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AN ABSTRACT DIFFERENTIAL EQUATION AND THE POTENTIAL BIFURCATION THEOREMS BY KRASNOSELSKII

Jan NEUMANN

Abstract: A proof of a certain generalization of two potential bifurcation theorems by M.A. Krasnoselskii (see [1]) based on a method used by I.V. Skrypnik to prove another bifurcation result (see [2] and [3], respectively) is given. A bifurcation solution lying on the sphere $S(0,\varrho)$[+)] (with a sufficiently small positive ϱ) in a Hilbert space is constructed as an accumulation value ($t \to \infty$) of a map $k: t \in \langle 0,\infty) \to S(0,\varrho)$ satisfying a certain initial value problem for an abstract ordinary differential equation. The main contribution of the article consists in a detailed study of properties of this differential problem.

Key words: Potential bifurcation theorems, abstract ordinary differential equations.

Classification: 35B32, 34A10, 34G20.

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1. Introduction. A proof method, with help of which a certain important bifurcation theorem has been shown by I.V. Skrypnik (see [2] - p. 161, Theorem 3.4 and [3] - p. 178, Theorem 12, respectively), is investigated. The Skrypnik's procedures are used to prove a generalization of two bifurcation results by M.A. Krasnoselskii (see [1], Theorems 1 and 2). The mentioned generalization is not, from the application point of view, essential. The contribution of this article consists in an elaboration of some Skrypnik's ideas, which leads to give precision to certain details of them.

The most meaningful results of this treatise are concentrated in Section 2 where the following differential equation is explored:

$$k'(t)=G(k(t))-(G(k(t)),k(t))\, k(t)/\|k(t)\|^2 ;$$

G is a continuous operator in a Hilbert space and k an abstract

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+) $a \in H, r>0 \quad S(a,r)=\{x \in H;\ \|x-a\|=r\}$

function of one real variable.

In Section 4, the theory constructed in Section 2 together with several simple assertions comprehended in Section 3, is utilized to prove our modification of the Krasnoselskii's theorems which reads as follows:

Theorem 1: Let H be a real Hilbert space and let $\rho > 0$. Let:

1. F be a real functional defined and Fréchet differentiable on $B(0,\rho)$ +) $\subset H$,
2. $F':B(0,\rho)\subset H \longrightarrow H$ be a completely continuous operator,
3. $F'(0)=0$,
4. the Fréchet differential $F''(0):H \longrightarrow H$ exist.

Then $\lambda \neq 0$ is a bifurcation point of the equation $\lambda \cdot x-F'(x)=0$ (with respect to the line of the zero solutions) if and only if λ is an eigenvalue of the linear operator $F''(0)$.

Remark: Krasnoselskii (see [1], Theorem 1) assumes, moreover, that the functional F is weakly continuous and uniformly Fréchet differentiable on $B(0,\rho)$ and the operator $F''(0)$ is selfadjoint and completely continuous. Our reduced assumptions guarantee the validity of the first, the third and the fourth from the conditions introduced (see [4] - p. 104, Theorem 8.2; [5] - p. 70, Theorem 5.11 and [4] - p.74, Theorem 4.7). In addition, from the proof it will be apparent that it is redundant to suppose the uniform differentiability of F.

From Theorem 1 the following assertion, being a special case of the potential bifurcation theorem by Skrypnik (see [2]- p.161, Theorem 3.4 and [3] - p. 178, Theorem 12), follows (see [4] - p. 99, Theorem 7.6).

Theorem 2: Let H be a separable real Hilbert space and let $\rho > 0$.
Let:

1. F be a real functional defined, weakly continuous and uniformly Fréchet differentiable on $B(0,\rho)\subset H$,
2. $F'(0)=0$,
3. the Fréchet differential $F''(0):H \longrightarrow H$ exist.

+) $a \in H, r > 0 \quad B(a,r)=\{x \in H;\ \|x-a\| < r\}$

Then $\lambda \neq 0$ is a bifurcation point of the equation $\lambda \cdot x - F'(x) = 0$ if and only if λ is an eigenvalue of the linear operator $F''(0)$.

The investigated Skrypnik's method may be also exploited to derive some interesting assertions on eigenvalues and bifurcation points of variational inequalities - an illustration example is the content of the author's following article (see [6]).

Note that for the sake of completeness we do not omit some standard and simple proofs in this paper.

2. Basic differential problem. Let H be a real Hilbert space and let ρ, $T > 0$. Consider a continuous operator $G: B(0,\rho) \subset H \longrightarrow H$ and $x \in B(0,\rho) \setminus \{0\}$. We shall look for an abstract function

(2.1) $k: I \longrightarrow B(0,\rho) \setminus \{0\} \subset H$ such that

(2.2) $k \in C^1(I,H)$,

(2.3) $k'(t) = G(k(t)) - (G(k(t)), k(t)) \cdot k(t) / \|k(t)\|^2$ for all $t \in I$,

(2.4) $k(0) = x$,

where $I = \langle 0,T \rangle$ and $\langle 0,\infty)$, respectively.

For the main results of this section see Lemma 6 and Example. Lemmas 3 - 5 serve not only for proving Lemma 6 but they are also applied directly in our proof of the bifurcation theorem.

Lemma 1: Let k be a solution of (2.1) - (2.4) with $I = \langle 0,T \rangle$. Then $\|k(t)\| = \|x\|$ for all $t \in \langle 0,T \rangle$.

Proof: Because $(d/dt)(\|k(t)\|^2)/2 = (k'(t),k(t)) = (G(k(t)) - (G(k(t)), k(t)) \cdot k(t)/\|k(t)\|^2, k(t)) = 0$ on $\langle 0,T \rangle$, the function $t \in \langle 0,T \rangle \longrightarrow \|k(t)\|$ is constant.

Lemma 2: Let $k: \langle 0,T \rangle \longrightarrow B(0,\rho) \setminus \{0\} \subset H$. Then the following assertions are equivalent:

1. k fulfils (2.2),(2.3) and (2.4) with $I = \langle 0,T \rangle$,
2. $k(t) = x + \int_0^t [G(k(\tau)) - (Gk(\tau),k(\tau)) \cdot k(\tau)/\|k(\tau)\|^2]\, d\tau$ +) for all $t \in \langle 0,T \rangle$,
3. $k(t) = [x + \int_0^t G(k(\tau)) \cdot \exp(\int_0^\tau (G(k(\xi)),k(\xi))/\|k(\xi)\|^2 d\xi) d\tau] \cdot$ $\cdot \exp(-\int_0^t (G(k(\xi)),k(\xi))/\|k(\xi)\|^2 d\xi)$ +) for all $t \in \langle 0,T \rangle$.

+) The sign $\int$ denotes the Bochner type integration.

