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Smoothing effect and regularity for evolution integrodifferential systems

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Abstract. A system of two partial quasilinear integrodifferential equations (hyperbolic and parabolic) is considered. Smoothing effect and regularity of higher order in t resp. t, x is studied.

Keywords: Rothe's method, evolution systems, smoothing effect, regularity

Classification: 65M20, 58D25

1. Introduction.

Character of linear parabolic equation guarantees that its solution for $t > 0$ belongs to a more regular space than the initial function. This fact is well-known as smoothing effect and it has been studied in [1]–[4], [10],.... In general it is not true for hyperbolic equations. In spite of this the smoothing effect for the parabolic part of the system (2.3) can be proved (see Theorem 3.3)

Regularity of weak solutions for linear parabolic and hyperbolic equations has been considered in [1], [3], [4], [6], [7], [9],..... The aim of the section 4 resp. 5 is to obtain higher order regularity of the weak solution of a linear evolution integrodifferential system in t resp. t, x. To this purpose we discretize the time variable and apply the technique of Rothe's method which allows to carry the regularity results from elliptic to parabolic (hyperbolic) equations.

The existence and uniqueness of the weak solution of our problem is considered in [11]. To prove it, we use the technique developed in [3]. Before reading this paper we recommend the reader to see [11].

2. Notations and preliminaries.

Let H, Y be real abstract Hilbert spaces with norms $|\cdot|, \|\cdot\|$, where $H \cap Y$ is dense in H and Y; H^*, Y^* be dual spaces to H, Y with norms $| \quad |_1, \| \quad \|_1$. Denote by S_t the interval $\langle -q, t\rangle$ for $t \in J, J = \langle 0, T\rangle$ where $T < \infty, q \in (0, \infty)$. The function spaces we use are rather familiar and we omit their definitions (see [5]). By $\hookrightarrow$ is denoted the continuous imbedding. Let $\langle z, w\rangle_H, \langle u, v\rangle_Y$ be the continuous pairings for $z \in H^*, w \in H, u \in Y^*, v \in Y$.

In the following we keep the notation from [11].

If X, Y are Banach spaces, $\alpha \in (0, 1\rangle$ then:

– By $\mathrm{Lip}_\alpha(X, Y)$ is denoted the set of all functions $g : X \to Y$ satisfying

$$\|g(u) - g(v)\|_Y \le C\|u - v\|_X^\alpha \qquad \forall u, v \in X.$$

For $\alpha = 1$ the notation $\mathrm{Lip}(X, Y) \equiv \mathrm{Lip}_1(X, Y)$ is used.

– By $\mathrm{Lip}(J \times X, Y)$ is denoted the set of all functions $g : J \times X \to Y$ satisfying

$$\|g(t,u) - g(t',v)\|_Y \le C\left(|t - t'|\,[1 + \|u\|_X + \|v\|_X] + \|u - v\|_X\right)$$
$$\forall t, t' \in J;\ \forall u, v \in X.$$

Definition 2.1. The operator $E : L_\infty(S_T, X) \to L_\infty(J, X)$ (X is a Banach space) is said to be a Volterra operator in X iff

$$[u(s) = v(s) \text{ for a.e. } s \in S_t, \quad t \in J] \Rightarrow$$
$$[E(u)(s) = E(v)(s) \text{ for a.e. } s \in \langle 0, t\rangle].$$

Let $E : \mathrm{Lip}(S_T, H) \to \mathrm{Lip}(S_T, H)$ resp. $F : \mathrm{Lip}(S_T, Y) \to \mathrm{Lip}(S_T, Y)$ be a Volterra operator in H resp. Y and $G : L_\infty(J, Y) \to L_\infty(J, Y), I : L_\infty(J, H) \to L_\infty(J, H)$ be in the form

$$R(z)(t) = \int_0^t K(t,s) z(s)\, ds, \qquad R = G, I; \quad K \in L_\infty(J \times J). \tag{2.2}$$

Let us fix $e \in \mathrm{Lip}(J \times Y \times H^3 \times Y^2, H^*), \mu : J \to Y^*, \nu : J \to H^*, \mu_1 : J \to Y^*$, and the continuous bilinear forms $p(t; z, w), a_1(t; u, v), a_2(t; u, v), b(t; u, v), d(t; z, v)$, $g(t; u, v), \rho(t; u, w), \vartheta(t; z, w)$ for $z, w \in H$ and $u, v \in Y$. The notation $r^{(k)}(t; x, y)$ is used for $\partial_t^k r(t; x, y)$.

We consider the following problem:

PC-1. To find u, v such that

(i) $u = \alpha, v = \beta, \partial_t v = \gamma$ in $S_0 = \langle -q, 0\rangle$ where $\alpha \in \mathrm{Lip}(S_0, H), \beta \in \mathrm{Lip}(S_0, Y \cap H)$ and $\gamma \in \mathrm{Lip}(S_0, H)$.
(ii) the identity (2.3) is satisfied

$$\begin{gathered} p(t; \partial_t u(t), \varphi) + a_1(t; u(t), \varphi) = d(t; (u + v + I(u+v))(t), \varphi) + \\ + g(t; G(u+v)(t), \varphi) + \rho(t; (u + G(u+v))(t), \varphi) + \\ + \vartheta(t; (u + v + I(u+v))(t), \varphi) + \langle \mu(t), \varphi\rangle_Y + \langle \nu(t), \varphi\rangle_H, \\ p(t; \partial_t^2 v(t), \phi) + b(t; \partial_t v(t), \phi) + a_2(t; v(t), \phi) = \\ = d(t; (v + I(u+v))(t), \phi) + g(t; G(v)(t), \phi) + \\ + \langle e(t, u(t), E(u)(t), E(v)(t), E(\partial_t v)(t), F(v)(t), G(u)(t)), \phi\rangle_H \\ \forall \varphi, \phi \in Y \cap H, \qquad \text{for a.e. } t \in J \end{gathered} \tag{2.3}$$

Remark 2.4. In general the symbols p, d, g, I, G, E may be different at any two places of their occurrence in (2.3).

