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Constructions of classes by transfinite induction in AST

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Abstract. Many constructions of classes in the Alternative set theory are made using a special type of transfinite induction. In this paper we are trying to formulate theorems describing the common core of the constructions in question so that these constructions can be essentially simplified using the proposed theorems.

Keywords: alternative set theory, transfinite induction

Classification: Primary 03E70, Secondary 03H15

In the alternative set theory (AST; see [V]) a lot of interesting classes were constructed by transfinite induction as union of an increasing sequence o. countable classes (cf.e.g. constructions of automorphisms and endomorphisms §1,2 ch. V [V] and constructions of automorphisms with particular properties [S], [Č–V]) however this type of construction was especially used in papers [**Ve 1**] and [**Ve 2**] where a series of remarkable endomorphic universes was constructed. In the mentioned articles A.Vencovská constructed endomorphic universes so that some in advance chosen sets were elements of the resulting endomorphic universe and others were not elements of it. To simplify her construction A.Vencovská states two general theorems, nevertheless it is obvious that by similar ideas even other interesting endomorphic universes can be constructed and that these ideas can be used even for construction of some other important types of classes. The authors of the present article tried to find some theorems describing the common core of these constructions.

The second theorem of the paper seems to be a sufficiently general description of the type of construction we are dealing with though this statement is rather complicated; thus we state at first a more simple theorem which is sufficient in almost all cases (Theorem 1). At the end of the paper we show a method how the first theorem can be applied even on some systems not fulfilling the conditions required in the applied theorem.

We use the notation usual in the alternative set theory (see [V]).

We say that a system of classes $\mathcal{I}$ is closed under countable monotonous unions if for every system $\{X_n; n \in FN\} \subseteq \mathcal{I}$ with

$$n \leq m \rightarrow X_n \subseteq X_m$$

the class $\bigcup\{X_n; n \in FN\}$ is an element of $\mathcal{I}$. We define the closeness under uncountable monotonous unions analogically (replacing the class FN by the class Ω). A system of classes is closed under monotonous unions if it is closed both under countable and uncountable monotonous unions.

Let us mention that the closeness under monotonous unions is equivalent neither to the closeness under countable monotonous unions nor to the closeness under uncountable monotonous unions–e.g. if C is a countable class then the system $\{X; \neg C \subseteq X\}$ is closed under uncountable monotonous unions however it is not closed under countable ones.

The symbol $\mathcal{I}^c$ denotes the system of all elements of $\mathcal{I}$ which are at most countable (i.e. $\mathcal{I}^c = \{C \in \mathcal{I}; C \preceq FN\}$).

A system of classes $\mathcal{I}$ is called a cut (with respect to $\subseteq$) if

$$(I' \subseteq I \;\&\; I \in \mathcal{I}) \rightarrow I' \in \mathcal{I}.$$

A system of classes $\mathcal{D}$ is said to be dense in a system of classes $\mathcal{I}$ if

$$(\forall I \in \mathcal{I})(\exists I' \in \mathcal{D}) I \subseteq I' \in \mathcal{I}.$$

Theorem 1. *Let $\mathcal{I}$ be a system of classes closed under monotonous unions and let for every $x \in K$, the system $\mathcal{D}_x$ be closed under uncountable monotonous unions and let*

$$\mathcal{D}_x^c \quad \textit{be dense in } \mathcal{I}^c.$$

Then

(a) *for every element C of $\mathcal{I}$ which is at most countable there is $I \in \mathcal{I}$ with*

$$C \subseteq I \;\&\; (\forall x \in K) I \in \mathcal{D}_x$$

(b) *if $\mathcal{I}$ is a cut then for every element C of $\mathcal{I}$ which is at most countable we can find a maximal element I of $\mathcal{I}$ (i.e $\neg(\exists I' \in \mathcal{I}) I \subset I'$) satisfying the property described in (a).*

Since we are able to prove a statement stronger than the assertion (a) of Theorem 1 (however also a more complicated one) we state and prove this stronger result first.