Proof: is very simple - therefore we introduce a proof of one implication only. Let us put:

(2.5) $\eta(t)=(G(k(t)),k(t))/\|k(t)\|^2$ and

(2.6) $l(t)=k(t)\exp(\int_0^t \eta(\tau)d\tau)$ for all $t\in\langle 0,T\rangle$.

From the equation 3 we have:

(2.7) $l(t)=x+\int_0^t G(k(\tau))\exp(\int_0^{\tau}\eta(\xi)d\xi)d\tau$,

and hence

(2.8) $l'(t)=G(k(t))\cdot\exp(\int_0^t \eta(\xi)d\xi)$.

From the definition of l we obtain:

(2.9) $k(t)=l(t)\cdot\exp(-\int_0^t \eta(\tau)d\tau)$.

Differentiating the last equation and using (2.6) and (2.8) we get:

(2.10) $k'(t)=l'(t)\cdot\exp(-\int_0^t \eta(\tau)d\tau)-l(t)\cdot\eta(t)\cdot\exp(-\int_0^t \eta(\tau)d\tau)=$
$=G(k(t))-\eta(t)\cdot k(t)$.

<u>Lemma 3</u>: Let the operator G be Lipschitz continuous on $B(0,\rho)$. Then for every $x\in B(0,\rho)\setminus\{0\}$ there exists the unique abstract function $k=k(\cdot,x)$ satisfying (2.1) - (2.4) with $I=\langle 0,\infty)$. The mapping $k:\langle 0,\infty)\times B(0,\rho)\setminus\{0\}\longrightarrow H$ is continuous, $\|k(t,x)\|=$ $=\|x\|$.

Proof: Denote:

(2.11) $M=\sup\{\|G(x)\|;x\in B(0,\rho)\}$ $(<\infty)$,

(2.12) $L=\sup\{\|G(x)-G(y)\|/\|x-y\|;\ x,y\in B(0,\rho),x\neq y\}$ $(<\infty)$.

1. Existence. Take an $x\in B(0,\rho)\setminus\{0\}$. Put $\sigma=\|x\|$ and choose a $\delta>0$ such that $0<\sigma-\delta<\sigma+\delta<\rho$. Put:

(2.13) $T=\min\{\delta/(2M),1/[4(L+2M/(\sigma-\delta))]\}$

and define the operator

(2.14) $W:\mathcal{D}=\{l\in C(\langle 0,T\rangle,H);\ l(t)\in\overline{B(x,\delta)}$ for all $t\in\langle 0,T\rangle\}\longrightarrow$
$\longrightarrow C(\langle 0,T\rangle,H)$ as:

(2.15) $(Wl)(t)=x+\int_0^t[G(l(\tau))-(G(l(\tau)),l(\tau))\cdot l(\tau)/\|l(\tau)\|^2]d\tau$
for all $l\in\mathcal{D}$ and $t\in\langle 0,T\rangle$.

Obviously $W(\mathcal{D})\subset\mathcal{D}$ and for all l and $\hat{l}\in\mathcal{D}$:

(2.16) $\sup\{\|(Wl)(t)-(W\hat{l})(t)\|;\ t\in\langle 0,T\rangle\}\leq(1/2)\cdot\sup\{\|l(t)-$
$-\hat{l}(t)\|;\ t\in\langle 0,T\rangle\}$.

Hence according to the Banach fixed point theorem and Lemma 2, there is k satisfying (2.1) - (2.4) with $I=\langle 0,T\rangle$. Making use of Lemma 1 we get the existence of k satisfying (2.1) - (2.4) with $I=\langle 0,\infty)$.

2. Uniqueness and continuous dependence on the initial condition. Choose $\varepsilon\in(0,\rho)$, $T>0$. Let l_1, l_2 fulfil (2.1) - (2.3) on $\langle 0,\infty)$. Let $\varepsilon\leq\|l_i(0)\|<\rho$ for i=1,2. Put:

(2.17) $c\doteq 4\cdot(L+2\,M/\varepsilon)$.

Then for all $t\in\langle 0,T\rangle$:

(2.18) $$\|l_1(t)-l_2(t)\| = \|l_1(0)-l_2(0)\|+(c/2)\cdot\int_0^t\|l_1(\tau)-l_2(\tau)\|\cdot\exp(-c\cdot\tau)\exp(+c\cdot\tau)d\tau \leq \|l_1(0)-l_2(0)\| +(c/2)\cdot\sup\{\|l_1(\tau)-l_2(\tau)\|\cdot\exp(-c\cdot\tau);\tau\in\langle 0,T\rangle\}\cdot\int_0^t\exp(c\cdot\xi)d\xi = \|l_1(0)-l_2(0)\|+(1/2)\cdot(\exp(ct)-1)\cdot\sup\{\|l_1(\tau)-l_2(\tau)\|\cdot\exp(-c.\tau);\tau\in\langle 0,T\rangle\}.$$

Hence we have that:

(2.19) $$\sup\{\|l_1(t)-l_2(t)\|\cdot\exp(-ct);\ t\in\langle 0,T\rangle\}\leq\|l_1(0)-l_2(0)\| + +(1/2)\cdot\sup\{\|l_1(t)-l_2(t)\|\cdot\exp(-ct);\ t\in\langle 0,T\rangle\}.$$

Accordingly, for all $t\in\langle 0,T\rangle$:

(2.20) $$\|l_1(t)-l_2(t)\|\leq 2\cdot\exp(cT)\cdot\|l_1(0)-l_2(0)\|.$$

<u>Lemma 4</u>: Let:

1. $\{H_n\}_{n=1}^{+\infty}$ be a sequence of closed linear subspaces of H,
2. $\{\varepsilon_n\}_{n=1}^{+\infty}$ be a sequence of positive numbers, $\lim\limits_{n\to\infty}\varepsilon_n=0$,
3. $\hat{G}:B(0,\rho)\subset H\longrightarrow H$ be a completely continuous operator,
4. $\{G_n:B(0,\rho)\cap H_n\longrightarrow H_n\}_{n=1}^{+\infty}$ be a sequence of continuous operators such that $\|G_ny-\hat{G}y\|\leq\varepsilon_n$ for all $n\in\mathbb{N}$ and $y\in B(0,\rho)\cap H_n$,
5. $\{x_n\}_{n=1}^{+\infty}\subset B(0,\rho)\setminus\{0\}$, $x_n\in H_n$ for all $n\in\mathbb{N}$, $\hat{x}\in B(0,\rho)\setminus\{0\}$, $\lim\limits_{n\to\infty}x_n=\hat{x}$,
6. k_n solve (2.1) - (2.4) with $I=\langle 0,\infty)$, $G=G_n$, $x=x_n$ for all $n\in\mathbb{N}$.

Then there exist an increasing sequence of positive integers $\{i_n\}_{n=1}^{+\infty}$ and an abstract function $\hat{k}$ satisfying (2.1) - (2.4) with $I=\langle 0,\infty)$, $G=\hat{G}$ and $x=\hat{x}$ such that $\{k_{i_n}\}_{n=1}^{+\infty}$ tends to $\hat{k}$ on $\langle 0,\infty)$ locally uniformly.