The following conditions are sufficient for our approach ($\forall t \in J; \omega, \lambda \geq 0$ will be determined; $\forall z, w \in H; \forall u, v \in Y; \forall y \in Y \cap H$):

$$(2.5) \qquad p(t; z, w) = p(t; w, z)$$

$$(2.6) \qquad p(t; z, z) \geq C_1 |z|^2$$

$$(2.7) \qquad |p^{(j)}(t; z, w)| \leq C|z|\,|w| \qquad j = 0, \dots, \lambda$$

$$(2.8) \qquad a_1(t; y, y) \geq C_1 \|y\|^2 - C|y|^2$$

$$(2.9) \qquad |a_1^{(j)}(t; u, v)| \leq C\|u\|\,\|v\| \qquad j = 0, \dots, \omega$$

$$(2.10) \qquad a_2(t; u, v) = a_2(t; v, u)$$

$$(2.11) \qquad a_2(t; y, y) \geq C_1 \|y\|^2 - C|y|^2$$

$$(2.12) \qquad |a_2^{(j)}(t; u, v)| \leq C\|u\|\,\|v\| \qquad j = 0, \dots, \omega$$

$$(2.13) \qquad b^{(1)}(t; u, v) = b^{(1)}(t; v, u)$$

$$(2.14) \qquad |b^{(j)}(t; u, v)| \leq C\|u\|\,\|v\| \qquad j = 0, \dots, \omega$$

$$(2.15) \qquad b(t; y, y) \geq -C|y|^2$$

$$(2.16) \qquad \exists \sigma \in (0,1) : \sigma a_2(t; y, y) + b^{(1)}(t; y, y) \geq -C|y|^2$$

$$(2.17) \qquad |g^{(j)}(t; u, v)| \leq C\|u\|\,\|v\| \qquad j = 0, \dots, \omega$$

$$(2.18) \qquad |d^{(j)}(t; z, u)| \leq C|z|\,\|u\| \qquad j = 0, \dots, \omega$$

$$(2.19) \qquad |\rho^{(j)}(t; u, z)| \leq C\|u\|\,|z| \qquad j = 0, \dots, \omega$$

$$(2.20) \qquad |\vartheta^{(j)}(t; w, z)| \leq C|w|\,|z| \qquad j = 0, \dots, \omega$$

$$(2.21) \qquad |E(x)(t) - E(x)(t')| \leq |t - t'| \theta(\|x\|_{C(S_t, H)})(1 + \|\partial_t x\|_{L_\infty(S_t, H)})$$
$$\forall t, t' \in J; t' < t; \theta \in C(R_+, R_+); \forall x \in \mathrm{Lip}(S_T, H)$$

$$(2.22) \qquad \|F(x)(t) - F(x)(t')\| \leq |t - t'| \theta(\|x\|_{C(S_t, Y)})(1 + \|\partial_t x\|_{L_\infty(S_t, Y)})$$
$$\forall t, t' \in J; t' < t; \theta \in C(R_+, R_+); \forall x \in \mathrm{Lip}(S_T, Y)$$

$$(2.23) \qquad D^\omega K \in L_\infty(J \times J)$$

$$(2.24) \qquad \mu \in H^\omega(J, Y^*)$$

$$(2.25) \qquad \mu_1 \in H^{\omega+1}(J, Y^*)$$

$$(2.26) \qquad \nu \in H^\omega(J, H^*)$$

(compatibility condition)
for $U_0 = \alpha(0), V_0 \beta(0), V_1 = \gamma(0) \in Y \cap H$ exist $U_1, V_2 \in H$ such that

$$(2.27) \qquad p(0; U_1, \varphi) + a_1(0; U_0, \varphi) = d(0, U_0 + V_0, \varphi) +$$
$$+ \rho(0; U_0, \varphi) + \vartheta(0; U_0 + V_0, \varphi) + \langle \mu(0), \varphi \rangle_Y + \langle \nu(0), \varphi \rangle_H,$$
$$p(0; V_2, \phi) + b(0; V_1, \phi) + a_2(0; V_0, \phi) = d(0; V_0, \phi) +$$
$$+ \langle e(0, U_0, E(\alpha)(0), E(\beta)(0), E(\gamma)(0), F(\beta)(0), 0), \phi \rangle_H$$
$$\forall \varphi, \phi \in Y \cap H.$$

Remark 2.28. The function θ may be different in both inequalities (2.21) and (2.22). Nonnegative constants C may stand for various constants in the same discussion (C does not depend on n).

For a given positive integer n the following notation is introduced ($i = 1, \dots, n$; $\tau = T/n; t_i = i\tau$):

$$w_i = w(t_i), \qquad \delta w_i = (w_i - w_{i-1})/\tau$$

(where w is an arbitrary function),

$$u_n(t) = \begin{cases} \alpha(t) & t \in S_0 \\ u_{i-1} + (t - t_{i-1})\delta u_i & t_{i-1} < t \le t_i; \quad i = 1, \dots, n \end{cases} \tag{2.29}$$

$$v_n(t) = \begin{cases} \beta(t) & t \in S_0 \\ v_{i-1} + (t - t_{i-1})\delta v_i & t_{i-1} < t \le t_i; \quad i = 1, \dots, n \end{cases} \tag{2.30}$$

$$\overline{u}_n(t) = \begin{cases} \alpha(t) & t \in S_0 \\ u_i & t_{i-1} < t \le t_i; \quad i = 1, \dots, n \end{cases} \tag{2.31}$$

$$\overline{v}_n(t) = \begin{cases} \beta(t) & t \in S_0 \\ v_i & t_{i-1} < t \le t_i; \quad i = 1, \dots, n \end{cases} \tag{2.32}$$

$$V_n^{(1)}(t) = \begin{cases} \gamma(t) & t \in S_0 \\ \delta v_{i-1} + (t - t_{i-1})\delta^2 v_i & t_{i-1} < t \le t_i; \quad i = 1, \dots, n \end{cases} \tag{2.33}$$

$$\overline{V}_n^{(1)}(t) = \begin{cases} \gamma(t) & t \in S_0 \\ \delta v_i & t_{i-1} < t \le t_i; \quad i = 1, \dots, n \end{cases} \tag{2.34}$$

$$\widetilde{u}_{i-1} = \widetilde{u}_{i-1,n}(t) = \begin{cases} \alpha(t) & t \in S_0 \\ U_0 = \alpha(0) & t \in \langle 0, \tau \rangle \\ u_{j-1} + (t - t_j)\delta u_j & t \in \langle t_j, t_{j+1} \rangle; \quad j = 1, \dots, i-1 \\ u_{i-1} & t \in \langle t_i, T \rangle. \end{cases} \tag{2.35}$$