Theorem 2. *Let $\mathcal{I}$ be a system of classes closed under monotonous unions. Let for every $\langle x, y\rangle \in K$, the system $\mathcal{E}_{x,y}$ be closed under uncountable monotonous unions and let for each $y \in \mathrm{dom}(K)$ the class*

$$Q_y = \{x; \mathcal{E}_{x,y} \textit{ is not closed under countable monotonous unions}\}$$

be at most countable.

Let for every z and every G which is at most countable function with

$$y \in \mathrm{dom}(G) \rightarrow G(y) \notin Q_y,$$

the system

$$\left(\bigcup\{\mathcal{E}_{x,z}; \langle x, z\rangle \in K\}\right)^c$$

be dense in the system

$$(\mathcal{I}\cap\bigcap\{\mathcal{E}_{x,y};\langle x,y\rangle\in G\})^c.$$

Then for each C element of $\mathcal{I}$ which is at most countable there is $I\in\mathcal{I}$ with

$$C\subseteq I\ \ \&\ \ (\forall y\in\mathrm{dom}(K))(\exists x)(\langle x,y\rangle\in K\ \ \&\ \ I\in\mathcal{E}_{x,y}).$$

PROOF: Let $\{y_\alpha;\alpha\in\Omega\}$ be an enumeration of the class $\mathrm{dom}(K)$ such that every element of $\mathrm{dom}(K)$ appears in it cofinally (i.e.

$$(\forall y\in\mathrm{dom}(K))(\forall\alpha\in\Omega)(\exists\beta\in\Omega)(\alpha<\beta\ \ \&\ \ y_\beta=y)).$$

We are going to construct by induction a monotonous system $\{I_\alpha;\alpha\in\Omega\}$ of elements of $\mathcal{I}$ which are at most countable classes so that

$$\begin{gathered}I_\alpha\in\bigcup\{\mathcal{E}_{x,y_\alpha};\langle x,y_\alpha\rangle\in K\}\ \ \&\\ \&\ \ (\forall\beta\in\alpha\cap\Omega)(\forall x)([I_\beta\in\mathcal{E}_{x,y_\beta}\ \ \&\ \ x\notin Q_{y_\beta}\ \ \&\\ \&\ \ (\forall\gamma\in\beta\cap\Omega)(\forall z)((I_\gamma\in\mathcal{E}_{z,y_\beta}\ \ \&\ \ y_\gamma=y_\beta)\rightarrow z\in Q_{y_\beta})\ \ \&\\ \&\ \ (\forall z)((I_\beta\in\mathcal{E}_{z,y_\beta}\ \ \&\ \ z\notin Q_{y_\beta}\rightarrow xWz)]\rightarrow I_\alpha\in\mathcal{E}_{x,y_\beta})\end{gathered}$$

where W is a fixed well–ordering of the universal class (the existence of a well–ordering of the universal class is guaranteed by the axiom of choice). (Informally: if $I_\beta\in\mathcal{E}_{x,y_\beta}\ \ \&\ \ x\notin Q_{y_\beta}$ and if x is "minimal" then $(\forall\gamma\geq\beta)I_\gamma\in\mathcal{E}_{x,y_\beta}$).

We put $I_0=C$. If the sequence $\{I_\beta;\beta\in\alpha\cap\Omega\}$ is constructed (where $0<\alpha\in\Omega$) then according to induction hypothesis the class

$$\bigcup\{I_\beta;\beta\in\alpha\cap\Omega\}$$

is at most countable. Put

$$\begin{aligned}G=\{\langle x,y\rangle\in K;(\exists\beta\in\alpha\cap\Omega)(y=y_\beta\ \ \&\ \ I_\beta\in\mathcal{E}_{x,y}\ \ \&\\ \&\ \ x\notin Q_y\ \ \&\ \ (\forall\gamma\in\beta\cap\Omega)(\forall z)((I_\gamma\in\mathcal{E}_{z,y}\ \ \&\ \ y=y_\beta)\rightarrow\\ \rightarrow z\in Q_y)\ \ \&\ \ (\forall z)((I_\beta\in\mathcal{E}_{z,y}\ \ \&\ \ z\notin Q_y)\rightarrow xWz).\end{aligned}$$

