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On positive solutions of semilinear elliptic problems

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Dedicated to the memory of Svatopluk Fučík

Abstract. In this paper we study the existence of positive solutions of semilinear elliptic equations. Our method is based on the use of the topological degree and the apriori estimates of Brézis and Turner.

Keywords: positive solution, semilinear equation, variational inequality, topological degree

Classification: 35J65, 47H15, 34B15

1. Introduction and main results. This paper deals with the existence of solutions of the problem

$$\begin{aligned} -\Delta u &= f(u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \\ u &> 0 && \text{in } \Omega \end{aligned} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $f : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function. Our existence results are based on the following three assumptions

(A1) f crosses the first eigenvalue λ_1 of the operator $-\Delta$ on $H_o^1(\Omega)$, i.e.

$$\limsup_{t\to 0+} \frac{f(t)}{t} < \lambda_1 < \liminf_{t\to +\infty} \frac{f(t)}{t}$$

(A2) $|f(t)| \leq C(1 + |t|^\alpha)$, where $\alpha < (N+1)/(N-1)$

(A3) $f \geq -\lambda$, where $\lambda > 0$ is "sufficiently" small (more precisely see Theorem 2)

and they can be easily extended e.g. to the problem

$$\begin{aligned} -Lu &= f(x, u, \nabla u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \\ u &> 0 && \text{in } \Omega \end{aligned} \tag{2}$$

where L is a general second-order elliptic operator with smooth coefficients.

If $f(0) \geq 0$, then the existence of solutions to (1) was proved under rather general assumptions by many authors (see P. L. Lions [6] for a survey). If $f(0) < 0$, then the existence of solutions to (1) in a general domain Ω was proved under the assumption (A3) in Smoller–Wasserman [8] for sublinear f and in Castro [3] for $f(u) = \lambda(u^q - 1)$, where $q \in (1, 2^* - 1)$, $2^* = 2N/(N-2)$. The proof of Castro is based on the mountain pass theorem and thus it can be used only in problems with variational structure. Our method is more general, on the other hand we require more restrictive growth condition (A2) on f. First we use the topological degree and the apriori estimates of Brézis–Turner [2] to prove that under the assumptions (A1) and (A2) the variational inequality

$$(3) \qquad u \in K^+ : \qquad \langle -\Delta u - f(u), v - u \rangle \geq 0 \qquad \forall v \in K^+$$

(where $K^+ = \{u \in H_o^1(\Omega); u \geq 0\}$ and $\langle \cdot, \cdot \rangle$ is the duality between $H^{-1}(\Omega)$ and $H_o^1(\Omega)$) has a nontrivial solution and then we use the maximum principle in order to show that under the additional assumptions (A3) any nontrivial solution of (3) is automatically a solution of (1).

To be more precise let us formulate our main results for the model problems (1),(3):

Theorem 1. *Let $f \in C(\mathbb{R}^+, \mathbb{R})$ satisfy (A1) and (A2). Then the inequality (3) has a nontrivial solution.*

Theorem 2.. *Let $E \in C(\mathbb{R}^+, \mathbb{R}^+)$ fulfil (A1) and (A2) with f replaced by E. Then there exists $\lambda = \lambda(E, \Omega) > 0$ such that for any $f \in C(\mathbb{R}^+, [-\lambda, +\infty))$ satisfying*

$$(4) \qquad f \leq E$$

and

$$(5) \qquad \liminf_{t \to +\infty} \frac{f(t)}{t} > \lambda_1$$

the problem (1) has a solution.

Note that using Theorem 2 one can easily prove the existence of solutions to (1) with $f(u) = \lambda(g(u) - 1)$ or $f(u) = g(u - \frac{1}{\lambda}) - \lambda$, where $\lambda \to 0+$, $g \in C(\mathbb{R}, \mathbb{R}^+)$ satisfies the growth condition (A2), $g(t) = 0$ for $t \leq 0$ and $\liminf_{t \to +\infty} \frac{g(t)}{t} = +\infty$ or $\liminf_{t \to +\infty} \frac{g(t)}{t} > \lambda_1$, respectively. Finally, let us remark that the growth condition (A2) can be weakened to

$$\limsup_{t \to +\infty} \frac{f(t)}{t^\beta} = 0 \qquad \text{with} \quad \beta = \frac{N+1}{N-1}$$

in Theorem 1 and Lemma 1 and that the existence of a nontrivial solution of (3) for $f(u) = u^q - 1$, $q \in (1, 2^* - 1)$, was proved by Szulkin [9,Theorem 5.1] using his version of the mountain pass theorem.