The functions $\widetilde{v}_{i-1}$ resp $\delta\widetilde{v}_{i-1}$ are defined analogously as $\widetilde{u}_{i-1}$ but instead of α will be β resp. γ.

3. Smoothing effect.

The aim of this section is to prove the smoothing effect for the parabolic part of our system in PC–1. We choose the following approximation scheme

$$\begin{gathered} p(t_i; \delta u_i, \varphi) + a_1(t_i; u_i \varphi) = d(t_i; u_{i-1} + v_{i-1} + I_i(u + v), \varphi) + \\ + g(t_i; G_i(u + v), \varphi) + \rho(t_i; u_{i-1} + G_i(u + v), \varphi) + \\ + \vartheta(t_i; u_{i-1} + v_{i-1} + I_i(u + v, \varphi) + \langle \mu_i, \varphi \rangle_Y + \langle \nu_i, \varphi \rangle_H, \\ p(t_i; \delta^2 v_i, \phi) + b(t_i; \delta v_i, \phi) + a_2(t_i; v_i, \phi) = \\ = d(t_i; v_{i-1} + I_i(u + v), \phi) + g(t_i; G_i v, \phi) + \langle e_i, \phi \rangle_H \\ \forall \varphi, \phi \in Y \cap H, \end{gathered} \tag{3.1}$$

where

$$R_i z = R(\widetilde{z}_{i-1})(t_i) \qquad \text{for } R = E, F, G, I$$

$$e_i = e(t_i, u_{i-1}, E_i u, E_i v, E_i \delta v, F_i v, G_i u)$$

and $u_0 = U_0, v_0 = V_0, \delta v_0 = V_1$.

Under the conditions of Theorem 3.3 the next a priori estimates can be obtained (see [**11**, Lemma 3.20]):

$$|\delta u_j| + \|\delta v_j\| + \|u_j\| + \sum_{i=1}^{j} \|\delta u_i\|^2 \tau + |\delta^2 v_j| \le C \tag{3.2}$$
$$\forall j = 1, \dots, n \text{ and } \tau \le \tau_0.$$

Theorem 3.3. *Let* $e \in \mathrm{Lip}(J \times Y \times H^3 \times Y^2, H^*)$, $E \in \mathrm{Lip}(C(S_T, H), C(J, H))$, $F \in \mathrm{Lip}(C(S_T, Y), C(J, Y))$, *(2.2) and* $\alpha(0), \beta(0), \gamma(0) \in Y \cap H$. *Moreover (2.5)–(2.27) (excluding (2.25)) are fulfilled for* $\omega = \lambda = 2$ *and*

$$a_1(t; z, w) = a_1(t; w, z)$$

hold true $\forall z, w \in Y$. *Then the solution* u, v *of PC–1 satisfies* $(\forall \varepsilon > 0, t \in (0, T))$:

$$t^{1/2}\partial_t u \in L_\infty(J, Y), \qquad t^{1+\varepsilon}\partial_t^2 u \in L_2(J, Y),$$
$$t\partial_t^2 u \in L_\infty(J, H), \qquad t^{1/2+\varepsilon}\partial_t^2 u \in L_2(J, H).$$

PROOF : Subtracting $(3.1)_1$ from $(3.1)_1$ for $i, i-1$; dividing by τ and applying the identity

$$[r(t_i; z_i, \varphi) - r(t_{i-1}; z_{i-1}, \varphi)]/\tau = r(t_i; \delta z_i, \varphi) + \delta r(t_i; z_{i-1}, \varphi)$$

for bilinear forms in $(3.1)_1$ it yields

$$\begin{gathered} p(t_i; \delta^2 u_i, \varphi) + a_1(t_i, \delta u_i, \varphi) = -\delta p(t_i; \delta u_{i-1}, \varphi) - \\ -\delta a_1(t_i; u_{i-1}, \varphi) + d(t_i; \delta(u_{i-1} + v_{i-1} + I_i(u+v)), \varphi) + \\ +\delta d(t_i; u_{i-2} + v_{i-2} + I_{i-1}(u+v), \varphi) + g(t_i; \delta G_i(u+v), \varphi) + \\ +\delta g(t_i; G_{i-1}(u+v), \varphi) + \rho(t_i; \delta(u_{i-1} + G_i(u+v)), \varphi) + \\ +\delta\rho(t_i; u_{i-2} + G_{i-1}(u+v), \varphi) + \vartheta(t_i; \delta(u_{i-1} + v_{i-1} + I_i(u+v)), \varphi) + \\ +\delta\vartheta(t_i; u_{i-2} + v_{i-2} + I_{i-1}(u+v), \varphi) + \langle \delta\mu_i, \varphi\rangle_Y + \langle \delta\nu_i, \varphi\rangle_H \\ \forall \varphi \in Y \cap H, \quad 2 \le i \le n. \end{gathered} \tag{3.4}$$