Proof: Denote:

(2.21) $M=\sup\{\|\hat{G}x\|;\ x\in B(0,\rho)\}$ $(<\infty)$,

(2.22) $\sigma=\inf\{\|x_n\|;\ n\in\mathbb{N}\}$ (>0),

(2.23) $\varphi_n(t)=(\hat{G}(k_n(t)),k_n(t))/\|k_n(t)\|^2$ and

(2.24) $\psi_n(t)=(G_n(k_n(t)),k_n(t))/\|k_n(t)\|^2$ for all $t\in\langle 0,\infty)$, $n\in\mathbb{N}$.

Choose a $T>0$. According to Lemma 2, for all $n\in\mathbb{N}$ and $t\in\langle 0,T\rangle$

(2.25) $k_n(t)=p_n(t)+q_n(t)$,

where

(2.26) $p_n(t)=\hat{x}\cdot\exp(-\int_0^t\varphi_n(\xi)d\xi)+\int_0^t\hat{G}(k_n(\tau))\cdot\exp(\int_t^\tau\varphi_n(\xi)d\xi)d\tau$,

(2.27) $q_n(t)=x_n\cdot[\exp(-\int_0^t\psi_n(\xi)d\xi)-\exp(-\int_0^t\varphi_n(\xi)d\xi)]+(x_n-\hat{x})\cdot$
$\cdot\exp(-\int_0^t\varphi_n(\xi)d\xi)+\int_0^t[G_n(k_n(\tau))-\hat{G}(k_n(\tau))]\cdot\exp(\int_t^\tau\psi_n(\xi)d\xi)d\tau+$
$+\int_0^t\hat{G}(k_n(\tau))\cdot[\exp(\int_t^\tau\psi_n(\xi)d\xi)-\exp(\int_t^\tau\varphi_n(\xi)d\xi)]d\tau$

are continuous functions.

Using the inequality $|\exp(a)-\exp(b)|\le\max\{\exp(a),\exp(b)\}\cdot$ $\cdot|b-a|$ and other simple estimates, we obtain that for all $n\in\mathbb{N}$ and $t\in\langle 0,T\rangle$

(2.28) $\|q_n(t)\|\le\exp(T(M+\varepsilon_n)/\sigma)\cdot[(2+TM/\sigma)T\varepsilon_n+\|x_n-\hat{x}\|]$.

Thus, $q_n\rightrightarrows 0$ on the interval $\langle 0,T\rangle$.

Obviously, $\hat{G}(B(0,\rho))$ is a relatively compact subset of H. For every positive integer n let us denote a finite 1/n - net of the set $\hat{G}(B(0,\rho))$ by $\mathcal{H}_n$. For all $x,y\in H$ and $n\in\mathbb{N}$ put $m_n(x,y)=$ $=\max(0,1/n-\|x-y\|)$. Define the sequence of the continuous operators $\{M_n:x\in\hat{G}(B(0,\rho))\longmapsto(\sum_{y\in\mathcal{H}_n}m_n(x,y)\cdot y)/(\sum_{y\in\mathcal{H}_n}m_n(x,y))\in$ $\in\mathcal{L}(\mathcal{H}_n)\}_{n=1}^{+\infty}$. A simple account gives: $\|M_nx-x\|\le 1/n$ for all $n\in\mathbb{N}$ and $x\in\hat{G}(B(0,\rho))$. For all $n,j\in\mathbb{N}$ and $t\in\langle 0,T\rangle$ put:

(2.29) $p_{nj}(t)=\hat{x}\cdot\exp(-\int_0^t\varphi_n(\xi)d\xi)+\int_0^t M_j\,\hat{G}(k_n(\tau))\cdot\exp(\int_t^\tau\varphi_n(\xi)d\xi)d\tau$.

Then for all n, $j\in\mathbb{N}$ and $t,t'\in\langle 0,T\rangle$

(2.30) $\|p_{nj}(t)\|\le[\|\hat{x}\|+T(M+1/j)]\cdot\exp(TM/\sigma)$ and

(2.31) $\|p_{nj}(t)-p_{nj}(t')\|=\|\hat{x}\cdot[\exp(-\int_0^t\varphi_n(\xi)d\xi)-\exp(-\int_0^{t'}\varphi_n(\xi)d\xi)]+$
$+\int_0^t M_j\hat{G}(k_n(\tau))\cdot[\exp(\int_t^\tau\varphi_n(\xi)d\xi)-\exp(\int_{t'}^\tau\varphi_n(\xi)d\xi)]d\tau+$

$$+\int_{t'}^{t} M_j\hat{G}(k_n(\tau))\cdot\exp\left(\int_{t'}^{\tau}\varphi_n(\xi)d\xi\right)d\tau\,\| \le \|\hat{x}\|\cdot\exp(TM/\sigma)\cdot(M/\sigma)\cdot$$

$$\cdot|t-t'|+T(M+1/j)\cdot\exp(TM/\sigma)(M/\sigma)\cdot|t-t'|+(M+1/j)\exp(TM/\sigma)\cdot$$

$$\cdot|t-t'|=[\|\hat{x}\|\cdot(M/\sigma)+(M+1/j)\cdot(1+TM/\sigma)]\cdot\exp(TM/\sigma)\cdot|t-t'|.$$

In virtue of the Arzela-Ascoli theorem for every positive integer j the set $\{p_{nj}\}_{n=1}^{+\infty}$ is relatively compact in $C(\langle 0,T\rangle, \mathfrak{L}(\mathcal{H}_j\cup\{\hat{x}\}))$ (and also in $C(\langle 0,T\rangle,H)$). Accordingly, the set $\{p_n\}_{n=1}^{+\infty}$ is relatively compact in $C(\langle 0,T\rangle,H)$ as well.

