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FEW COLORED CUTS OR CYCLES IN EDGE COLORED GRAPHS

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Abstract: Put $g(m,n)=\max\{k;$ there is a graph on n vertices with m-colored edges containing no at most k-colored cycle or edge cut$\}$ and $g(m)=\limsup_{n\to\infty} g(m,n)$. We prove that $g(m,n)<1/2+\sqrt{(2m-7/4)}$ and by a probabilistic method we show that $g(m)\geq(1/60)\sqrt{m}/\log m$ if m is sufficiently large. This answers the problem of A. Bachem and J. Nešetřil.

Key words: Graph, cycle, cut set.

Classification: 05C38, 05C40, 05C80

The symbol $|M|$ will denote the cardinality of a set M. $[x,y]$ is an ordered pair with components x and y. By a graph we shall mean an undirected graph without loops and multiple edges. For a graph G, the symbol V(G) will denote the set of its vertices and E(G) the set of its edges. The edge joining vertices u and v will be denoted (u,v). An m-coloration of E(G) is a mapping of E(G) into the set $\{1,2,\dots,m\}$ (without any further conditions).

We say that a graph G has the property CC(k,m) (CC stands for cut/cycle) if there exists an m-coloration of E(G) such that G contains neither a cycle nor an (edge) cut colored by k or less different colors. We shall assume that some such m-coloration is chosen for every G with the property CC(k,m). We put

$$g(m,n)=\max\{k;\ \text{there exists a graph G on n vertices with CC(k,m)}\}$$
$$g(m)=\limsup_{n\to\infty} g(m,n).$$

This definition is motivated by the following result (known as Minty Lemma, see [2]): Let the edge of a graph be colored by the colors 1,2 and 3 and let (u,v) be an edge of color 1. Then one can find either a cycle colored only by colors 1, 2 containing (u,v), or an edge cut colored only by colors 1, 3 separating u from v. A. Bachem and J. Nešetřil asked for a generalization of this result for more colors.

Theorem 1: $g(m,n) < 1/2+\sqrt{(2m-7/4)}$.

Proof: Suppose that G on n vertices has the property CC(k,m). Every vertex of G must have degree greater than k, hence $|E(G)| \geq (k+1)n/2$. Divide the colors into disjoint groups $C_1, C_2, \ldots, C_q$, each containing at most k colors. Consider a subgraph of G containing only the edges colored by colors from some group C_i; it must be a tree and hence at most n-1 edges are colored by the colors from C_i. The groups may be chosen so that q is not greater than (m+k-1)/k, and we get $|E(G)| \leq (n-1)(m+k-1)/k$. Combining the two inequalities for $|E(G)|$, we obtain the statement of Theorem 1. □

Lemma 1: If some graph G on n vertices has the property CC(k,m), then there exists a graph G' on n^2 vertices with the property CC(k,m).

Proof: Choose some vertex $v \in V(G)$. Let $V(G')=V(G) \times V(G)$. Two vertices $[x,y]$ and $[x',y']$ will be joined by an edge colored by color i if one of the following conditions holds:

(i) $x=x'$ and y and y' are joined by an edge of color i in G or

(ii) $y=y'=v$ and x and x' are joined by an edge of color i in G. The graph G' arises from G by replacing every vertex of G by a copy of G.It is easy to see that a q-colored cycle (cut) in G' enforces the presence of at most q-colored cycle (cut) in G, so that the property CC(k,m) is preserved. □

Corollary: For all n, $g(m) \geq g(n,m)$.

Theorem 2: For sufficiently large m $g(m) \geq (1/60)\sqrt{m}/\log m$.

Proof: By Lemma 1, it is sufficient to prove the inequality $g(m,n) \geq (1/60)\sqrt{m}/\log m$ for some n. We shall use a probabilistic (nonconstructive) argument (for introduction and numerous examples of probabilistic methods see e.g. [11]).

Let G(p,n,m) denote a random variable whose values are graphs on $\{1,2,\ldots,n\}$ with m-colored edges. For every edge (i,j) the probability that (i,j) belongs to E(G(p,n,m)) is equal to p and these probabilities are independent for different edges. Every edge present is colored independently and equiprobably by one of colors $1,2,\ldots,m$.