Setting $\varphi = \delta u_i - \delta u_{i-1}$ in (3.4); summing up for $i = r+1, \dots, s$; using a priori estimates (3.2), the following can be obtained (the same way as in [**11**, Lemma 3.20]

$$\sum_{i=r+1}^{s} |\delta^2 u_i|^2 \tau + \|\delta u_s\|^2 \le C(1 + \|\delta u_r\|^2) \tag{3.5}$$

from which

$$\|\delta u_s\|^2 \le C(1 + \|\delta u_r\|^2) \qquad \text{for } s > r.$$

Multiplying by τ, summing up for $r = i_0, \dots, 2i_0$ it follows

$$\|\delta u_{2i_0}\|^2 \le C/(i_0\tau).$$

Thus from (3.5) we conclude

$$\sum_{i=r+1}^{s} |\delta^2 u_i|^2\tau + \|\delta u_s\|^2 \le C/(i_0\tau), \qquad \forall s \ge 2i_0. \tag{3.6}$$

Subtracting (3.4) form (3.4) for $i, i-1$; setting $\varphi = \delta^2 u_i$ and summing up for $i = r+1, \dots, s$; successively we estimate (analogously as above)

$$|\delta^2 u_s|^2 + \sum_{j=r+1}^{s} \|\delta^2 u_j\|^2\tau \le C\left(1 + |\delta^2 u_r|^2 + \sum_{j=r+1}^{s} |\delta^2 u_j|^2\tau\right). \tag{3.7}$$

Omitting the second term on the left–hand side and using Gronwall's lemma it is easy to see that

$$|\delta^2 u_s|^2 + \sum_{j=r+1}^{s} \|\delta^2 u_j\|^2\tau \le C\left(1 + |\delta^2 u_r|^2\right), \qquad \forall s > r.$$

In the same way as in (3.5) by virtue of (3.6) we get

$$|\delta^2 u_s|^2 + \sum_{j=r+1}^{s} \|\delta^2 u_j\|^2\tau \le C(i_0\tau)^{-2}, \qquad \forall s \ge 3i_0 \tag{3.8}$$

For given ε $(0 < \varepsilon < T)$ we can choose $i_0 = i_0(n)$ such that

$$\varepsilon/2 < 3i_0\tau < \varepsilon \quad \forall n \ge n_0.$$

Using $\delta^j u_i$ $(i \ge i_0; j = 0, 1, 2)$, the functions $U_n^{(j)}(t), \overline{U}_n^{(j)}(t)$ can be constructed (analogously as in (2.29) and (2.31) where $u_n(t) \equiv U_n^{(0)}(t), \overline{u}_n(t) \equiv \overline{U}_n^{(0)}(t)$). These functions are defined in (ε, T). The estimates (3.6) and (3.8) can be rewritten in this way

$$\|\overline{U}_n^{(1)}(t)\|^2 \le C\varepsilon^{-1} \qquad |\overline{U}_n^{(2)}(t)|^2 \le C\varepsilon^{-2} \qquad \forall t \in (\varepsilon, T)$$
$$\int_\varepsilon^T |\partial_t U_n^{(1)}(t)|^2\, dt \le C\varepsilon^{-1} \qquad \int_\varepsilon^T \|\overline{U}_n^{(2)}(t)\|^2\, dt \le C\varepsilon^{-2}$$

from which we deduce

$$\|\partial_t u\|_{L_\infty(\langle\varepsilon,T\rangle,Y)} \le C\varepsilon^{-1/2}, \qquad \|\partial_t^2 u\|_{L_\infty(\langle\varepsilon,T\rangle,H)} \le C\varepsilon^{-1} \tag{3.9}$$

$$\int_\varepsilon^T \|\partial_t^2 u\|^2\, dt \le C\varepsilon^{-2}, \qquad \int_\varepsilon^T |\partial_t^2 u|^2\, dt \le C\varepsilon^{-1}. \tag{3.10}$$

If any function g satisfies

$$\int_\varepsilon^T g(s)\,ds \le C\varepsilon^{-p},$$

then for $q > p$ and $\varepsilon \to 0$ it is easy to see that

$$\int_0^T s^q g(s)\,ds \le C.$$

From this consideration applying (3.10) we conclude

$$\int_0^T t^{2+\varepsilon}\|\partial_t^2 u\|^2\,dt \le C, \qquad \int_0^T t^{1+\varepsilon}|\partial_t^2 u|^2\,dt \le C.$$

The rest of the proof is a consequence of (3.9). ■

4. Regularity in t.

To obtain some higher order regularity results we consider this slightly modified problem of PC–1 (the right–hand sides of our system being linear):

$$\begin{gathered}
L_P(t;u,\varphi) \equiv p(t;\partial_t u,\varphi) + a_1(t;u,\varphi) - [d(t;u+v+I(u+v)+\partial_t v,\varphi)+\\
+g(t;v+G(u+v),\varphi) + \rho(t;u+v+G(u+v),\varphi)+\\
+\vartheta(t;u+v+I(u+v)+\partial_t v,\varphi) + \langle\mu,\varphi\rangle_Y + \langle\nu,\varphi\rangle_H] = 0,\\
L_H(t;v,\varphi) \equiv p(t;\partial_t^2 v,\phi) + b(t;\partial_t v,\phi) + a_2(t;v,\phi)-\\
-[d(t;v+I(u+v),\phi) + g(t;Gv,\phi) + \vartheta(t;u+v+I(u+v)+\delta_t v,\phi)+\\
+\rho(t;u+v+G(u+v),\phi) + \langle\mu_1,\phi\rangle_Y + \langle\nu,\phi\rangle_H] = 0\\
\forall\varphi,\phi \in Y\cap H
\end{gathered}$$

and the initial functions are taken from PC–1 (variable t is omitted).

The compatibility conditions are in the form ($k \ge 0$, integer)

$$\begin{array}{lll}
\exists U_p \in Y\cap H & 0 \le p \le k & U_{1+k} \in H\\
\exists V_p \in Y\cap H & 0 \le p \le k+1 & V_{2+k} \in H
\end{array}$$

$$\begin{array}{lll}
\partial_t^p L_p(t;u(t),\varphi)|_{t=0} = 0 & p = 0,\dots,k & \forall\varphi \in Y\cap H\\
\partial_t^p L_H(t;v(t),\phi)|_{t=0} = 0 & p = 0,\dots,k & \forall\phi \in Y\cap H
\end{array} \tag{4.2}$$

where

$$\left.\frac{d^\alpha}{dt^\alpha}u(t)\right|_{t=0} := U_\alpha, \qquad \left.\frac{d^\alpha}{dt^\alpha}v(t)\right|_{t=0} := V_\alpha$$

for admissible α.