2. Proofs of Theorem 1 and 2 (for $N > 1$). We shall write briefly $\int u$ instead of $\int_\Omega u\,dx$ and we put $\|u\| := (\int |\nabla u|^2)^{1/2}$. By c and C we shall denote various constants which depend only on Ω and f (in Lemma 1 and Theorem 1) or on Ω and E (in Lemma 2 and Theorem 2). First we prove some apriori estimates for the solutions of (3). The following Lemma 1 is based on the results of Brézis–Turner [2] (see also de Figueiredo [5]) and so its proof is just sketched.

Lemma 1. *Let $f \in C(\mathbb{R}^+, \mathbb{R})$ satisfy (A1),(A2). Then there exists a constant $C = C(f, \Omega) > 0$ such that for any $s \geq 0$ and for any solution u of the inequality*

$$u \in K^+ : \qquad \langle -\Delta u - f(u) - s\Phi, v - u\rangle \geq 0 \qquad \forall v \in K^+ \tag{6}$$

we have $\|u\| < C$ and $\|u\|_{L^\infty} < C$.
Here Φ is the positive eigenfunction of $-\Delta$ on $H^1_o(\Omega)$ corresponding to the eigenvalue λ_1 .

PROOF : Let $s \geq 0$ and let u be a solution of (6). Putting $v = u + \Phi$ in (6) and using (A1) we get

$$\lambda_1 \int u\Phi \quad \geq \quad \int f(u)\Phi + s\int \Phi^2 \quad \geq \quad (\lambda_1 + \varepsilon)\int u\Phi - C\int \Phi + s\int \Phi^2 \quad ,$$

hence $\int u\Phi \leq C$, $\int f(u)\Phi \leq C$, $s \leq C$.
Putting $v = 2u$ and $v = 0$ in (6) we get $\langle -\Delta u - f(u) - s\Phi, u\rangle = 0$, thus

$$\|u\|^2 \leq \int f(u)u + C \quad . \tag{7}$$

Putting $\gamma = 2/(N+1), \beta = (N+1)/(N-1)$, using the estimate $f(t) \leq \varepsilon t^\beta + C_\varepsilon$ and the Hardy–Sobolev inequality we obtain

$$\begin{aligned} \int f(u)u \quad &\leq \quad \Big(\int f(u)\Phi\Big)^\gamma \Big(\int \frac{f(u)}{\Phi^{\gamma/(1-\gamma)}} u^{1/(1-\gamma)}\Big)^{1-\gamma} \\ &\leq \quad C\Big(\int \frac{(\varepsilon u^\beta + C_\varepsilon)u^{1/(1-\gamma)}}{\Phi^{\gamma/(1-\gamma)}}\Big)^{1-\gamma} \quad \leq \quad C\varepsilon^{1-\gamma}\|u\|^2 + C_\varepsilon \|u\| \end{aligned}$$

which together with (7) implies $\|u\| < C$. Now the regularity results for variational inequalities (Brézis [1]) imply

$$\|u\|_{W^{2,p}} \leq C\|f(u) + s\Phi\|_{L^p} \leq C(\|u\|^\beta_{L^{p\beta}} + 1) \qquad \text{for any} \quad p \geq 2 \tag{8}$$

which enables us to use a bootstrap argument to conclude $\|u\|_{L^\infty} < C$. ∎

Lemma 2. *Let $E \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfy (A1) and (A2) with f replaced by E. Then there exists $c = c(E, \Omega) > 0$ such that for any $f \in C(\mathbb{R}^+, \mathbb{R})$ satisfying $f \leq E$ and for any nontrivial solution u of the inequality (3) we have $\|u\| > c$, $\|u\|_{L^\infty} > c$.*

Proof : Without loss of generality we may suppose $\alpha > 1$.
Let u be a nontrivial solution of (3) with $f \leq E$. Since $E(t) \leq (\lambda_1 - \varepsilon)t$ for $t \leq t_o$, we have

$$(9) \quad \|u\|^2 = \int f(u)u \leq (\lambda_1 - \varepsilon)\int u^2 + \int_{\{u > t_o\}} C(u^{\alpha+1} + 1)\,.$$