Choose an increasing sequence $\{i_n(T)\}_{n=1}^{+\infty}$ of positive integers. Then there exists a sequence $\{j_n(T)\}_{n=1}^{+\infty}$ chosen from $\{i_n(T)\}_{n=1}^{+\infty}$ such that $\{p_{j_n(T)}\}_{n=1}^{+\infty}$ is convergent in $C(\langle 0,T\rangle,H)$. Hence $\{k_{j_n(T)}\}_{n=1}^{+\infty}$ tends to a $k^{0,T}$ in $C(\langle 0,T\rangle,H)$. Obviously, the set $\mathcal{U}=\{k_{j_n(T)}(t);n\in\mathbb{N},t\in\langle 0,T\rangle\}\cup\{k^{0,T}(t);t\in\langle 0,T\rangle\}$ is compact in H and for all $x\in\mathcal{U}$: $\varrho>\|x\|\ge\sigma$. Thus, the operator $\hat{G}$ and the functional $\Phi:\xi\in\mathcal{U}\longmapsto(\hat{G}\xi,\xi)/\|\xi\|^2$ are uniformly continuous on $\mathcal{U}$. Since $k_{j_n(T)}\rightrightarrows k^{0,T}$ on $\langle 0,T\rangle$, $\hat{G}\circ k_{j_n(T)}\rightrightarrows\hat{G}\circ k^{0,T}$ and $\Phi\circ k_{j_n(T)}=$ $=\varphi_{j_n(T)}\rightrightarrows\Phi\circ k^{0,T}$ on $\langle 0,T\rangle$. Hence it is easy to see that

$$k^{0,T}(t)=\lim_{n\to\infty}k_{j_n(T)}(t)=\lim_{n\to\infty}[p_{j_n(T)}(t)+q_{j_n(T)}(t)]=[\hat{x}+$$
$$+\int_0^t\hat{G}(k^{0,T}(\tau))\cdot\exp\left(\int_0^{\tau}\Phi(k^{0,T}(\xi))d\xi\right)d\tau]\cdot$$
$$\cdot\exp\left(-\int_0^t\Phi(k^{0,T}(\xi))d\xi\right)\text{ for all }t\in\langle 0,T\rangle. \tag{2.32}$$

By virtue of Lemma 2 we have: $k^{0,T}$ satisfies (2.1) - (2.4) with $I=\langle 0,T\rangle$, $G=\hat{G}$ and $x=\hat{x}$.

Obviously, there exists a system of increasing sequences of positive integers $\{\{j_n(N)\}_{n=1}^{+\infty}\}_{N=1}^{+\infty}$ with the following properties.

1. For all $N\in\mathbb{N}$ the sequence $\{j_n(N+1)\}_{n=1}^{+\infty}$ is chosen from the sequence $\{j_n(N)\}_{n=1}^{+\infty}$.
2. For all $N\in\mathbb{N}$ the sequence $\{k_{j_n(N)}\}_{n=1}^{+\infty}$ tends to a mapping $k^{0,N}$ satisfying (2.1) - (2.4) with $I=\langle 0,N\rangle$, $G=\hat{G}$ and $x=\hat{x}$ in $C(\langle 0,N\rangle,H)$.

Define $\hat{k}:\langle 0,\infty)\longrightarrow S(0,\|\hat{x}\|)$; $\hat{k}(t)=k^{0,N}(t)$ on the interval

$\langle N-1,N)$ for all $N \in \mathbb{N}$. Obviously, $\hat{k}$ satisfies (2.1) - (2.4) with $I=\langle 0,\infty)$, $G=\hat{G}$ and $x=\hat{x}$. At the same time the sequence $\{k_{i_n}\}_{n=1}^{+\infty}$, where $i_n=j_n(n)$ for all $n \in \mathbb{N}$, tends to $\hat{k}$ on $\langle 0,\infty)$ locally uniformly.

Lemma 5: Let:

1. $\tilde{H}$ be a finite dimensional subspace of H,
2. $G:B(0,\varrho)\subset H \longrightarrow H$ be a completely continuous operator,
3. $\sigma \in (0,\varrho)$.

Then there exist a sequence $\{H_n\}_{n=1}^{+\infty}$ of finite dimensional subspaces of H and a sequence $\{G_n:\overline{B(0,\sigma)} \cap (\tilde{H}+H_n) \longrightarrow H_n\}_{n=1}^{+\infty}$ of Lipschitz continuous operators such that $\|G_n y-Gy\| \leq 1/n$ for all $n \in \mathbb{N}$ and $y \in \overline{B(0,\sigma)} \cap (\tilde{H}+H_n)$.

Proof: For every positive integer n let us denote a finite $1/(2\cdot n)$ - net of the set $G(\overline{B(0,\sigma)})$ by $\mathcal{H}_n$. Take a sequence $\{M_n:G(\overline{B(0,\sigma)}) \longrightarrow H_n=\mathcal{L}(\mathcal{H}_n)\}_{n=1}^{+\infty}$ of continuous operators such that $\|M_n x-x\| \leq 1/(2\cdot n)$ for all $n \in \mathbb{N}$ and $x \in G(\overline{B(0,\sigma)})$.

Choose an $n \in \mathbb{N}$. Let $\{e_1,\dots,e_p\}$ and $\{e_1,\dots,e_r\}$ $(r \geq p)$ be an orthonormal basis of H_n and $\tilde{H}+H_n$ respectively.

(2.33) For all $z \in (\tilde{H}+H_n) \cap \overline{B(0,\sigma)}$: $M_n Gz=\sum_{i=1}^{r} f_i((z,e_1),\dots,(z,e_r))\cdot e_i$ where f_i $(i=1,2,\dots,p)$ is a real function of r real variables defined and continuous on $B=\{\eta=(\eta_1,\dots,\eta_r) \in \mathbb{R}^r; \sum_{i=1}^{r} \eta_i^2 \leq \sigma^2\}$;

(2.34) $f_i(\eta)=(M_n G(\sum_{j=1}^{r} \eta_j\cdot e_j),e_i)$ for all $\eta \in B$.

For every $i=1,2,\dots,p$ there exists a real polynomial P_i such that for all $\eta \in B$: $|f_i(\eta)-P_i(\eta)| \leq 1/(2\cdot n\cdot p^{1/2})$ (see [7]). Define the continuous mapping $G_n:\overline{B(0,\sigma)} \cap (\tilde{H}+H_n) \longrightarrow H_n$ as:

(2.35) $G_n(z)=\sum_{i=1}^{r} P_i((z,e_1),\dots,(z,e_r))\cdot e_i$ for all $z \in \overline{B(0,\sigma)} \cap (\tilde{H}+H_n)$

and denote

(2.36) $L_{ij}=\sup\{|(\partial P_i/\partial\eta_j)(\xi)|; \xi \in B\}$ $(<\infty)$ for all $i=1,\dots,p$ and $j=1,\dots,r$.

Then:

(2.37) for all $z \in \overline{B(0,\sigma)} \cap (\tilde{H}+H_n)$: $\|Gz-G_n z\| \leq \|Gz-M_n Gz\|+$

$+ \|M_nGz-G_nz\| = \|Gz-M_nGz\| + (\sum_{i=1}^{n}(f_i((z,e_1),\dots,(z,e_r))-$

$-P_i((z,e_1),\dots,(z,e_r)))^2)^{1/2} \leq 1/(2\cdot n)+(\sum_{i=1}^{n}(1/(2\cdot n\cdot p^{1/2}))^2)^{1/2}=$

$=1/n$ and

(2.38) for all $z,z' \in \overline{B(0,\sigma)} \cap (\tilde{H}+H_n): \|G_nz-G_nz'\|^2 = \sum_{i=1}^{n}(P_i((z,e_1),$

$\dots,(z,e_r))-P_i((z',e_1),\dots,(z',e_r)))^2 \leq \sum_{i=1}^{n}(\sum_{j=1}^{n} L_{ij}|(z-z',$

$e_j)|)^2 \leq (\sum_{i=1}^{n}\sum_{j=1}^{n} L_{ij}^2)\cdot\|z-z'\|^2.$

<u>Lemma 6</u>: Let the operator G be completely continuous on $B(0,\rho)$. Then for every $x \in B(0,\rho)\setminus\{0\}$ there exists at least one solution k of the problem (2.1) - (2.4) with $I=\langle 0,\infty)$. In addition, $\|k(t)\|=\|x\|$ for all $t \in \langle 0,\infty)$.

