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A sequential approach to a construction of measures

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Abstract. This paper deals with measures in the Alternative Set Theory. First of all σ–additive measures are constructed. Then measures, "depending on the way of measurement", are obtained. It is proved that the measure of a given class can, in the dependence on the way of measurement, be an arbitrary nonnegative real number.

Keywords: Alternative set theory, measure, σ–additivity, way of measurement, observable class

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The idea of developing the measure theory in the Alternative Set Theory has originated in Prague seminar on Set Theory (see the notes in [**Č 1976**]). M.Raškovič, in his paper [**R 1981**], has re–constructed Loeb measure in the framework of AST. Further results, concerning the measurability of projective semisets, are due to K.Čuda [**Č 1986**]. A different approach is due to A. Tzouvaras [**Tz 1987**], where he has used the notion of cuts of classes to the construction of a measure.

In this paper a new approach is developed. Both classical measures (i.e. σ–additive and nondecreasing) and measures, "depending on the way of measurement" are obtained.

1. Preliminaries.

The reader is assumed to be familiar with [V]. The notions, results and conventions from it will be used freely without any reference. Some modifications and supplements are stated below.

1.1. The letters b, c, d (possibly indexed) and m, n will always denote natural numbers (i.e. the elements of the class N); i, j, k, will be reserved for finite natural numbers (i.e. for the elements of the class FN) and a will denote a fixed infinite natural number (i.e. an element of $N \setminus FN$).

The indiscernibility equivalence $\doteq$ of infinitesimal nearness on the class Q of all rational numbers is defined by

$$p \doteq q \equiv ((\exists k)(\forall i > 0)(|p| < k \ \& \ |p-q| < 1/i) \vee (\forall k)$$
$$(p > k \ \& \ q > k) \vee (p < -k \ \& \ q < -k)).$$

For each $q \in Q$ denote $\operatorname{mon}(q) = \{s \in Q; s \doteq q\}$.

R will denote the class of all real numbers. Denote

$$\infty = \{q \in Q; (\forall i)(q > i)\} \text{ and } -\infty = \{q \in Q; (\forall i)(q < -i)\}.$$

∞ and $-\infty$ are assumed to be real numbers, too. The letter r (possibly indexed) will be reserved for real numbers.

The countable sum of nonnegative real numbers r_i is defined by the following – let $b_i \in r_i$. Prolong the sequence $\{b_i; i \in FN\}$ onto a set $\{b_n; n \in a\}$. Then, since the numbers r_i are nonnegative, there exists a $d \leq a, d \notin FN$, such that for all $c \leq d, c \notin FN$, there holds

$$\sum_{n=0}^{c} b_n \doteq \sum_{n=0}^{d} b_n.$$

We put $\sum_{i \in FN} r_i = r$, where $r \in R$ is such that $\sum_{n=0}^{d} b_n \in r$.

1.2.. Further we state some modifications of notions and results from [**K–Z 1988**] and [**K–Z 1989**]

Let X be a class. Then $\underline{X}$ will denote its lower cut (i.e. $\underline{X} = \{n; (\exists u)(u \subseteq X \;\&\; n \hat{<} u)\}$) and $\overline{X}$ its upper cut (i.e. $\overline{X} = \{n; (\forall u)(u \supseteq X \Rightarrow n \hat{<} u)\}$). If $\underline{X} = \overline{X}$, then the common value will be denoted by $|X|$ and called the cut of X. The order $\leq$ on the family of all cuts is given by inclusion.

Further, if C, D are arbitrary cuts, then we shall denote

$$C/FN = \{n; (\forall i)(n \cdot i \in C)\}; \quad C \cdot FN = \{n; (\exists m \leq C)(\exists i)(n < m \cdot i)\};$$
$$C + D = \{c; (\exists n \leq C, m \leq D)(c < n + m)\};$$
$$C - D = \{n; (\forall m \leq D)(m + n \in C)\};$$
$$\mathrm{int}(C) = C - C/FN; \quad \mathrm{cl}(C) = C + C/FN.$$

We define an equivalence $\approx$ on the family of all cuts by $C \approx D \equiv \mathrm{int}(C) \leq D \leq \mathrm{cl}(C)$.