Moreover,

$$(10) \qquad \int u^2 \leq \frac{1}{\lambda_1}\|u\|^2\,,$$

$$(11) \int_{\{u>t_o\}} Cu^{\alpha+1} \leq C\|u\|^{\alpha+1} < \frac{\varepsilon}{3\lambda_1}\|u\|^2 \qquad \text{if } \|u\| \leq c\,,$$

$$\operatorname{meas}\{u > t_o\}\, t_o^2 \leq \int_{\{u>t_o\}} u^2 < \left(\int u^{\alpha+1}\right)^{2/(\alpha+1)} \left(\int_{\{u>t_o\}} 1\right)^{(\alpha-1)/(\alpha+1)}$$
$$< C\|u\|^2 \left(\operatorname{meas}\{u > t_o\}\right)^{(\alpha-1)/(\alpha+1)}\,,$$

hence

$$(12) \quad \int_{\{u>t_o\}} C = C \operatorname{meas}\{u > t_o\} < \frac{\varepsilon}{3\lambda_1}\|u\|^2 \qquad \text{if } \|u\| \leq c\,.$$

From (9)—(12) it follows that $\|u\| > c$, $\|u\|_{L^\infty} > t_o$. ∎

Proof of Theorem 1: Let H be the Hilbert space $H_o^1(\Omega)$ with the scalar product $((u,v)) := \int \nabla u \nabla v$ and let P be the projection in H onto K^+. Then the inequality (3) is equivalent to the equation

$$u \in H : \qquad u - PF(u) = 0\,,$$

where $F : H \to H$ is a compact map defined by $((F(u), v)) := \int f(u)v$.
Putting $B_c := \{u \in H; \|u\| \leq c\}$, using Lemma 2 and the homotopy $H(t,u) := u - tPF(u)$ we get

$$\deg(I - PF, 0, B_c) = \deg(I, 0, B_c) = 1\,,$$

where deg is the Leray–Schauder degree and I is the identity in H. Hence to prove the existence of a nontrivial solution of (3) it is sufficient to show $\deg(I - PF, 0, B_C) = 0$ for some C. According to Lemma 1 and the homotopy invariance property of the degree we have

$$\deg(I - PF, 0, B_C) = \deg(I - PF_s, 0, B_C)\,,$$

where $((F_s(u),v)) := \int (f(u)+s\Phi)v$. Thus it is sufficient to show that the inequality (6) has no solution for s sufficiently large, which follows from the proof of Lemma 1 (we have shown $s \le C$ under the assumption of solvability of (6)). ∎

PROOF of Theorem 2: Let $f \in C(\mathbb{R}^+, [-\lambda, +\infty))$ satisfy (4) and (5) and let u be the nontrivial solution of (3) whose existence is guaranteed by Theorem 1. It follows from (8) and Lemma 1 that $u \in W^{2,p}(\Omega)$ for any $p \ge 2$, hence $u \in C^{1,\nu}(\overline{\Omega})$ for any $\nu < 1$. Moreover,

$$\|u\|_{C^{1,\nu}} \le C\|u\|_{W^{2,p}} \le C\|f(u)\|_{L^p} \le C\|u\|^\alpha_{L^\infty}, \tag{13}$$

where $\nu = 1 - N/p$. In what follows choose p such that $\alpha \le (N+\nu)/(N-1)$ and choose $x_o \in \Omega$ such that $K := u(x_o) = \max_{x\in\Omega} u(x)$. According to Lemma 2 we have $K \ge c > 0$. First we shall prove that for $\varepsilon = CK^{-1/(N-1)}$ (with suitable $C > 0$) we have

$$\begin{aligned} &B_\varepsilon(x_o) := \{x \in \mathbb{R}^N; |\, x - x_o \,| < \varepsilon\} \subset \Omega \\ &u(x) \ge \frac{K}{2} \qquad \text{for any} \quad x \in B_\varepsilon(x_o)\,. \end{aligned} \tag{14}$$

