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Global branching for discontinuous problems

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In memory of Svatopluk Fučík and Rudolf Švarc

Abstract. The paper deals with a class of elliptic problems with discontinuous nonlinearities both in bounded and unbounded domains. The existences of a global branch of positive solutions is proved and an application to Plasma Physics is discussed.

Keywords: Global bifurcation, elliptic free boundary problems, plasma physics

Classification: 35B32, 35R35, 82A45

1. Introduction. The main purpose of this paper is to estabilish the existence of global branch of positive solutions for some elliptic problems with discontinuous nonlinearities .

In §**2** we deal with problems like

$$(1a) \qquad -\Delta u = f(u-a) \quad \text{in} \quad \Omega,\ u=0 \quad \text{on} \quad \partial\Omega$$

where Δ is the Laplace operator, $a \in \mathbf{R}$, Ω is a bounded domain in $\mathbf{R}^N$ with suitable symmetry and $f = f(s)$ has a discontinuity at $s = 0$.

Taking a as the bifurcation parameter, we will prove (Theorems 1 and 1') that there is a global branch S of pairs (a, u) with $u > 0$ solution of $(1a)$, bifurcating from $a = 0,\ u = 0$ and having a turning point at some value $a_0 > 0$, in such a way that $(1a)$ has at least two positive solutions for all $a \in\,]0, a_0[$.

In §**3** we study problems like

$$(2a) \qquad -\Delta u = f(u-a) \quad \text{in} \quad \mathbf{R}^N,\ u(x) \to 0 \quad \text{as} \quad |x| \to \infty$$

with $N \geq 3$.

Problem $(2a)$ is approximated by Dirichlet problems such as $(1a)$ with $\Omega = B(R) = \{x \in \mathbf{R}^N \ :\ |x| < R\}$ where $|x|$ denotes the Euclidean norm.

By a limiting procedure based upon an a priori estimate of the free boundary $\{u = a\}$, we show (Theorem 8) that $(2a)$ possesses an unbounded branch S of positive solutions bifurcating from $(0, 0)$.

Moreover, we prove (Theorem 10) that for all $a > 0$ there is a solution u of $(2a)$ such that $(a, u) \in \mathbf{S}$.

It is well known that several problems in Plasma Physics give rise to equations with discontinuous nonlinearities, such as $(1a)$ or $(2a)$ (see [2], [8]). As an example, an application in this frame is shortly discussed in the sequel.

We also recall that problems with discontinuous nonlinearities have been studied by several points of view : see e.g. [1], [5], [7], [13] for a general set up and [4], [6], [9] for problems in Vortex Theory.

2. Problems in a bounded domain. In this section we study problem $(1a)$ with Ω bounded domain in $\mathbf{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$.

Even if our results hold true for any Steiner symmetric set Ω (see Theorem 1' below) we prefer to start dealing with the case in which Ω is the ball $B(R)$. Actually, this is the kind of results we need in the following section where we study problem $(2a)$ on all $\mathbf{R}^N$.

Let us denote by $\lambda(R)$ the first eigenvalue of $-\Delta$ on $B(R)$ with zero Dirichlet boundary condition and let ϕ be such that

$$-\Delta\phi = \lambda(R)\phi, \int_{B(R)} \phi^2 = 1, \ \phi > 0.$$

Consider the boundary value problem

$$(3a\text{–}R) \qquad -\Delta u = f(u-a) \quad \text{in} \quad B(R), \ u = 0 \quad \text{on} \quad \partial B(R)$$

with $a \in \mathbf{R}$ and $f : \mathbf{R} \longrightarrow \mathbf{R}$ satisfying

(f1) $f(s) = 0 \ \forall s \leq 0$;
(f2) $f \in C^{0,\alpha}(\mathbf{R}^+, \mathbf{R}^+)$ and is non-decreasing;
(f3) there exists $d > 0$, $c_1 < \lambda(R)$ and $c_2 \leq 0$, such that $d \leq f(s) \leq c_1 s + c_2$ for all $s > 0$.

By a positive solution of $(3a\text{–}R)$ we mean an $u(x) \in C_0^1(\overline{B(R)})$ such that: (i) $u > 0$ in $B(R)$; (ii) letting $T(a) = \{x \in B(R) \ : \ u(x) = a\}$, $u \in C^2(B(R) - T(a))$; and (iii) u solves pointwise $(3a\text{–}R)$ on $B(R) - T(a)$.