G is a function according to the definition and moreover

$$(\alpha\cap\Omega)\preceq FN\rightarrow G\preceq FN.$$

For every $\langle x,y\rangle\in G$ there is $\beta\in\alpha\cap\Omega$ such that

$$(\forall\gamma\in\alpha\cap\Omega)(\beta\leq\gamma\rightarrow I_\gamma\in\mathcal{E}_{x,y})$$

and thus closeness of $\mathcal{I}$ and $\mathcal{E}_{x,y}$ under countable monotonous unions implies

$$\bigcup\{I_\gamma;\gamma\in\alpha\cap\Omega\}\in\mathcal{I}\cap\mathcal{E}_{x,y}$$

(In more detail: since $\alpha \cap \Omega$ is countable we can find a sequence $\{\beta_n; n \in FN\} \subseteq \alpha \cap \Omega$ such that

$$\beta_0 = \beta \ \& \ (\forall n, m)(n \leq m \rightarrow \beta_n \leq \beta_m) \ \& \\ \& \ (\forall \gamma \in \alpha \cap \Omega)(\exists n)(\beta \leq \beta_n).$$

The sequence $\{I_{\beta_n}; n \in FN\}$ is a monotonous sequence of elements of $\mathcal{I} \cap \mathcal{E}_{x,y}$ and thus

$$\bigcup\{I_\gamma; \gamma \in \alpha \cap \Omega\} = \bigcup\{I_{\beta_n}; n \in FN\} \in \mathcal{I} \cap \mathcal{E}_{x,y}).$$

Hence using the assumption of the density stated in the theorem we can find $I_\alpha \in \mathcal{I}$ so that

$$\bigcup\{I_\beta; \beta \in \alpha \cap \Omega\} \subseteq I_\alpha \ \& \ I_\alpha \preceq FN \ \& \\ \& \ (\forall x, y)(\langle x, y\rangle \in G \rightarrow I_\alpha \in \mathcal{E}_{x,y}) \ \& \ I_\alpha \in \bigcup\{\mathcal{E}_{x,y_\alpha}; \langle x, y_\alpha\rangle \in K\}.$$

(In more detail: the prolongation axiom makes possible to code countable classes by sets and thus the axiom of choice enables us the precise choice of the class I_α).

Putting

$$I = \bigcup\{I_\beta; \beta \in \Omega\}$$

we get $C \subseteq I$ and the closeness of $\mathcal{I}$ under monotonous unions implies $I \in \mathcal{I}$ immediately. Let $y \in \mathrm{dom}(K)$ be given. If there is y and $\alpha \in \Omega$ such that

$$I_\alpha \in \mathcal{E}_{x,y} \ \& \ x \notin Q_y \ \& \ y = y_\alpha$$

then

$$(\forall \beta \in \Omega)(\alpha \leq \beta \rightarrow I_\beta \in \mathcal{E}_{\overline{x},y})$$

for a convenient $\overline{x}$ and thus I is an element of $\mathcal{E}_{\overline{x},y}$ because $\mathcal{E}_{\overline{x},y}$ is closed under uncountable unions. If there are no y, α with above required properties then

$$(\forall \alpha \in \Omega)((I_\alpha \in \mathcal{E}_{x,y} \ \& \ y = y_\alpha) \rightarrow x \in Q_y).$$

The class

$$\{\alpha \in \Omega; y = y_\alpha\}$$

is uncountable by our assumption concerning the enumeration in question while the class Q_y is countable. Therefore there is $x \in Q_y$ so that

$$\{\alpha \in \Omega; y = y_\alpha \ \& \ I_\alpha \in \mathcal{E}_{x,y}\}$$

is uncountable (because according to our construction

$$(\forall \alpha \in \Omega) I_\alpha \in \bigcup\{\mathcal{E}_{x,y_\alpha}; \langle x, y_\alpha\rangle \in K\}$$

holds). Hence

$$I = \bigcup\{I_\alpha : \alpha \in \Omega \ \& \ y = y_\alpha \ \& \ I_\alpha \in \mathcal{E}_{x,y}\}$$

and thence $I \in \mathcal{E}_{x,y}$ is a consequence of the closeness of $\mathcal{E}_{x,y}$ under uncountable monotonous unions. ∎