A cut D is additive if $D + D = D$. A cut D is nonadditive if it is not additive.

Let $\{A_i; i \in FN\}$ be a sequence of cuts. Then we shall denote

$$\underset{\cdot}{\sum}\{A_i; i \in FN\} = \{n; (\exists j)(n \in A_0 + A_1 + \cdots + A_j)\},$$

$$\overset{\cdot}{\sum}\{A_i; i \in FN\} = \{n; (\forall f)((N \supseteq \mathrm{dom}(f) \subseteq FN) \;\&$$
$$\& \;(\mathrm{rng}(f) \subseteq N) \;\&\; (\forall i)(f(i) \notin A_i) \Rightarrow n < \sum f)\}.$$

1.2.1. Theorem. *Let $\{A_i; i \in FN\}, \{B_i; i \in FN\}$ be sequences of cuts such that $(\forall i)(A_i \approx B_i)$. Then*

$$\underset{\cdot}{\sum}\{A_i; i \in FN\} \approx \overset{\cdot}{\sum}\{B_i; i \in FN\}.$$

1.2.2. Theorem. *Let $\{X_i; i \in FN\}$ be a sequence of pairwise disjoint classes. Denote $X = \bigcup\{X_i; i \in FN\}$. Then*

$$\underset{\cdot}{\sum}\{\underline{X_i}; i \in FN\} \leq \underline{X} \leq \overline{X} \leq \overset{\cdot}{\sum}\{\overline{X_i}; i \in FN\}.$$

Denote $\mathcal{B}$ the smallest σ-ring of semisets such that $V \subset \mathcal{B}$. The elements of $\mathcal{B}$ will be called Borel semisets.

1.2.3. Theorem.
Let $X \in \mathcal{B}$. Then each of $\underline{X}$ and $\overline{X}$ is either π or σ and $\underline{X} \approx \overline{X}$.

We define an equivalence $\overset{b}{\approx}$ on the family $\mathcal{B}$ by the following

$$(\forall B, C \in \mathcal{B})(B \overset{b}{\approx} C \equiv \underline{B} \approx \underline{C}).$$

1.2.4. Theorem. *Let $X \in \mathcal{B}$. Then $\underline{X} = \overline{X}$ or there exists an additive cut $A < \underline{X}$ such that for each $n \in \overline{X} \setminus \underline{X}$ there holds $\underline{X} = n - A$ and $\overline{X} = n + A$.*

1.3. Remind that a family of classes $\mathcal{A}$ is said to be codable if there exists a pair of classes $\langle X, S \rangle$ such that

$$(1) \qquad (\forall Y \in \mathcal{A})(\exists y \in X)(S''y = Y) \ \& \ (\forall y \in X)(S''y \in \mathcal{A})$$

and the pair $\langle X, S \rangle$, having Property (1) is said to be the coding pair of $\mathcal{A}$.

1.3.1. Axiom. Each codable family of classes $\mathcal{A}$ is extensionally codable, i.e. there exists such a coding pair $\langle X, S \rangle$ of $\mathcal{A}$, for which

$$(2) \qquad (\forall x, y \in X)(S''x = S''y \equiv x = y)$$

holds.

Troughout the whole paper, if $\mathcal{A}$ is a codable family and $\langle X, S \rangle$ its coding pair, then we shall assume Property (2) to hold for $\langle X, S \rangle$.

1.3.2. Remark. Since $\mathcal{B}$ is the smallest σ-ring such that $V \subset \mathcal{B}$, obviously $\mathcal{B}$ is codable.