We must use another approximation of operators G, I to obtain suitable a priori estimates for differences of higher order. This scheme is taken from [**3**]. (F means G resp. I)

$$F_i z = F_\tau(\overline{z}_n)(t) + P_i(Z_0, \dots, Z_k, \tau)$$

for $i = 1, \dots, n-k; n \geq n_0 > k; 0 \leq t \leq t_{n-k}$;

$$F_\tau(z)(t) = \int_0^t K(t + k\tau, s + k\tau) z(s)\, ds$$

and P_i are polynomials in τ of degree k with coefficients depending on $Z_0, \dots, Z_k$ and on the integral kernel K.

Let

$$M_\alpha(z) = \frac{d^\alpha}{dt^\alpha}\left(\int_0^t K(t,s)z(s)\,ds\right)\Bigg|_{t=0}$$

$$K_j^{(i)} = \tau^{-1}\int_{t_{j-1}}^{t_j} K(t_i + k\tau, s + k\tau)\,ds$$

$$\delta_{pd}K_j^{(i)} = (K_j^{(i)} - K_j^{(i-1)})\tau^{-1} \quad \text{(partial difference).}$$

Then

$$F_i z = \sum_{j=1}^{i} K_j^{(i)} z_j \tau + P_i \qquad 1 \leq i \leq n-k \tag{4.3}$$

$$\delta^p F_1 z = \left(\delta^{p-1} F_1 z - M_{p-1}(z)\right)\tau^{-1} \tag{4.4}$$

$$\delta^p F_i z = \sum_{j=1}^{i-p} \delta^p K_j^{(i)} z_j \tau + \sum_{j=0}^{p-1} \delta^j \left(\delta_{pd}^{p-j-1} K_{i+j-p+1}^{(i)} z_{i+j-p+1}\right) + \delta^p P_{i,p} \tag{4.5}$$

$p = 1, \dots, k+1; i = 2, \dots, n-k$ and the elements with the nonpositive indices vanish. Here $P_{i,p} \equiv P_i$ for $i \geq p$ and in $P_{i,p}$ for $i < p$ those terms of P_i are missing which have been used in the second sum of (4.5).

The estimate

$$\| |\delta^p F_i z\| | \leq C\left(1 + \sum_{j=1}^{i-p} \| |z_j\| |\tau + \sum_{j=1}^{p-1}\sum_{\ell=0}^{j} \| |\delta^\ell z_{i-p+1-\ell}\| |\right) \tag{4.6}$$

holds true for $\forall p = 0, \dots, k+1; i = 1, \dots, n-k$ where $\| |\cdot\| | \equiv \|\cdot\|$ for $F = G$ and $\| |\cdot\| | \equiv |\cdot|$ for $F = I$.

We consider the approximation scheme of (4.1) in this form

$$
\begin{gathered}
(4.7)\quad p(t_i;\delta u_i,\varphi)+a_1(t_i;u_i,\varphi)=d(t_i;u_{i-1}+v_{i-1}+I_i(u+v)+\delta v_{i-1},\varphi)+\\
+g(t_i;v_{i-1}+G_i(u+v),\varphi)+\vartheta(t_i;u_{i-1}+v_{i-1}+I_i(u+v)+\delta v_{i-1},\varphi)+\\
+\rho(t_i;u_{i-1}+v_{i-1}+G_i(u+v),\varphi)+\langle\mu_i,\varphi\rangle_Y+\langle\nu_i,\varphi\rangle_H,\\
p(t_i;\delta^2 v_i,\phi)+b(t_i;\delta v_i,\phi)+a_2(t_i;v_i,\phi)=\\
=d(t_i;v_{i-1}+I_i(u+v),\phi)+g(t_i;G_iv,\phi)+\\
+\vartheta(t_i;u_{i-1}+v_{i-1}+I_i(u+v)+\delta v_{i-1},\phi)+\langle\mu_{1,i},\phi\rangle_Y+\langle\nu_i,\phi\rangle_H+\\
+\rho(t_i;u_{i-1}+v_{i-1}+G_i(u+v),\phi)\\
\forall\varphi,\phi\in Y\cap H;i=1,\dots,n-k.
\end{gathered}
$$

Using the compatibility conditions (4.2) and the identity

$$
\delta_j^k r(t_j;z_j,\varphi)=\sum_{\alpha=0}^{k}\binom{k}{\alpha}\delta^{k-\alpha}r(t_j;\delta^\alpha z_{j+\alpha-k},\varphi)
$$

the difference of higher order can be made in (4.7) ($\pi:=i+\alpha-p,\kappa:=\pi-1$)

$$
\begin{gathered}
(4.8_1)\qquad p(t_i;\delta^{p+1}u_i,\varphi)+a_1(t_i;\delta^p u_i,\varphi)=\\
=d(t_i;\delta^p(u_{i-1}+v_{i-1}+I_i(u+v)+\delta v_{i-1}),\varphi)+g(t_i;\delta^p(v_{i-1}+G_i(u+v)),\varphi)+\\
+\vartheta(t_i;\delta^p(u_{i-1}+v_{i-1}+I_i(u+v)+\delta v_{i-1}),\varphi)+\langle\delta^p\mu_i,\varphi\rangle_Y+\langle\delta^p\nu_i,\varphi\rangle_H+\\
+\rho(t_i;\delta^p(u_{i-1}+v_{i-1}+G_i(u+v)),\varphi)+\\
+\sum_{\alpha=0}^{p-1}\binom{p}{\alpha}\delta^{p-\alpha}\left[-p(t_i;\delta^{\alpha+1}u_\pi,\varphi)-a_1(t_i;\delta^\alpha u_\pi,\varphi)+\right.\\
+d(t_i;\delta^\alpha(u_\kappa+v_\kappa+I_\pi(u+v)+\delta v_\kappa),\varphi)+g(t_i;\delta^\alpha(v_\kappa+G_\pi(u+v)),\varphi)+\\
\left.+\vartheta(t_i;\delta^\alpha(u_\kappa+v_\kappa+I_\pi(u+v)+\delta v_\kappa),\varphi)+\rho(t_i;\delta^\alpha(u_\kappa+v_\kappa+G_\pi(u+v)),\varphi)\right]+\\
+\sum_{\alpha=0}^{p}KP_\alpha(U_\alpha,V_\alpha,\varphi),
\end{gathered}
$$