In the sequel we shall assume that n is large enough for all estimations to be valid. We shall use the o, O notation for asymptotic comparing of functions; the variable is always n.

We introduce some parameters dependent on n:

$f=20 \log n$

$p=f/n$

$k=(1/60)\log n/(\log \log n)^2$

$m=(\log n)^2/(\log \log n)^2$

$d= (1/26)\log n/\log \log n$

(for k, m and d, take the nearest integer).

We shall write G instead of G(n,p,m). We say that G has some property typically if the probability that G has this property is 1-o(1).

Let C denote the set of all edges of G belonging to cycles of length less than d. We shall show that the graph $G^-=(V(G),E(G)\setminus C)$ has typically the property CC(k,m), which establishes Theorem 2. We start with cycles in G^-.

Lemma 2. Typically G contains no cycle longer than d-1 colored by k or less colors.

Proof: A cycle of length t may be chosen in less than n^t ways. There is $\binom{m}{k}$ of ways to choose k colors of m colors. For every cycle of length t, the probability that it is contained in G and colored by given k colors is $(p.k/m)^t$. The probability that G contains a cycle longer than d-1 colored by k or less colors is less than

$$\binom{m}{k} \sum_{t=d}^{n} (n,p,k/m)^t=o(1),$$

as a straightforward calculation shows.

Lemma 3. Typically $|C|<n^{1/25}$.

Proof: By a similar reasoning as in the proof of Lemma 2, we find that the expected number of edges in C is bounded by

$$\sum_{t=3}^{d-1} t.(np)^t < d^2 f^d < n^{1/26+o(1)}$$

and so the probability that the number of edges in C is greater than $n^{1/25}$ is o(1). □

Now we shall investigate the edge cuts of G^-. Consider a partition of V(G) into two parts R and S, $1 \leq |R|=r$, $|S|=n-r$ $(r \leq n/2)$. Let F denote the set of edges $\{(u,v);u \in R,v \in S\}$ and let $E=C \cap F$. Let H be the bipartite graph spanned by E and let $A=V(H)\cap R$, $B=V(H)\cap S$. Let $Q \subset B$ be the set of vertices of B incident to only one edge of E. Put $a=|A|$, $b=|B|$, $e=|E|$, $q=|Q|$ and $Q=\{v_1,v_2,\dots,v_q\}$. Since $e\geq q+2(b-q)$, we have $q\geq 2b-e$.

The edge cut determined by the partition (R,S) violates the condition of $CC(k,m)$ if the edges of $(E(G)\cap F)\setminus E$ are colored by k or less colors. We shall bound the probability that there exists such a cut.

Given the number r, we can choose the partition (R,S) in $\binom{n}{r}$ ways. Then given a, b and e, there is at most $\binom{r}{a}\binom{n-r}{b}\binom{ab}{e}$ possible choices of the set E, at the same time determining the sets A, B and Q. When E is fixed, each of the $r(n-r)-e$ edges of $F\setminus E$ must be either missing in G or colored by some of k chosen colors; the probability of this is $\binom{m}{k}(1-(1-k/m)p)^{r(n-r)-e}$.

The edges of E must be present in G (probability p^e) and finally for each vertex $v_i\in Q$ there must be a path P_i of length less than d in G, joining v_i to some other vertex of B. Denote $\alpha(b,q)$ the probability that for a fixed B and Q there exist paths $P_1,\dots,P_q$ of length at most d in G, where each P_i joins the vertex $v_i\in Q$ to some other vertex of B. Let

$\beta(b,e)=\max\{\alpha(b,q);\ 0\leq q\leq b,\ q\geq 2b-e\}$.