2. Basic notions.

Let $\{s_n; n \in FN\}$ be any sequence. By $\bigcup\bigcap\{s_n; n \in FN\}$ we shall denote $\bigcup_{i \in FN} \bigcap_{j \geq i} s_j$ and by $\bigcap\bigcup\{s_n; n \in FN\}$ we shall denote $\bigcap_{i \in FN} \bigcup_{j \geq i} s_j$.

A sequence $\{s_n; n \in a\}$ of natural numbers is said to be an approximating sequence of a pair of cuts $\langle A, B \rangle$ if

$$\bigcup\bigcap\{s_n; n \in FN\} = A \text{ and } \bigcap\bigcup\{s_n; n \in FN\} = B.$$

2.1. Lemma. *Let A, B be any cuts. There exists an approximating sequence of the pair $\langle A, B \rangle$ iff $A \leq B$ and each of the cuts A, B is π or σ.*

PROOF: Let $A \leq B$ and each of them be π or σ. Then there exists monotone sequences $\{b_n; n \in a\}, \{c_n; n \in a\}$ such that $\bigcup\bigcap\{b_n; n \in FN\} = A$ and $\bigcap\bigcup\{c_n; n \in FN\} = B$. Obviously the sequence $\{s_n; n \in a\}$, such that $s_n = b_n$ for even n and $s_n = c_n$ for odd n, is an approximating sequence of the pair $\langle A, B \rangle$.

On the other hand, if $\{s_n; n \in a\}$ is any sequence of natural numbers, then obviously $\bigcup\bigcap\{s_n; n \in FN\} \subseteq \bigcap\bigcup\{s_n; n \in FN\}$ and each of the cuts $\bigcup\bigcap\{s_n; n \in FN\}$ and $\bigcap\bigcup\{s_n; n \in FN\}$ is π or σ, as they are real classes. ■

A sequence $\{s_n; n \in a\}$ of natural numbers is said to be an approximating sequence of a class X if it is an approximating sequence of the pair $\langle \underline{X}, \overline{X} \rangle$.

2.2. Lemma. *Let B be a Borel semiset. Then there exists and approximating sequence of B.*

PROOF: follows immediately from 1.2.3 and 2.1. ■

2.3.Remark. For each set u the sequence $\{b_n; n \in a\}$, such that for each $n \in a \quad b_n = |u|$, is an approximating sequence of u.

2.4. Lemma. *Let $\langle X, S\rangle$ be a coding pair of $\mathcal{B}$. Then there exists a map F with $\operatorname{Dom} F = X$ such that for each $x \in X \quad F(x)$ is an approximating sequence of $S''x$.*

PROOF: Using transfinite construction we can get the function F. ■

2.5. Agreement. We shall consider $\mathcal{B}$ to be the domain of the above mentioned map F and for each $B \in \mathcal{B}$ by $F(B)$ we shall denote the value $F(x)$, where $x \in X$ is such that $S''x = B$ ($\langle X, S\rangle$ being the coding pair of $\mathcal{B}$).

Any map F which assigns to each semiset $A \in \mathcal{B}$ a sequence, approximating A, will be called the Borel approximating function (BAF, to be short).

Let F be a BAF, $s = \{s_n; n \in a\}$ any approximating sequence of a nonempty Borel semiset and $B \in \mathcal{B}$. Let $F(B) = \{b_n; n \in a\}$. The semiset B will be called s, F–observable if there exists a $d < a, d \notin FN$, such that for all $m < d, m \notin FN$ and $n < d, n \notin FN$ it holds $b_n/s_n \doteq b_m/s_m$.

The system of all s, F–observable semisets will be denoted by $O(s, F)$.

Let F be a BAF and $s = \{s_n; n \in a\}$ any approximating sequence of a nonempty Borel semiset. We define a measure $m_{s,F} : O(s, F) \to R$ by the following: $m_{s,F}(B) = r$ iff there exists a $d < a, d \notin FN$ such that for all $n \leq d$, $n \notin FN \quad b_n/s_n \in r$ holds, where $B \in \mathcal{B}$ and $F(B) = \{b_n; n \in a\}$.