Lemma. *If systems of classes $\mathcal{I}$ and $\mathcal{D}_n (n \in FN)$ are closed under countable monotonous unions and if*

$$\mathcal{D}_n^c \text{ is dense in } \mathcal{I}^c$$

for every $n \in FN$ then

$$(\bigcap\{\mathcal{D}_n; n \in FN\})^c \text{ is dense in } \mathcal{I}^c$$

PROOF: Let $\{i_n; n \in FN\}$ be an enumeration of FN s.t. each finite natural number appears in it cofinally. If $C \in \mathcal{I}^c$ is given then we can construct a monotonous sequence $\{I_n; n \in FN\}$ of elements of $\mathcal{I}^c$ such that $I_n \in \mathcal{D}_{i_n}$ and such that $I_0 = C$. In fact, if I_n is constructed then using density of $\mathcal{D}^c_{i_{n+1}}$ in $\mathcal{I}^c$ we can choose $I_{n+1} \in \mathcal{D}^c_{i_{n+1}} \cap \mathcal{I}$ with $I_n \subseteq I_{n+1}$. For every $n \in FN$ we have

$$\bigcup\{I_m; m \in FN\} \in \mathcal{D}_n^c \cap \mathcal{I}$$

because of closeness of $\mathcal{I}$ and $\mathcal{D}_n^c$ under countable monotonous unions and because the equality

$$\bigcup\{I_m; m \in FN\} = \bigcup\{I_{i_m}; i_m = n\}$$

holds. ■

PROOF: of the first theorem

(a) Apply the second theorem to the systems

$$\widetilde{\mathcal{I}} = \mathcal{I}$$
$$\widetilde{K} = \{0\} \times K$$
$$\mathcal{E}_{0,x} = \mathcal{D}_x.$$

For every $x \in K$ the class $Q_x \subseteq \{0\}$ is at most countable and assumption of density required in the statement in question is a consequence of the last lemma.

(b) Assuming $\mathcal{I}$ is a cut we can apply the just proved statement to

$$\widetilde{K} = K \times \{0\} \cup V \times \{1\}$$
$$\widetilde{\mathcal{D}}_{\langle x,0\rangle} = \mathcal{D}_x$$
$$\widetilde{\mathcal{D}}_{\langle x,1\rangle} = \{I \in \mathcal{I}; x \in I \vee I \cup \{x\} \notin \mathcal{I}\}$$

because under the mentioned assumption for every x, we are able to show that $\widetilde{\mathcal{D}}_{\langle x,1\rangle}$ is closed under uncountable monotonous unions—we have even

$$(I \in \widetilde{\mathcal{D}}_{\langle x,1\rangle} \ \& \ I \subseteq I' \in \mathcal{I}) \rightarrow I' \in \widetilde{\mathcal{D}}_{\langle x,1\rangle}.$$

It is also easy to prove density of $\widetilde{\mathcal{D}}^c_{\langle x,1\rangle}$ in $\mathcal{I}^c$ for each x, since if

$$I \preceq FN \ \& \ I \in \mathcal{I}$$

then either $I \cup \{x\} \notin \mathcal{I}$ and then $I \in \mathcal{D}^c_{\langle x,1\rangle}$ or $I \cup \{x\} \in \mathcal{I}$ and then

$$I \subseteq I \cup \{x\} \in \mathcal{D}^c_{\langle x,1\rangle}.$$

Therefore there is $I \in \mathcal{I}$ with

$$(\forall x \in K) I \in \mathcal{D}_x \ \ \& \ \ (\forall x \in V)(x \in I \vee I \cup \{x\} \notin \mathcal{I}).$$

To finish the proof it is sufficient it realize that the second part of the last formula implies the maximality of I in $\mathcal{I}$. In fact, if there were $I' \in \mathcal{I}$ with $I \subset I'$ then there would exist $x \in I' - I$ and for this x we would get

$$x \in I \vee I \cup \{x\} \notin \mathcal{I}.$$

The first possibility is excluded by our choice of x and the second one contradicts the assumption $\mathcal{I}$ is a cut because

$$I \cup \{x\} \subseteq I' \in \mathcal{I}. \qquad \blacksquare$$

The above formulated results can be applied even to systems which are not themselves dense in the system in question—one can conveniently extend these systems onto dense systems at first. The proof of the following theorem exemplifies this technique—in the theorem we assume that for a given f the system $\mathcal{D}^c_{\langle x,f\rangle}$ is dense in the system of all elements of $\mathcal{I}^c$ which contain the class $f''FN$ (which is at most countable) and we do not require the density in the whole system $\mathcal{I}^c$.