Putting everything together, we find that the probability of the existence of a bad cut is at most

$$\sum_{r,a,b,e} T(r,a,b,e),$$

where

$$T(r,a,b,e)=\binom{n}{r}\binom{r}{a}\binom{n-r}{b}\binom{ab}{e}\binom{m}{k}(1-(1-k/m)p)^{r(n-r)-e}p^e\beta(b,e)$$

and the sum is over all r,a,b,e with $1\leq r\leq n/2$, $0\leq a\leq r$, $0\leq b\leq n-r$, $a,b\leq e\leq$ $\leq ab$ and $e<n^{1/25}$ (in view of Lemma 3). Since the number of terms in this sum is $o(n^2)$, it suffices to show that each term is less than n^{-2}. We compute $\log T(r,a,b,e)$ and use some simple estimations (e.g. $\binom{x}{y}\leq x^y$, $-2x\leq\log(1-x)\leq -x$ for $0\leq x\leq 1/2$, $1/2\leq(1-k/m)\leq 1$, $1/2\leq(n-r)/n\leq 1$), getting the bound

$\log T(r,a,b,e)\leq -2\log n-\log n.\,[(23/25-o(1))e-b)]+\log\beta(b,e)$.

Now if $e\geq 3b/2$, the expression in square brackets is nonnegative and $\log T(r,a,b,e)\leq -2\log n$ as required. One can easily verify that the following estimation suffices to handle the case $e<3b/2$ (then $q\geq 2b-e>b/2$):

Lemma 4: If $q\geq b/2$, then $\alpha(b,q)\leq n^{-b/10}$.

Proof: We may assume that the paths $P_1,\dots,P_q$ meet B only by their endvertices. Let $J=P_1\cup P_2\cup\dots\cup P_q$, $j=|V(J)\setminus B|$, $h=|E(J)|$ and $z=|V(J)\cap B|$. Since J has at most $z/2$ connected components, we have $h\geq j+z-z/2\geq j+q/2\geq j+b/4$. We shall estimate the number of possible graphs J for given h and j.

For each P_i we choose v_i as its starting vertex. Then we traverse successively $P_1, P_2, \ldots, P_q$ from their starting vertices to the other endpoints, we order the vertices of J according to the order of their first appearance during this traversal and we do the same for the edges of J. This traversal also defines an orientation of edges (the tail (head) of an edge is its vertex encountered first (second, respectively) during the first traversal of this edge). We say that a vertex u is new for an edge (u,v) if u does not belong to B and u is incident only with edges following (u,v) and possibly with the edge immediately preceding (u,v) (otherwise u is old for (u,v)).

The number of edges with a new head is j, so the number of edges with an old head is h-j. Each edge (u,v) with a new head and an old tail (except possibly at most q such edges) can be assigned to the last edge with a new tail and an old head preceding (u,v) and this assignment is one-to-one, hence the number of edges with a new head and an old tail is at most $h-j+q$.

Imagine that we build the graph J as follows: first we choose the j vertices of $V(J) \setminus B$ together with their ordering (at most n^j possibilities). Then we select the subsets of edges of type old head-new tail and new head-old tail in the ordering, which determines also the position of edges of two remaining types in the ordering. Using the above estimations of the number of such edges, we see that there is less than $h^{2(h-j)+q}$ possibilities. Finally we choose edges one by one according to their ordering. Note that the new vertices of each edge are uniquely determined by the ordering of vertices, so only old vertices can be chosen and there is at most $\max(|V(J)|, b) < b.d$ possibilities for each old vertex (excluding the starting vertices of the paths). In view of the above, we can make at most $3(h-r)$ such choices. Therefore the number of different graphs J is bounded by $n^j(bd)^{5(h-j)+q}$. Each of them appears in G with the probability p^h, so

$$\alpha(b,q) \leq \sum_{0 \leq j \leq h-b/4 \leq bd} p^h n^j (bd)^{5(h-j)+q} = o(n^{-b/10}). \quad \square$$

Our probabilistic proof of Theorem 2 could perhaps be simplified using some known results on random graphs.

Careful balancing of numerical factors would improve the constant in Theorem 2, but more sophisticated methods are needed to bridge the gap between the lower bound and the upper bound for g(m).

An anonymous referee of an earlier version of this paper pointed out that using the results of [3] and [4] one could obtain explicit graphs on n vertices with the property CC(k,m) for $k=\log n/(\log\log n)^2$ and $m=(\log n)^2$.

This is only slightly weaker than our result and maybe it could be further improved.

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