3. Classical measures.

Throughout this section $s = \{s_n; n \in a\}$ will denote a fixed approximating sequence of a Borel semiset having nonadditive cuts.

3.1. Proposition. *Let $\{b_n; n \in a\}$ be an arbitrary approximating sequence of a nonempty Borel semiset B. Then the cuts of B are nonadditive iff there exists an $n < a$, $n \notin FN$, such that for all $m \leq n$, $m \notin FN \quad b_m/b_n \doteq 1$ holds.*

PROOF: Let the cuts of B be additive. Then by Theorem 1.2.4 $\underline{B} = \overline{B}$. Because of the additivity of $|B|$ for each i there exists a $j > i$ such that $b_j/b_i > 2$ or $b_i/b_j > 2$, hence for each $n < a$, $n \notin FN$, there exists an $m < n$, $m \notin FN$, such that $b_m/b_n \not\doteq 1$.

If the cuts of B are nonadditive, then by Theorem 1.2.3 there exists a $d \in N$ such that $\operatorname{int}(d) \leq \underline{B} \leq \overline{B} \leq \operatorname{cl}(d)$, hence for each $k > 1$ there exists an i such that for each $j > i$ there holds

$$\frac{d - d/k}{d + d/k} = \frac{1 - 1/k}{1 + 1/k} < a_j/a_i < \frac{1 + 1/k}{1 - 1/k} = \frac{d + d/k}{d - d/k}.$$

Prolongation Axiom implies the assertion of this proposition. ■

3.2. Agreement. For each $B \in \mathcal{B}$, having nonadditive cuts, we shall assume that for its approximating sequence $\{b_n; n \in a\}$ there holds $b_m/b_n \doteq 1$ for all $m, n < a$, $m, n \notin FN$.

3.3. Theorem. *Let F be a BAF. Then $O(s, F) = \mathcal{B}$. If G is any other BAF, then $m_{s,F} = m_{s,G}$. If $b \in \mathcal{B}$ has an additive cut, then*

$$m_{s,F}(B) = \begin{cases} 0, & \text{if } |B| \subset \bigcup\bigcap\{s_n; n \in FN\} \\ \infty, & \text{if } |B| \supset \bigcap\bigcup\{s_n; n \in FN\}. \end{cases}$$

PROOF: Denote $F(B) = \{b_n; n \in a\}$. If B has nonadditive cuts, then by Proposition 3.1 for all $n, m < a \quad n, m \notin FN \quad b_n/b_m \doteq s_n/s_m \doteq 1$ holds. This implies $B \in O(s, F)$.

If B has an additive cut, then there are two possibilities.

i./ $|B| \subset \bigcup\bigcap|s_n; n \in FN\}$. Then there exists an i such that for all $j > i$ there holds $s_j \notin |B|$. Hence there exists an i such that for all $j > i \quad s_j > b_j$. And, since $|B|$ is additive, for each k there exists an i such that for all $j > i \quad k \cdot b_j < s_j$ holds. Hence $B \in O(s, F)$ and $m_{s,F}(B) = 0$.

ii./ $|B| \supset \bigcap\bigcup\{s_n; n \in FN\}$. Similarly one can prove that $B \in O(s, F)$ and $m_{s,F}(B) = \infty$.

Let F, G be two different Borel approximating functions and let $F(B) = \{b_n; n \in a\}$ and $G(B) = \{c_n; n \in a\}$ for a $B \in \mathcal{B}$. Define a new BAF H by $H(B) = \{d_n; n \in a\}$, where $d_n = b_n$ for even n and $d_n = c_n$ for odd n. Since $B \in O(s, H)$, $m_{s,H}(B)$ is defined and by the definition of H $m_{s,H}(B) = m_{s,H}(B) = m_{s,G}(B)$, which was to be proved. ∎

In the remainder of this section F will denote a fixed BAF.