Theorem 3. *Let $\mathcal{I}$ be a cut which is closed under monotonous unions and let for every $\langle x,f\rangle \in K$, the system $\mathcal{D}_{\langle x,f\rangle}$ be closed under uncountable monotonous unions. Let*

$$\mathcal{D}^c_{\langle x,f\rangle} \textit{ be dense in } (\{I \in \mathcal{I}; f''FN \subseteq I\})^c.$$

Then for every element C of $\mathcal{I}$ which is at most countable there is a maximal element I of $\mathcal{I}$ with

$$C \leq I \ \ \& \ \ (\forall x,f)((\langle x,f\rangle \in K \ \ \& \ \ f''FN \subseteq I) \rightarrow I \in \mathcal{D}_{\langle x,f\rangle}).$$

PROOF: Put

$$\begin{aligned} \widetilde{\mathcal{I}} &= \mathcal{I} \\ \widetilde{K} &= K \cap \{\langle x,y\rangle; \mathrm{Fnc}(y)\} \\ \widetilde{\mathcal{D}}_{\langle x,f\rangle} &= \{I; I \in \mathcal{D}_{\langle x,f\rangle} \vee I \cup f''FN \notin \mathcal{I}\} \end{aligned}$$

If $\{I_\alpha; \alpha \in \Omega\} \subseteq \widetilde{\mathcal{D}}_{\langle x,f\rangle}$ is a monotonous sequence then either

$$(\exists \beta \in \Omega)(I_\beta \cup f''FN \notin \mathcal{I})$$

and then

$$\bigcup\{I_\alpha;\alpha\in\Omega\}\cup f''FN\notin\mathcal{I}$$

because $\mathcal{I}$ is supposed to be a cut or

$$(\forall\alpha\in\Omega)I_\alpha\in\mathcal{D}_{\langle x,f\rangle}$$

and then

$$\bigcup\{I_\alpha;\alpha\in\Omega\}\in\mathcal{D}_{\langle x,f\rangle}$$

because of the closeness of $\mathcal{D}_{\langle x,f\rangle}$ under uncountable unions. We have proved that the system $\widetilde{\mathcal{D}}_{\langle x,f\rangle}$ is closed under uncountable monotonous unions for every $\langle x,f\rangle\in\widetilde{K}$.

To prove the density of $\widetilde{\mathcal{D}}^c_{\langle x,f\rangle}$ in $\mathcal{I}^c$ let $C'\in\mathcal{I}$ with $C'\preceq FN$ be given. If $C'\cup f''FN\notin\mathcal{I}$ then $C'\in\widetilde{\mathcal{D}}_{\langle x,f\rangle}$ and we are done. Assuming $C'\cup f''FN\in\mathcal{I}$ we have $C'\cup f''FN\in(\{I\in\mathcal{I};f''FN\subseteq I\})^c$ and thus there is $I\in\mathcal{D}^c_{\langle x,f\rangle}$ (and consequently $I\in\widetilde{\mathcal{D}}^c_{\langle x,f\rangle}$) with $C'\cup f''FN\subseteq I\in\mathcal{I}$.

According to the first theorem there is a maximal element I of $\mathcal{I}$ with

$$\langle x,f\rangle\in\widetilde{K}\rightarrow\dot{I}\in\widetilde{\mathcal{D}}_{\langle x,f\rangle}.$$

It remains to prove the implication

$$f''FN\subseteq I\in\mathcal{I}\rightarrow I\in\mathcal{D}_{\langle x,f\rangle}$$

for every $\langle x,f\rangle\in K$, however to do it is sufficient to realize that the possibility

$$I\cup f''FN=I\notin\mathcal{I}$$

is excluded. ■

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