Proof: Let an $x \in B(0,\rho)\setminus\{0\}$ be given. Choose a $\sigma \in (\|x\|,\rho)$. Lemma 5 guarantees the existence of a sequence $\{H_n\}_{n=1}^{+\infty}$ of finite dimensional subspaces in H and a system $\{G_n : \overline{B(0,\sigma)} \cap (\mathcal{L}\{x\}+H_n) \to H_n\}_{n=1}^{+\infty}$ of Lipschitz continuous operators such that for all $n \in \mathbb{N}$ and $y \in \overline{B(0,\sigma)} \cap (\mathcal{L}\{x\}+H_n): \|G_ny-Gy\| \leq 1/n$. Further, according to Lemma 3 for all $n \in \mathbb{N}$ a $k_n : \langle 0,\infty) \to \mathcal{L}\{x\}+H_n$ satisfying (2.2) - (2.4) with $I=\langle 0,\infty)$ and $G=G_n$ has to exist. Finally, in virtue of Lemma 4 we get that there exist an increasing sequence $\{i_n\}_{n=1}^{+\infty} \subset \mathbb{N}$ and a k satisfying (2.1) - (2.4) with $I=\langle 0,\infty)$ such that $\{k_{i_n}\}_{n=1}^{+\infty}$ tends to k on $\langle 0,\infty)$ locally uniformly.

The following example shows that the complete continuity of G does not guarantee the uniqueness of the solution of the problem (2.1) - (2.4) for all initial conditions $x \in B(0,\rho)\setminus\{0\}$ ($x \in B(0,\sigma)\setminus\{0\}$ with arbitrary $\sigma \in (0,\rho)$, respectively). A potential operator G with a potential F satisfying the assumptions of Theorems 1 and 2 is chosen.

<u>Example</u>: Consider $F: \mathbb{R}^2 \to \mathbb{R}^1$, $F(x,y)=x^2+y^2+|x|^{3/2}\cdot y^2$ for all $x,y \in R^1$. The functional F is Fréchet differentiable on $\mathbb{R}^2$ and uniformly differentiable on every ball in $\mathbb{R}^2$; $F': \mathbb{R}^2 \to \mathbb{R}^2$, $F'(x,y)=(2\cdot x+(3/2)\cdot \mathrm{sign}(x)\cdot|x|^{1/2}\cdot y^2,\ 2\cdot y+2\cdot|x|^{3/2}y)^T$ for all $x,y \in \mathbb{R}^1$. The Fréchet differential $F''(0,0)$ exists; $F''(0,0)=$ $=\mathrm{diag}(2,2)$. The equation (2.3) is represented by the differential system:

(2.39) $k_1'(t)=[-2\cdot|k_1(t)|^{5/2}\cdot k_2^2(t)+(3/2)\cdot|k_1(t)|^{1/2}\cdot k_2^4(t)]\cdot$

$\cdot\,\mathrm{sign}(k_1(t))/[k_1^2(t)+k_2^2(t)]$,

$k_2'(t)=[2\cdot|k_1(t)|^{7/2}\cdot k_2(t)-(3/2)\cdot|k_1(t)|^{3/2}\cdot k_2^3(t)]/$

$/[k_1^2(t)+k_2^2(t)]$

for $k=(k_1,k_2)^T$.

Take $f:\langle 0,\sqrt{3/7})\longrightarrow \mathbb{R}^1$; $f(z)=\int_0^z[2\,d\xi/[\xi^{1/2}\cdot(1-\xi^2)\cdot(3-7\cdot\xi^2)]]$ and $\sigma>0$. Then for every $\delta\in\langle 0,\infty\rangle$ the mapping k_δ; $k_\delta(t)=(0,\sigma)^T$ for all $t\in\langle 0,\delta)$, $k_\delta(t)=(\pm\sigma\cdot f^{-1}(\sigma^{3/2}\cdot(t-\delta)),\sigma\cdot(1-[f^{-1}(\sigma^{3/2}\cdot(t-\delta))]^2)^{1/2})^T$ for all $t\in\langle\delta,\infty)$; is a solution of the system (2.39) with the initial condition $k_1(0)=0$, $k_2(0)=\sigma$. The problem (2.39), $k_1(0)=\omega$, $k_2(0)=(\sigma^2-\omega^2)^{1/2}$, $0<|\omega|<\sigma\cdot\sqrt{3/7}$ has the unique solution $k(t)=(\sigma\cdot\mathrm{sign}(\omega)\cdot f^{-1}(\sigma^{3/2}\cdot t+f(\sigma^{-1}\cdot|\omega|)),\sigma(1-[f^{-1}(\sigma^{3/2}\cdot t+f(\sigma^{-1}\cdot|\omega|))]^2)^{1/2})^T$.

Remark: Although the functional F defined in the foregoing example fulfils the assumptions of Theorem 1 (and Theorem 2, respectively), it is obvious that every mapping $k:\langle 0,\infty)\times S(0,\sigma)\longrightarrow S(0,\sigma)$, where $\sigma>0$ and $k(\cdot,x)$ solves (2.1) - (2.4) with $H=\mathbb{R}^2$, $I=\langle 0,\infty)$ and $G=F'$, is not continuous.

3. Auxiliary assertions.

In this section several simple and mostly well known assertions are summarized.

Lemma 7: Let H be a Hilbert space and let $\rho>0$. Consider a continuous operator $G:B(0,\rho)\subset H\longrightarrow H$ and define the operator $D:B(0,\rho)\setminus\{0\}\longrightarrow H$ by the formula:

(3.1) $D(x)=G(x)-(G(x),x)\cdot x/\|x\|^2$ for every $x\in B(0,\rho)\setminus\{0\}$. Take $\sigma\in(0,\rho)$, $x,y\in H$, $y\neq 0$ and $\{x_n\}_{n=1}^{+\infty}\subset S(0,\sigma)$ such that $x_n\longrightarrow x$, $G(x_n)\longrightarrow y$ and $D(x_n)\longrightarrow 0$ in H. Then $x_n\longrightarrow x$ in H and $(y,x)\cdot x/\sigma^2-G(x)=0$.