where $KP_\alpha(z,w,\varphi)$ is a linear combination of $\delta^i r(\xi;y,\varphi),r^{(i)}(0;y,\varphi)$ for $r=p,a_1,d,g,\vartheta,\rho;\xi=0,t_1;y=z,w;i=0,\dots,p$.

$$
\begin{gathered}
(4.8_2)\qquad p(t_i;\delta^{p+2}v_i,\phi)+b(t_i;\delta^{p+1}v_i,\phi)+a_2(t_i;\delta^p v_i,\phi)=\\
=d(t_i;\delta^p(v_{i-1}+I_i(u+v)),\phi)+g(t_i;\delta^p G_iv,\phi)+\\
+\vartheta(t_i;\delta^p(u_{i-1}+v_{i-1}+I_i(u+v)+\delta v_{i-1}),\phi)+\langle\delta^p\mu_{1,i},\phi\rangle_Y+\\
+\langle\delta^p\nu_i,\phi\rangle_H+\rho(t_i;\delta^p(u_{i-1}+v_{i-1}+G_i(u+v)),\phi)+\\
+\sum_{\alpha=0}^{p-1}\binom{p}{\alpha}\delta^{p-\alpha}\left[-p(t_i;\delta^{\alpha+2}v_\pi,\phi)-b(t_i;\delta^{\alpha+1}v_\pi,\phi)-\right.
\end{gathered}
$$

$$-a_2(t_i;\delta^\alpha v_\pi,\phi)+d(t_i;\delta^\alpha(v_\kappa+I_\pi(u+v)),\phi)+g(t_i;\delta^\alpha G_\pi v,\phi)+$$
$$+\vartheta(t_i;\delta^\alpha(u_\kappa+v_\kappa+I_\pi(u+v)+\delta v_\kappa),\phi)+\rho(t_i;\delta^\alpha(u_\kappa+v_\kappa+G_\pi(u+v)),\phi)]+$$
$$+\sum_{\alpha=0}^{p}\sum_{\beta=0}^{p+1}KH_{\alpha\beta}(U_\alpha,V_\beta,\phi),$$

where $KH_{\alpha\beta}(z,w,\phi)$ is the same type as KP_α but for $r=p,a_2,b,d,g,\vartheta,\rho$ (the terms with nonpositive indices vanish).

Theorem 4.9. *Let $k\geq 0$ be an integer and the conditions (2.2), (2.5)-(2.9) for $\omega=\lambda=k+1$, (2.10)-(2.13) and (2.15)-(2.23) and (4.10) for $\omega=k+2$, (2.24)-(2.26) for $\omega=k+1$, (4.2) are satisfied.*

$$(4.10)\qquad |b^{(j)}(t;z,w)|\leq c|z|\,|w|\qquad \forall z,w\in H\quad j=1,\ldots,\omega$$

Then the solution u,v of (4.1) satisfies

$$\partial_t^\alpha u\in L_\infty(\langle 0,T'\rangle,Y)\cap L_\infty(\langle 0,T'\rangle,H),$$
$$\partial_t^\alpha v\in L_\infty(\langle 0,T'\rangle,Y)\cap C(\langle 0,T'\rangle,H),\quad \partial_t^{k+2}v\in L_\infty(\langle 0,T'\rangle,H)$$

for $\alpha=0,\ldots,k+1;\forall T'<T$ and the estimates

$$(4.11)\qquad \|\delta^p u_i\|+|\delta^{p+1}u_i|+\|\delta^{p+1}v_i\|+|\delta^{p+2}v_i|+\sum_{i=1}^{n}\|\delta^{p+1}u_i\|^2\tau\leq C$$

hold true for $p=0,\ldots,k;\forall n\geq n_0;i=1,\ldots,n-k$.

PROOF : By multiple application of $z_j=\sum_{i=1}^{j}\delta z_i\tau+z_0$, the following can be obtained

$$\delta^\alpha z_i=L_\tau(Z_\alpha,\ldots,Z_{p-1})+\sum_{j=1}^{i}\tau q_j(\tau)y_j,$$

where $y_i=\delta^p z_i;0\leq\alpha\leq p-1;q_j(\tau)$ being polynomials in $\tau;L_\tau$ being a polynomial in τ the coefficients of which are depending on $Z_\alpha,\ldots,Z_{p-1}$.

The assertion of the theorem can be proved using the same technique as in [**11**]. ■

5. Regularity in t,x.

Let $\Omega\subset R^N$ be a bounded domain with Lipschitz continuous boundary. In this section we prove the regularity in t,x variables in the interior of the domain in concrete function spaces. The regularity results for the hyperbolic part of the system are comparable with those in [3] and the ones for the parabolic part are worse than in [3]. This fact is due to coupling the both types of equations (hyperbolic and parabolic).