3.4. Theorem. *Let $B, C \in \mathcal{B}$ be such that $B \overset{b}{\approx} C$. Then $m_{s,F}(B) = m_{s,F}(C)$.*

PROOF: If B, C have an additive cut, then the equality $m_{s,F}(B) = m_{s,F}(C)$ is implied by Theorem 3.3.

If B, C have nonadditive cuts, then by Theorem 1.2.3 there exists d such that $\text{int}(d) \leq \underline{B} \approx \underline{C} \approx \overline{B} \approx \overline{C} \leq \text{cl}(d)$. Denote $F(B) = \{b_n; n \in a\}$ and $F(C) = \{c_n; n \in a\}$. Then for each $k > 1$ there exists i such that for each $j > i$ there hold

$$(d - d/k)/d = 1 - 1/k < b_j/d < 1 + 1/k = (d + d/k)/d$$

and

$$(d - d/k)/d = 1 - 1/k < c_j/d < 1 + 1/k = (d + d/k)/d,$$

hence for each $n < a$, $n \notin FN \quad b_n/c_n \doteq 1$, hence $b_n/s_n \doteq c_n/s_n$, which was to be proved ∎

Theorem 3.4 has the following trivial, but important consequences. Their proofs are left to the reader.

3.5. Corollary. *Let $B \in \mathcal{B}$ have nonadditive cuts and d be such that $d \overset{b}{\approx} B$. Then $m_{s,F}(B) = \operatorname{mon}(d/s_n)$ for any $n < a$, $n \notin FN$.*

3.6. Corollary. *$m_{s,F}$ is an additive measure.*

3.7. Theorem. *$m_{s,F}$ is σ-additive, nondecreasing and nonnegative measure.*

PROOF: Obviously $m_{s,F}$ is nonnegative and nondecreasing. We shall prove its σ-additivity.

Let $\{B_i; i \in FN\}$ be a countable system of pairwise disjoint Borel semisets. Without loss of generality we can assume the measure of each B_i to be finite (and hence $\overline{B_i} < \bigcup\bigcap\{s_n; n \in FN\} \cdot FN$ for each i). Since the semisets $B_i, i \in FN$, are pairwise disjoint, there holds

$$\underset{\cdot}{\sum}\{\underline{B_i}; i \in FN\} \leq \underline{\bigcup\{B_i; i \in FN\}} \approx \overline{\bigcup\{B_i; i \in FN\}} \leq$$

$$\leq \underset{\cdot}{\sum}\{\overline{B_i}; i \in FN\}$$

and

$$\underset{\cdot}{\sum}\{\underline{B_i}; i \in FN\} \approx \dot{\sum}\{\overline{B_i}; i \in FN\}.$$

Denote $X = \{i \in FN; \underline{B_i} + \underline{B_i} \neq \underline{B_i}\}$ and for each $i \in X$ take a d_i such that $d_i \overset{b}{\approx} B_i$. Then

$$\underset{\cdot}{\sum}\{\underline{B_i}; i \in FN\} \approx \underset{\cdot}{\sum}\{d_i; i \in X\} + \underset{\cdot}{\sum}\{|B_i|; i \in FN \setminus X\},$$

$$\underset{\cdot}{\sum}\{d_i; i \in X\} \approx \dot{\sum}\{d_i; i \in X\}$$

and, since for all $i \in FN \setminus X$ $|B_i|$ are additive and $m_{s,F}(B_i) < \infty$,

$$\underset{\cdot}{\sum}\{|B_i|; i \in FN \setminus X\} \leq \bigcup\bigcap\{s_n; n \in FN\}/FN.$$