To prove this let us choose $x_1 \in \Omega$ such that $|\, x_o - x_1 \,| = \min\{|\, x_o - \bar{x} \,|; u(\bar{x}) = K/2\}$ and $u(x_1) = K/2$. Using (13) we get

$$\begin{aligned} \frac{K}{2} &= |\, u(x_1) - u(x_o) \,| \le \int_0^1 |\, Du(x_o + t(x_1 - x_o))(x_1 - x_o) \,|\, dt \\ &\le C\int_0^1 t^\nu \,|\, x_1 - x_o \,|^{1+\nu} K^\alpha dt = C \,|\, x_1 - x_o \,|^{1+\nu} K^\alpha\,, \end{aligned}$$

hence $|\, x_1 - x_o \,| \ge CK^{(1-\alpha)/(1+\nu)} \ge CK^{-1/(N-1)}$, which implies (14).
Now let us denote by z the unique solution of the problem

$$\begin{aligned} -\Delta z &= f^-(u) && \text{in } \Omega \\ z &= 0 && \text{on } \partial\Omega \end{aligned}$$

where $f^- = \max(0, -f)$. By the maximum principle we get $z \ge 0$ in Ω and using standard regularity theory we obtain

$$\|z\|_{C^1} \le C \max f^- \le C\lambda\,. \tag{15}$$

Putting $w := u + z$ we get $-\Delta w = f^+(u) + \mu$, where μ is a nonnegative measure, thus w is superharmonic and positive in Ω, $w \ge u$.
Choose $\delta = \delta(\Omega) > 0$ such that $\Omega_\delta := \{x \in \Omega; \mathrm{dist}(x, \partial\Omega) > \delta\}$ is connected and $\partial B_\delta(x) \cap \Omega_\delta \ne \emptyset$ for any $x \in \Omega$.

Let $y_o \in \partial B_\delta(x_o) \cap \Omega_\delta$ be fixed and let u_y be the solution of the problem

$$\begin{aligned} -\Delta u_y &= 0 \qquad \text{in } B_\delta(y_o) \\ u_y &= w \qquad \text{on } \partial B_\delta(y_o)\,. \end{aligned}$$

Then $w \geq u_y$ on $B_\delta(y_o)$ and (14) implies

$$\begin{aligned} u_y(y_o) \quad &= \quad C\int_{\partial B_\delta(y_o)} w \quad \geq \quad C\int_{\partial B_\delta(y_o)} u \\ &\geq \quad C\frac{K}{2}\,\mathrm{meas}_{N-1}(\partial B_\delta(y_o)\cap B_\varepsilon(x_o)) \quad \geq \quad C\,, \end{aligned}$$

where $C = C(\delta) > 0$ does not depend on u and f.
Using the Harnack's inequality we get $u_y(x) \geq C$ for any $x \in B_{\delta/5}(y_o)$, hence $w \geq C$ on $B_{\delta/5}(y_o)$.
Choose a fixed covering $\bigcup_{i=1}^m B_{\delta/5}(y_i)$ of $\overline{\Omega_\delta}$ with $y_i \in \Omega_\delta$. Without loss of generality we may assume $y_o \in B_{2\delta/5}(y_1)$, hence

$$w(x) \;\geq\; C\int_{B_{4\delta/5}(x)} w \;\geq\; C\int_{B_{\delta/5}(y_o)} w \;\geq\; C \qquad \text{for any } x \in B_{\delta/5}(y_1).$$

Repeating this argument m-times we get $w \geq C$ on Ω_δ.
Let u_δ be the solution of the problem

$$\begin{aligned} -\Delta u_\delta &= 0 \qquad \text{in } \Omega\setminus\overline{\Omega_\delta} \\ u_\delta &= 0 \qquad \text{on } \partial\Omega \\ u_\delta &= C \qquad \text{on } \partial\Omega_\delta\,. \end{aligned}$$

By the strong maximum principle we have $u_\delta(x) \geq C\,\mathrm{dist}(x,\partial\Omega)$ for any $x \in \Omega\setminus\overline{\Omega_\delta}$ and some $C > 0$. Hence $w(x) \geq C\,\mathrm{dist}(x,\partial\Omega)$ for some $C > 0$ and any $x \in \Omega$. Now (15) implies $u = w - z > 0$ in Ω, provided λ is small enough.
Note that the decomposition $u = w - z$ was used also by Castro [3]. ■

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