Proof: Evidently

(3.2) $\lim_{n\to\infty}[(Gx_n,x_n)\cdot x_n/\|x_n\|^2]=\lim_{n\to\infty}[Gx_n-Dx_n]=y\neq 0$

and consequently

$$(3.3)\quad |(y,x)| = \lim_{n\to\infty} |(Gx_n,x_n)| = \lim_{n\to\infty} [\sigma\cdot\|(Gx_n,x_n)\cdot x_n/\|x_n\|^2\|] = \\ = \sigma\cdot\|y\| \neq 0.$$

Thus, $(Gx_n,x_n)\neq 0$ for arbitrary sufficiently large $n\in\mathbb{N}$ and

$$(3.4)\quad x_n=(Gx_n-Dx_n)\cdot\sigma^2/(Gx_n,x_n) \longrightarrow y\cdot\sigma^2/(y,x)=x \text{ for } n\to\infty .$$

Then obviously $Gx_n \to Gx=y$ and $Dx_n \to Dx=0$ for $n\to\infty$; $\|x\|=\sigma$.
Finally, $(y,x)\cdot x/\sigma^2-Gx=(Gx,x)\cdot x/\|x\|^2-Gx= -Dx=0$.

<u>Definition</u>: Let R be a metric space.

1. Let $M_1,M_2\subset R$. Suppose that a continuous mapping $f:M_1\times\langle 0,1\rangle\to R$ such that $f(x,0)=x$ for all $x\in M_1$ and $f(M_1,1)=M_2$ exists. Then we say that the set M_2 is a continuous deformation of the set M_1 within R.
2. Let $M\subset R$. We say that the set M is contractible within R if there exists an $a\in R$ such that $\{a\}$ is a continuous deformation of M within R.

<u>Lemma 8</u>: Let R be a metric space. Let $M_2\subset R$ being contracttible within R be a continuous deformation of $M_1\subset R$ within R.Then M_1 is contractible within R.

Proof is obvious.

<u>Lemma 9</u>: Let H_1 be a finite dimensional subspace of a Hilbert space H and let P_1 be the orthogonal projection of H onto H_1. Put $R=\{x\in H;P_1x\neq 0\}$. Then:

1. For every $\rho>0$ the set $S_\rho=S(0,\rho)\cap H_1$ is not contractible within R.
2. Every subset $\mathcal{U}$ of R such that $P_1(\mathcal{U})\cap\mathcal{L}\{x_0\}=\emptyset$ for an $x_0\in H_1\setminus\{0\}$ is contractible within R.

Proof: 1. Assume the existence of $\rho>0$ such that the set S_ρ is contractible within R. Then an $x_0\in R$ and a continuous mapping $f:S_\rho\times\langle 0,1\rangle\to R$ satisfying the relations $f(x,0)=x$ and $f(x,1)=x_0$ for every $x\in S_\rho$ have to exist. Consider the continuous mapping $g:\overline{B(0,\rho)}\cap H_1\to S_\rho$ given as:

$$(3.5)\quad g(x)= -\rho\cdot P_1f(\rho\cdot x/\|x\|,1-\|x\|/\rho)/\|P_1f(\rho\cdot x/\|x\|,1-\|x\|/\rho)\|$$

for all $x\in\overline{B(0,\rho)}\cap H_1\setminus\{0\}$; $g(0)= -\rho\cdot P_1x_0/\|P_1x_0\|$.

According to the Brouwer fixed point theorem there exists $x_\rho \in S_\rho$ such that $x_\rho = g(x_\rho) = -f(x_\rho, 0) = -x_\rho$. However, it is impossible.

2. Define the mapping $f: \mathcal{U} \times \langle 0,1\rangle \rightarrow H$ by the formula $f(x,t) = (1-t)\cdot x + t\cdot x_0$ for all $x \in \mathcal{U}$ and $t \in \langle 0,1\rangle$. Obviously, this mapping is continuous; $f(x,0) = x$ and $f(x,1) = x_0$ for all $x \in \mathcal{U}$. The existence of $(t_1, x_1) \in (0,1) \times \mathcal{U}$ such that $P_1 f(x_1, t_1) = 0$ implies $P_1 x_1 = t_1 \cdot x_0/(t_1 - 1) \in P_1(\mathcal{U}) \cap \mathcal{L}\{x_0\}$, which contradicts our assumptions.

4. Proof of the bifurcation theorem. We confine ourselves to a proof of the fact that every eigenvalue $\lambda \neq 0$ of $F''(0)$ is a bifurcation point of $\lambda \cdot x - F'(x) = 0$. A proof of the converse implication is obvious.

1. Choose an eigenvalue $\lambda_0 > 0$ of $F''(0)$. Put $H_0 = \mathrm{Ker}(\lambda_0 \cdot I - F''(0))$, $H_1 = \mathcal{L}(\bigcup_{\lambda \geq \lambda_0} \mathrm{Ker}(\lambda \cdot I - F''(0)))$ and $H_2 = H_1^{\perp}$. Obviously, the spaces H_0 and H_1 are finite dimensional and the one H_2 is closed. Denote P_i the orthogonal projection of H onto H_i for $i=1,2$ and put $R = \{x \in H; P_1 x \neq 0\}$. Without loss of generality we may assume that $F(0) = 0$. Let us define the functions $\omega, \hat{\omega}: (0,\rho) \rightarrow \mathbb{R}^1$ as:

$$(4.1) \quad \omega(\sigma) = \sup\{|F(x) - (F''(0)x,x)/2|/\|x\|^2;\ x \in \overline{B(0,\sigma)} \setminus \{0\}\},$$

$$(4.2) \quad \hat{\omega}(\sigma) = \sup\{\|F'(x) - F''(0)x\|/\|x\|;\ x \in \overline{B(0,\sigma)} \setminus \{0\}\} \text{ on } (0,\rho).$$

Evidently, the functions $\omega, \hat{\omega}$ are increasing and $\lim_{\sigma \to 0+} \omega(\sigma) = \lim_{\sigma \to 0+} \hat{\omega}(\sigma) = 0$.

Further, denote Λ the set of all eigenvalues of $F''(0)$; put $\lambda_1 = \sup \Lambda$ and $\lambda_2 = \sup[\Lambda \cap (0, \lambda_0) \cup \{0\}]$. Choose a $\sigma_0 \in (0,\rho)$ such that $\lambda_0 \cdot (\lambda_0 - \lambda_2 - 4\cdot\omega(\sigma_0))/(\lambda_1 - \lambda_2) - \hat{\omega}(\sigma_0) > 0$. Put:

$$(4.3) \quad \delta_1 = (\lambda_0 - \lambda_2 - 4\cdot\omega(\sigma_0))/(\lambda_1 - \lambda_2),\ \delta_2 = \lambda_0 \cdot \delta_1 - \hat{\omega}(\sigma_0).$$

Obviously, the constants δ_1, δ_2 are positive. Denote:

$$(4.4) \quad D(x) = F'(x) - (F'(x),x)\cdot x/\|x\|^2 \text{ for all } x \in B(0,\rho) \setminus \{0\}.$$

Finally, choose an arbitrary $\sigma \in (0, \sigma_0)$.