Hence, by Theorem 3.3 $m_{s,F}(\underset{\cdot}{\sum}\{|B_i|; i \in FN \setminus X\}) = m_{s,F}(\bigcup\{B_i; i \in FN \setminus X\}) = 0$ and hence, by Corollary 3.6,

$$m_{s,F}(\bigcup\{B_i; i \in FN\}) = m_{s,F}(\underset{\cdot}{\sum}\{d_i; i \in X\}) +$$

$$+ m_{s,F}(\underset{\cdot}{\sum}\{|B_i|; i \in FN \setminus X\} = m_{s,F}(\underset{\cdot}{\sum}\{d_i; i \in X\}).$$

Hence, without loss of generality, we can assume $X = FN$. Then

$$m_{s,F}(\bigcup\{B_i; i \in FN\}) = m_{s,F}(\underset{\cdot}{\sum}\{d_i; i \in FN\}) =$$

$$= m_{s,F}(\overset{\cdot}{\sum}\{d_i; i \in FN\}),$$

hence for each $n \in FN$ and each $\{d_m; m \leq \theta\}$, where $\theta \in N \setminus FN$, there holds

$$\sum_{i=0}^{n} m_{s,F}(d_i) = m_{s,F}(d_0 + \cdots + d_n) \leq m_{s,F}(\underset{\cdot}{\sum}\{d_i; i \in FN\}) =$$

$$= m_{s,F}(\bigcup\{B_i; i \in FN\}) = m_{s,F}(\overset{\cdot}{\sum}\{d_i; i \in FN\}) \leq$$
$$\leq m_{s,F}(d_0 + \cdots + d_\theta) = \operatorname{mon}((d_0 + \cdots + d_\theta)/s_b)$$

for any $b \notin FN$, and the σ-additivity follows. ∎

4. Measures depending on the way of measurement.

Throughout this section $s = \{s_n; n \in a\}$ will denote a fixed approximating sequence of a nonempty Borel semiset having an additive cut.

4.1. Lemma. *Let F be any BAF and $b \in \mathcal{B}$. Then*

i.) *if $\bigcap\bigcup\{s_n; n \in FN\} \subsetneq \underline{B}$ then $B \in O(s, F)$ and $m_{s,F}(B) = \infty$*

ii.) *if $\bigcup\bigcap\{s_n; n \in FN\} \supsetneq \overline{B}$, then $B \in O(s, F)$ and $m_{s,F}(B) = 0$.*

PROOF: The proofs of i.) and ii.) are very similar, therefore we shall prove only ii.)

Let $F(B) = \{b_n; n \in a\}$ and $d \in \bigcup\bigcap\{s_n; n \in FN\} \setminus \overline{B}$. Then there exists an i such that for all $j > i$ $b_j < d$. Since $S = \bigcup\bigcap\{s_n; n \in FN\}$ is an additive cut and $d \in S$, for each k there exists an i such that for all $j > i$ there holds $s_j > d \cdot k$, hence $B \in O(s, F)$ and $m_{s,F}(B) = 0$. ∎

4.2. Theorem. *Let $B \in \mathcal{B}$, $|B| = \bigcup\bigcap\{s_n; n \in FN\}$. Then*

i.) *for each $0 \leq r \leq \infty$ there exists a BAF F such that $B \in O(s, F)$ and $m_{s,F}(B) = r$*

ii.) *there exists a BAF G such that $B \notin 0(s, G)$.*

PROOF: We shall restrict our attention to the case $|B|$ being σ. Assertion i.) will be proved in three steps.

a.) Let $0 < r < \infty$ and $q \in r$. Put $b_n = \lceil q \cdot s_n \rceil$ ($\lceil\ \rceil$ being the integer part). The equality $|B| = \bigcup\bigcap\{s_n; n \in FN\}$ and the additivity of $|B|$ imply $\{b_n; n \in a\}$ to be an approximating sequence of B. Put $F(B) = \{b_n; n \in a\}$. Then obviously $B \in O(s, F)$ and $m_{s,F}(B) = r$.