Let $r>s\geq 0,Y\hookrightarrow H,\overset{\circ}{W}{}_2^s\subset H\subset W_2^s,\overset{\circ}{W}{}_2^r\subset Y\subset W_2^r$, (the notation W_p^k being adopted for $W_p^k(\Omega)$)

$$a_1(t;z,w) = \sum_{|i|,|j|\le r} \int_\Omega a_{1,ij}(x,t)D^j z D^i w\,dx,$$

$$a_2(t;z,w) = \sum_{|i|,|j|\le r} \int_\Omega a_{2,ij}(x,t)D^j z D^i w\,dx,$$

$$g(t;z,w) = \sum_{|i|,|j|\le r} \int_\Omega g_{ij}(x,t)D^j z D^i w\,dx,$$

$$p(t;u,v) = \sum_{|i|,|j|\le s} \int_\Omega p_{ij}(x,t)D^j u D^i v\,dx,$$

$$b(t;u,v) = \sum_{|i|,|j|\le s} \int_\Omega b_{ij}(x,t)D^j u D^i v\,dx,$$

$$\vartheta(t;u,v) = \sum_{|i|,|j|\le s} \int_\Omega \vartheta_{ij}(x,t)D^j u D^i v\,dx,$$

$$d(t;u,w) = \sum_{|i|\le r}\sum_{|j|\le s} \int_\Omega d_{ij}(x,t)D^i w D^j u\,dx,$$

$$\rho(t;w,u) = \sum_{|i|\le r}\sum_{|j|\le s} \int_\Omega \rho_{ij}(x,t)D^i w D^j u\,dx,$$

$\forall u,v \in W_2^s; \forall z,w \in W_2^r$.

We consider $Q, \overline{Q} \in L_2(J, L_2(\Omega))$ such that

(5.1) $$\langle \mu(t),\varphi\rangle_Y + \langle \nu(t),\varphi\rangle_H = \int_\Omega Q(x,t)\varphi(x)\,dx \equiv (Q(t),\varphi)$$

(5.2) $$\langle \mu_1(t),\varphi\rangle_Y + \langle \nu(t),\varphi\rangle_H = \int_\Omega \overline{Q}(x,t)\varphi(x)\,dx \equiv (\overline{Q}(t),\varphi)$$

$\forall\varphi \in \mathcal{D}(\Omega)$ for a.e. $t \in J$ ($\mathcal{D}(\Omega)$ being the set of all C^∞–smooth functions with compact support in Ω).

Now we state a well–known regularity result from the theory of linear elliptic equations (see [8, Chap.4, Th.1.2]) which we use to obtain regularity results in x–variable.

Theorem 5.3. *Let $\alpha \ge 1$ be an integer and $t \in J$ be fixed. Let $a(t;u,u) \ge C_1\|u\|^2 - C_2|u|^2_{L_2}$, $|a(t;u,v)| \le C_3\|u\|\,\|v\|, \forall u,v \in \overset{\circ}{W}{}_2^r$. Suppose that (5.4)–(5.6) are satisfied where*

(5.4) $$a_{ij}(x,t) \in C^{q_i,1}(\overline{\Omega}) \qquad q_i = \max\{0, |i| + \alpha - r - 1\}$$

$\forall |i|,|j| \le k$; *($C^{q,1}$ is the set of all $v \in C^q(\overline{\Omega})$ for which $\partial_x^i v$ is Lipschitz continuous in $\overline{\Omega}, \forall |i| \le q$)*

(5.5) $$a(t;u,v) = \mathcal{R}(v) \qquad \forall v \in \mathcal{D}(\Omega) \text{ where } \mathcal{R} \in \left(\overset{\circ}{W}{}_2^{r-1}\right)^*$$

(5.6) $$\partial_x^i \mathcal{R} \in W_2^{-r+1}, \qquad \forall |i| \le \alpha - 1 \quad \left(W_2^{-r+1} \equiv (\overset{\circ}{W}{}_2^{r-1})^*\right).$$

Then $u \in W^{r+\alpha}_{2,\mathrm{loc}}$ *and the estimate*

$$\|u\|_{W_2^{r+\alpha}(\Omega')} \le C(\Omega') \left(\|u\| + \sum_{|i|\le \alpha-1} \|\partial_x^i \mathcal{R}\|_{W_2^{-r+1}} \right)$$

holds $\forall \Omega' \subset \Omega$ *with* $\overline{\Omega}' \subset \Omega$. *(The constant* $C(\Omega')$ *depends only on* $\Omega', C_1 - C_3$ *and the norm of* a_{ij} *in* $C^{q_i,1}(\overline{\Omega})$.*)*.

For coefficients of bilinear forms $a_1, a_2, p, b, g, d, \rho, \vartheta$ we shall assume

$$a_{1,ij}; a_{2,ij}; g_{ij}; \rho_{ij} \in C^{q_{i,p},1;p,1}(\overline{J \times \Omega}) \tag{5.7}$$

where $q_{i,p} = |i| + \lambda_p - r + 1$ (λ_p will be determined) and $C^{\alpha,1;\beta,1}(\overline{J \times \Omega})$ is the set of all $v \in C(\overline{J \times \Omega})$ for which $D_t^i D_x^j v$ are Lipschitz continuous in $\overline{J \times \Omega}$ $\forall |j| \le \alpha,\ \ 0 \le i \le \beta$.

$$\begin{aligned} p_{ij}, b_{ij}, d_{ij}, \vartheta_{ij} &\in C^{m_{i,p},1;p,1}(\overline{J \times \Omega}) \\ m_{i,p} &= |i| + \lambda_p - s - 1, \qquad 0 \le p \le k. \end{aligned} \tag{5.8}$$

Theorem 5.9. *Let the assumptions of Theorem 4.9 be fulfilled. Suppose (5.7), (5.8) for* $\lambda_p = (k+1-p)(r-s), 0 \le p \le k$. *Denote* $\alpha_p = \lambda_p - r, \beta_p = \lambda_p + r$. *If* $U_p \in W_2^{\beta_0}$ *for* $0 \le p \le k, V_p \in W_2^{\beta_0}$ *for* $0 \le p \le k+1$ *and*

$$\partial_t^p Q, \partial_t^p \overline{Q} \in L_\infty(J, W_2^{\alpha_p}) \qquad \textit{for } 0 \le p \le k,$$

then

$$\partial_t^p u, \partial_t^p v \in L_\infty(\langle 0, T' \rangle, W_{2,\mathrm{loc}}^{\beta_p}) \textit{ for } p = 0, \dots, k; \quad \forall T' < T$$

where u, v *is the solution of (4.1).*

PROOF : We transform the identity (4.8) in this way:

1. Sum up $(4.8)_1$ and $(4.8)_2$.
2. On the right–hand side, there is $z_i (z = u, v)$ only in terms of the type $\delta^p F_i z (F = G, I$ – see (4.5)) i.e.

$$\begin{aligned} &\text{if } p = 0 \qquad \text{in } K_i^{(i)} z_i \tau \\ &\text{if } p \ge 1 \qquad \text{in } K_i^{(i)} \delta^{p-1} z_i \left(= K_i^{(i)} \left[Z_{p-1} + \sum_{j=1}^{i} \delta^p z_j \tau \right] \right). \end{aligned}$$

3. We replace all the terms of the type $\delta^\alpha F_\beta z$ using (4.4), (4.5) and we put those members which include $\delta^p z_i$ on the left–hand side and the others on the right–hand side.

In this way the following can be obtained

$$(5.10)\qquad A_\tau(t_i;[\delta^p u_i,\delta^p v_i],[\varphi,\phi]) = -p(t_i;\delta^{p+1}u_i,\varphi) - p(t_i;\delta^{p+2}v_i,\phi) - b(t_i;\delta^{p+1}v_i,\phi) + (\delta^p Q,\varphi) + (\delta^p\overline{Q},\phi) + \sum_{\alpha=0}^{k}\sum_{\beta=0}^{k+1} K_{\alpha\beta}(U_\alpha,V_\beta,\varphi,\phi) + \sum_{\ell=1}^{i-1}\tau\sum_{j=0}^{p} L_j^{(\ell)}(t_i,\delta^j u_\ell,\delta^j v_{\ell,\varphi,\phi}) \equiv \mathcal{R}_i^{(p)}([\varphi,\phi]),\qquad \forall\varphi,\phi\in\mathcal{D}(\Omega),$$

where

$$A_\tau(t_i;[\delta^p u_i,\delta^p v_i],[\varphi,\phi]) := a_1(t_i;\delta^p u_i,\varphi) + a_2(t_i;\delta^p v_i,\phi) - \tau K_i^{(i)}\{d(t_i;\delta^p(u_i+v_i),\varphi+\phi) + \vartheta(t_i;\delta^p(u_i+v_i),\varphi+\phi) + \rho(t_i;\delta^p(u_i+v_i),\varphi+\phi) + g(t_i;\delta^p v_i,\varphi+\phi) + g(t_i;\delta^p u_i,\varphi)\}$$

and

$K_{\alpha\beta}(u,v,x,y)$ is a linear combination of $\delta^i r(t_1;z,w),\delta^j r(0;z,w),r^{(i)}(0;z,w)$ for $i=0,\dots,p; j=0,\dots,p+1; z=u,v; w=x,y; r=a_1,a_2,b,p,d,g,\vartheta,\rho$

and

$L_j^{(\ell)}(t,u,v,x,y)$ is a linear combination of $\delta^m r(t;z,w)$ for $0\le m\le p+1; z=u,v; w=x,y; r=a_1,a_2,b,d,p,g,\vartheta,\rho$.

Let us denote $\delta^p z_i := [\delta^p u_i,\delta^p v_i]$ for $0\le p\le k$ and $X_p := W_{2,\mathrm{loc}}^{r+p(r-s)}\times W_{2,\mathrm{loc}}^{r+p(r-s)}$ with summation norm $|\|\cdot|\|$.

Setting $p=k$ in (5.10), using Gronwall's lemma, (4.11) it is easy to see that for

$$f([\varphi,\phi]) = p(t_i;\delta^{p+1}u_i,\varphi) + p(t_i;\delta^{p+2}v_i,\phi) + b(t_i;\delta^{p+1}v_i,\phi)$$

we have

$$\partial_x^\omega f\in W_2^{-r}\times W_2^{-r}\qquad\forall\omega,|\omega|\le r-s.$$

For $i=1$, applying the regularity of $Q,\overline{Q},U_\alpha,V_\beta$ it yields

$$\partial_x^\omega\mathcal{R}_1^{(0)}\in W_2^{-r}\times W_2^{-r}$$

and so by virtue of the theorem 5.3 we get $\delta^k z_1\in X_1$. Moreover $\delta^p z_1\in X_1$ for $0\le p\le k$ because

$$\delta^p z_i = Z_p + \sum_{j=1}^{i}\delta^{p+1}z_j\tau.$$

In the case $i=2$ we proceed in the same way as for $i=1$ using $\delta^p z_1\in X_1$ for $0\le p\le k$. We obtain $\delta^k z_2\in X_1$. Successively for $i=3,\dots,n-k$ can be obtained

$$(5.11)\qquad |\|\delta^k z_i|\|_1\le C(\Omega')\qquad\forall n\ge n_0; i=1,\dots,n-k.$$

Analogously as for $p=k$, this procedure can be repeated for $p=k-1$ in order to reach

$$(5.12)\qquad |\|\delta^{k-1}z_i|\|_2\le C(\Omega')\qquad\forall n\ge n_0;\quad i=1,\dots,n-k.$$

Gradually the following can be obtained

$$(5.13)\qquad |\|\delta^p z_i|\|_{k+1-p}\le C(\Omega'),\qquad 0\le p\le k;\forall n\ge n_0;\quad i=1,\dots,n-k,$$

from which we conclude the proof. ■

Remark 5.14. In fact the domain $\Omega'(=\Omega'_{p,i})$ of the regularity of $z_i(z=u,v)$ changes in each step $i=1,\ldots,n$ and $p=0,\ldots,k$. But in the end we can take $\Omega'\subset\bigcap_{p=0}^{k}\bigcap_{i=1}^{n}\Omega'_{p,i}$.

Remark 5.15. For the case $H\hookrightarrow Y$ we refer the reader to [**3**, Remark 5.12].

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