2. Choose an $x \in S(0,\sigma) \cap H_1$ and consider k satisfying

(2.1) - (2.4) with $I=\langle 0,\infty)$ and $G=F'$ (the existence of k is guaranteed by Lemma 6).

(4.5) For all $t\in\langle 0,\infty):\|k'(t)\|^2=(F'(k(t))-(F'(k(t)),k(t))\cdot$

$\cdot k(t)/\|k(t)\|^2,k'(t))=(F'(k(t)),k'(t))=(d/dt)F(k(t))$

and therefore

(4.6) $F(k(t))=F(x)+\int_0^t \|k'(\tau)\|^2 d\tau \geq F(x)\geq (F''(0)x,x)/2 -$

$-\omega(\sigma)\cdot\|x\|^2\geq[\lambda_0/2-\omega(\sigma)]\cdot\|k(t)\|^2.$

Further,

(4.7) for all $t\in\langle 0,\infty):F(k(t))\leq(F''(0)k(t),k(t))/2+\omega(\sigma)\cdot$

$\|k(t)\|^2=(F''(0)P_1k(t),P_1k(t))/2+(F''(0)P_2k(t),P_2k(t))/2+$

$+\omega(\sigma)\cdot\|k(t)\|^2\leq\lambda_1\cdot\|P_1k(t)\|^2/2+\lambda_2\cdot\|P_2k(t)\|^2/2+\omega(\sigma)\cdot$

$\cdot\|k(t)\|^2=(\lambda_1-\lambda_2)\cdot\|P_1k(t)\|^2/2+\lambda_2\cdot\|k(t)\|^2/2+\omega(\sigma)\cdot$

$\cdot\|k(t)\|^2.$

From (4.6) and (4.7) we obtain that

(4.8) $\|P_1k(t)\|^2\geq\sigma_1\cdot\|k(t)\|^2$ for all $t\in\langle 0,\infty)$.

3. We shall show that for every $T>0$ a k satisfying the conditions (2.1) - (2.3) ($I=\langle 0,\infty)$, $G=F'$), $k(0)\in H_1\cap S(0,\sigma)$ and $k(T)\in H_0+H_2$ exists.

According to Lemma 5 a sequence $\{\tilde{H}_n\}_{n=1}^{+\infty}$ of finite dimensional subspaces of H and a system $\{G_n:\overline{B(0,\sigma_0)}\cap(H_1+\tilde{H}_n)\longrightarrow\tilde{H}_n\}_{n=1}^{+\infty}$ of Lipschitz continuous operators such that for all $n\in\mathbb{N}$ and $y\in\overline{B(0,\sigma_0)}\cap(H_1+\tilde{H}_n):\|G_ny-F'y\|\leq 1/n$, exist. Further, according to Lemma 3 for every $n\in\mathbb{N}$ and $x\in S(0,\sigma)\cap H_1$ there exists the unique $k_n(\cdot,x)$ satisfying (2.1) - (2.4) with $I=\langle 0,\infty)$ and $G=G_n$. Moreover, for every $n\in\mathbb{N}$ the mapping $k_n:\langle 0,\infty)\times S(0,\sigma)\cap H_1\longrightarrow$ $\longrightarrow S(0,\sigma)\cap(H_1+\tilde{H}_n)$ is continuous.

Choose $T>0$. Let $\{t_n\}_{n=1}^{+\infty}\subset\langle 0,T\rangle$, $\{x_n\}_{n=1}^{+\infty}\subset S(0,\sigma)\cap H_1$ and $\{p_n\}_{n=1}^{+\infty}\subset\mathbb{N}$, $p_n\nearrow\infty$, such that for all $n\in\mathbb{N}:P_1k_{p_n}(t_n,x_n)=0$, exist. Without loss of generality we may assume that $\{t_n\}_{n=1}^{+\infty}$ and $\{x_n\}_{n=1}^{+\infty}$ converges to a $\hat{t}\in\langle 0,T\rangle$ and an $\hat{x}\in S(0,\sigma)\cap H_1$, respectively. For the sake of brevity write l_n instead of $k_{p_n}(\cdot,x_n)$. According to Lemma 4 there exist an increasing sequence $\{r_n\}_{n=1}^{+\infty}$ of positive integers and a k satisfying (2.1) - (2.4) with $I=\langle 0,\infty)$, $G=F'$ and $x=\hat{x}$ such that $\{l_{r_n}\}_{n=1}^{+\infty}$ tends to k on $\langle 0,\infty)$

locally uniformly. Thus:

(4.9) $\|P_1k(\hat{t})\| \leq \|P_1[k(\hat{t})-k(t_{r_n})]\| + \|P_1[k(t_{r_n})-l_{r_n}(t_{r_n})]\| \leq$

$$\leq \|k(\hat{t})-k(t_{r_n})\| + \sup\{\|k(t)-l_{r_n}(t)\|;\ t\in\langle 0,T\rangle\}.$$

Passing to the limit ($n\to\infty$) we obtain that $P_1k(\hat{t})=0$. However, from (4.8) we have: $\|P_1k(\hat{t})\| \geq \delta_1^{1/2}\cdot\|k(\hat{t})\| = \delta_1^{1/2}\cdot\sigma > 0$.

Hence there exists an $n_0=n_0(T)\in \mathbb{N}$ such that for all positive integers $n\geq n_0$, $t\in\langle 0,T\rangle$ and $x\in S(0,\sigma)\cap H_1: k_n(t,x)\in R$. Evidently, for all $n\geq n_0: k_n(T,S(0,\sigma)\cap H_1)$ is a continuous deformation of $S(0,\sigma)\cap H_1$ within R. According to Lemma 8 and the first part of Lemma 9 for all $n\geq n_0: k_n(T,S(0,\sigma)\cap H_1)$ is not contractible within R. Further, in virtue of the second part of Lemma 9 we have that for every positive integer $n\geq n_0$ an $\hat{x}_n\in S(0,\sigma)\cap H_1$ such that $P_1k_n(T,\hat{x}_n)\in H_0$ - i.e. $k_n(T,\hat{x}_n)\in H_0+H_2$ - has to exist. Finally, according to Lemma 4 there exist an increasing sequence $\{u_n\}_{n=1}^{+\infty}$ of positive integers greater than $n_0=n_0(T)$ and a $\hat{k}$ satisfying (2.1) - (2.3) with $I=\langle 0,\infty)$ and $G=F'$ such that $\{k_{u_n}(\cdot,\hat{x}_{u_n})\}_{n=1}^{+\infty}$ tends to $\hat{k}$ on $\langle 0,\infty)$ locally uniformly. Thus, $\hat{k}(0)=\lim_{n\to\infty} k_{u_n}(0,\hat{x}_{u_n})\in S(0,\sigma)\cap H_1$ and $\hat{k}(T)=\lim_{n\to\infty} k_{u_n}(T,\hat{x}_{u_n})\in H_0+H_2$.