b.) Let $r = \infty$. Put $b_n = n \cdot s_n$. Then $\{b_n; n \in a\}$ is an approximating sequence of B and for $F(B) = \{b_n; n \in a\}$ $B \in O(s, F)$ and $m_{s,F}(B) = \infty$.

c.) Let $r = 0$. Obviously each of the sequences $|\{\lceil s_n/k \rceil; n \in a\}, k \neq 0$, approximates B. Since $|B|$ is σ, without loss of generality $s_i \in |B|$ can be assumed for all i. Define a sequence $\{n(k); 0 \neq k \in FN\}$ by the following – $n(1) = 0$ and for all $k \geq 2$ put $n(k) = i$, where i is the least number such that $i > n(k-1)$ and for all $j \geq i \quad \lceil s_j/k \rceil \geq s_k$. Since $|B|$ is additive, such an i does exist. For $n(k) \leq j < n(k+1)$ put $b_j = \lceil s_j/k \rceil$. Then obviously $\cap\cup\{b_j; j \in FN\} = |B|$ and hence any prolongation $\{b_n; n \in a\}$ of $\{b_n; n \in FN\}$ is an approximating sequence of B and if we put $F(B) = \{b_n; n \in a\}$, then $B \in O(s, F)$ and $m_{s,F}(B) = 0$.

ii.) Assertion i.) implies the existence of BAF–s F, H such that $m_{s,F}(B) \neq m_{s,H}(B)$. Denote $F(B) = \{b_n; n \in a\}$, $H(B) = \{c_n; n \in a\}$ and define G by $G(B) = \{d_n; n \in a\}$ such that $d_n = b_n$ for even n and $d_n = c_n$ for odd n. Then obviously $\{d_n; n \in a\}$ approximates B and $B \notin O(s, H)$. ■

4.3. Remark. From the physical point of view, Theorem 4.2 can be interpreted as the dependence of a measure (or of a result of an experiment) on the way of measurement.

Using transfinite construction, from Lemma 4.1 and Theorem 4.2. we get the following

4.4. Theorem. *Let $\mathcal{O} \subseteq \mathcal{B}$ be any class of semisets such that*

$$(\forall B \in \mathcal{B})(\overline{B} \overset{\subset}{\neq} \cup\cap\{s_n; n \in FN\} \vee \underline{B} \overset{\supset}{\neq} \cap\cup\{s_n; n \in FN\}) \Rightarrow B \in \mathcal{O}$$

and let $\lambda : \mathcal{O} \to R$ be any nonnegative real-valued function such that

$$(\forall B \in \mathcal{O})(\overline{B} \overset{\subset}{\neq} \cup\cap\{s_n; n \in FN\} \Rightarrow \lambda(B) = 0) \ \& $$
$$\& \ (\underline{B} \overset{\supset}{\neq} \cap\cup\{s_n; n \in FN\} \Rightarrow \lambda(B) = \infty).$$

Then there exists a BAF F such that $O(s, F) = \mathcal{O}$ and $m_{s,F} = \lambda$.

References

[Č] Čuda K., *A nonstandard set theory*, Comment.Math.Univ.Carolinae **17** (1976), 647-663.

[Č] Čuda K., *The consistency of the measurability of projective semisets*, Comment.Math. Univ.Carolinae **27** (1986), 103-121.

[K-Z] Kalina M., Zlatoš P., *Arithmetic of cuts and cuts of classes*, Comment.Math.Univ. Carolinae **30** (1988).

[K-Z] Kalina M., Zlatoš P., *Cuts of Borel classes*, to be published.

[R] Rašković M., *Measure and integration in the Alternative Set Theory*, Publications de l'Institut mathématique **29(43)** (1981), 191-197.

[Tz] Tzouvaras A., *A notion of measure for classes in AST*, Comment.Math.Univ.Carolinae **28** (1987), 449-455.

[V] Vopěnka P., "Mathematics in the Alternative Set Theory," Teubner Texte, Leipzig, 1979; russian translation Mir, Moscow.

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