_k(x)-u_k(y)}{|x-y|^{\alpha}}\right|\leq C,$$

where C does not depend on k. Letting $k\to\infty$ we obtain the conclusion of the theorem.

5. Open problems

a) Is the estimate $a<\frac{n-1}{n-2}$ in Lemma 3.4 sharp ?

b) It is a well known fact that in the case n=2 or m=1 is the system (2.1) regular. The case n=2 is the consequence of the theorem 3.1, but our condition on $\frac{\lambda_0}{\lambda_1}$ does not take into account the number of the equations m. It would be better to have conditions in the form

$$\frac{\lambda_0}{\lambda_1}>K(m,n).$$

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