4. Choose an increasing sequence $\{T_n\}_{n=1}^{+\infty}$ of positive numbers such that $\lim_{n\to\infty} T_n=\infty$. Then for all $n\in\mathbb{N}$ there exists a $\hat{k}_n$ satisfying the conditions (2.1) - (2.3) ($I=\langle 0,\infty)$, $G=F'$), $\hat{k}_n(0)\in S(0,\sigma)\cap H_1$ and $\hat{k}_n(T_n)\in H_0+H_2$. According to Lemma 4 there exist an increasing sequence $\{v_n\}_{n=1}^{+\infty}\subset\mathbb{N}$ and a $\hat{k}_\infty$ satisfying (2.1) - (2.3) with $I=\langle 0,\infty)$ and $G=F'$ such that $\{\hat{k}_{v_n}\}_{n=1}^{+\infty}$ tends to $\hat{k}_\infty$ on $\langle 0,\infty)$ locally uniformly. Obviously, $\hat{k}_\infty(0)\in S(0,\sigma)\cap H_1$.

For every $T>0$:

(4.10) $F(\hat{k}_\infty(T))-F(\hat{k}_\infty(0)) = \int_0^T (F'(\hat{k}_\infty(t)),\hat{k}'_\infty(t))dt =$

$$= \int_0^T \|\hat{k}'_\infty(t)\|^2dt = \int_0^T \|D\hat{k}_\infty(t)\|^2dt.$$

Because the operator F' is completely continuous on $B(0,\rho)$, the functional F is bounded on $B(0,\rho)$ and hence $\int_0^\infty \|D\hat{k}_\infty(t)\|^2dt<\infty$. Accordingly, there exist a sequence of positive numbers $\{\hat{t}_n\}_{n=1}^{+\infty}$ and $x,y\in H$ such that $\hat{t}_n\nearrow\infty$ and $D(\hat{k}_\infty(\hat{t}_n))\to 0$, $\hat{k}_\infty(\hat{t}_n)\to x$,

$F'(\hat{k}_\infty(\hat{t}_n)) \longrightarrow y$ in H. For every $n \in \mathbb{N}$:

$$(4.11) \quad (F'(\hat{k}_\infty(\hat{t}_n)), P_1\hat{k}_\infty(\hat{t}_n)) \geq (F''(0)\hat{k}_\infty(\hat{t}_n), P_1\hat{k}_\infty(\hat{t}_n)) - \hat{\omega}(\sigma)\cdot \|\hat{k}_\infty(\hat{t}_n)\|^2 \geq \lambda_0 \cdot \|P_1\hat{k}_\infty(\hat{t}_n)\|^2 - \hat{\omega}(\sigma)\cdot\|\hat{k}_\infty(\hat{t}_n)\|^2 \geq \geq \delta_2 \cdot \|\hat{k}_\infty(\hat{t}_n)\|^2 = \delta_2 \cdot \sigma^2.$$

The last inequality follows from (4.8). Further, passing to the limit ($n \longrightarrow \infty$) in (4.11) we get:

$$(4.12) \quad (y, P_1 x) \geq \delta_2 \cdot \sigma^2.$$

Thus, $y \neq 0$. According to Lemma 7 $\hat{k}_\infty(\hat{t}_n) \longrightarrow x \in S(0,\sigma)$ and

$$(4.13) \quad \lambda \cdot x - F'(x) = 0, \text{ where } \lambda = (y,x)/\sigma^2 = (F'(x),x)/\sigma^2.$$

5. For every $n \in \mathbb{N}$ put $\hat{l}_n = \hat{k}_{v_n}, \tau_n = T_{v_n}$. Since the functional F is continuous on $\dot{B}(0,\rho)$, for every $n \in \mathbb{N}$ there exists an $m_0 = m_0(n) \in \mathbb{N}$ such that for all $m \geq m_0$: $F(\hat{k}_\infty(\hat{t}_n)) \leq F(\hat{l}_m(\hat{t}_n)) + \sigma^3$. Take an $n \in \mathbb{N}$. Choose a positive integer $m_1 \geq m_0(n)$ satisfying $\tau_{m_1} \geq \hat{t}_n$. Then for all $m \geq m_1$: $F(\hat{l}_m(\tau_m)) - F(\hat{l}_m(\hat{t}_n)) = \int_{\hat{t}_m}^{\tau_m} \|\hat{l}_m'(\xi)\|^2 d\xi \geq 0$ and thus $F(\hat{k}_\infty(\hat{t}_n)) \leq F(\hat{l}_m(\tau_m)) + \sigma^3$. Since $\hat{l}_m(\tau_m) \in H_0 + H_2$, $F(\hat{k}_\infty(\hat{t}_n)) \leq (F''(0)\hat{l}_m(\tau_m), \hat{l}_m(\tau_m))/2 + \omega(\sigma)\sigma^2 + \sigma^3 \leq$

$\leq [\lambda_0/2 + \omega(\sigma) + \sigma] \cdot \sigma^2$. Passing to the limit ($n \longrightarrow \infty$) in the last inequality we get:

$$(4.14) \quad F(x) \leq [\lambda_0/2 + \omega(\sigma) + \sigma] \cdot \sigma^2.$$

Further, it is obvious that for all $n \in \mathbb{N}$: $F(\hat{k}_\infty(\hat{t}_n)) \geq \geq F(\hat{k}_\infty(0)) \geq (F''(0)\hat{k}_\infty(0), \hat{k}_\infty(0))/2 - \omega(\sigma) \cdot \sigma^2 \geq [\lambda_0/2 - \omega(\sigma)] \cdot \sigma^2$. Hence passing to the limit ($n \longrightarrow \infty$) we have:

$$(4.15) \quad F(x) \geq [\lambda_0/2 - \omega(\sigma)] \cdot \sigma^2.$$

The inequalities (4.14) and (4.15) imply: $|F(x) - \lambda_0 \sigma^2/2| \leq (\sigma + \omega(\sigma)) \cdot \sigma^2$. Accordingly:

$$(4.16) \quad |\lambda_0 - \lambda| = |(F'(x),x)/\sigma^2 - \lambda_0| \leq \sigma^{-2} \cdot [\,|(F'(x) - F''(0)x, x)| + |2 \cdot F(x) - (F''(0)x,x)| + |2 \cdot F(x) - \lambda_0 \sigma^2|\,] \leq \hat{\omega}(\sigma) + 4\,\omega(\sigma) + 2 \cdot \sigma .$$

The proof is finished.

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