Notice that, by (f1) and the maximum principle, any positive solution u which is $\not\equiv 0$ is strictly positive: in fact, $u(x) > a$ for some $x \in B(R)$; moreover the discontinuity is not influent for $a \leq 0$ and, taking into account (f3), $(3a\text{–}R)$ has nontrivial, positive, classical solutions for such range of parameters.

Set $r = |x|$ $(x \in \mathbf{R}^N)$, $E = C(\overline{B(R)})$ with norm $| \cdot |_{\infty,R}$; $V(b) = \{u \in E : |u|_{\infty,R} < b\}$; for u radial, $u' = \dfrac{du}{dr}$; $\Sigma(R) = \{(a,u) \in \mathbf{R}^+ \times E \ : \ u$ is a positive solution of $(3a\text{–}R)\}$; $\Sigma_0(R) = \{u \ : \ u$ is a positive solution of $(3a\text{–}R)$ for $a = 0\}$.

Theorem 1. *Let* (f1-2-3) *hold. Then there exists a global branch* $S(R) \subset cl(\Sigma(R))$ *(i.e. a closed, connected component of* $cl\Sigma(R)$*) such that:*

(i) $(0,0) \in S(R)$ *and if* $(a,0) \in S(R)$ *then* $a = 0$*;*
(ii) $S(R)$ *is bounded in* $\mathbf{R}^+ \times E$ *and* $S(R) \cap \Sigma_0(R) \neq \emptyset$*;*

(iii) *if* $(a,u) \in S(R)$, *with* $0 < |u|_{\infty,R}$ *small, then* $a > 0$. *As a consequence, there is* $a_0 > 0$ *such that for all* $a \in]0, a_0[$ *(3a–R) has at least two distinct positive solutions with* $(a,u) \in S(R)$.

(iv) *any* $u \in S(R)$ *is radial ,* $u'(r) < 0\ \forall r > 0$ *and* $|T(a)| = meas[T(a)] = 0$.

PROOF : The proof is divided in 2 steps. For simplicity of notations the dependence on R is understood and omitted during all the proof, but where a precisation is worthwhile.

Step 1. (Smooth approximation of (3a–R)). For $\epsilon > 0$, let $f_\epsilon \in C^{0,\alpha}(\mathbf{R},\mathbf{R})$ be defined by setting

$$f_\epsilon(s) = f(s) \quad \text{for} \quad s \le -\epsilon \quad \text{and} \quad s > 0; f_\epsilon(s) = f(0+)(\frac{s}{\epsilon} + 1) \quad \text{for} \quad -\epsilon \le s \le 0;$$

and consider the smooth problems

$$(4a\text{–}\epsilon) \qquad -\Delta u = f_\epsilon(u-a)\ in\ B(R),\ u = 0\ on\ \partial B(R)$$

Let $G = (-\Delta)^{-1}$ with zero Dirichlet boundary conditions, set $\Phi(\epsilon,a,u) = u - Gf_\epsilon(u-a)$ and denote by $\Sigma(\epsilon)$ the set of $(a,u) \in \mathbf{R}^+ \times E$ such that $\Phi(\epsilon,a,u) = 0$. If $(a,u) \in \Sigma(\epsilon)$, then, by regularity theory, u is a positive, classical solution of (4a–ϵ).

First of all we note that $\exists\ a^* = a^*(R) > 0$ and $b^* = b^*(R) > 0$ such that $\Sigma(\epsilon) \subset [0,a^*] \times V(b^*)$ for all $\epsilon > 0$. In fact, by (f3) there is $a^* > 0$ such that $f_\epsilon(u-a) < c_1 u$ for all $a \ge a^*$. If u is any (positive) solution of (4a–ϵ) with $a > a^*$, one has:

$$\lambda(R) \int_{B(R)} u\phi = \int_{B(R)} f_\epsilon(u-a)\phi < c_1 \int_{B(R)} u\phi$$

a contradiction.

We need:

Lemma 2. *There is a global branch* $S(\epsilon) \subset \Sigma(\epsilon)$ *such that* $(\epsilon, 0) \in S(\epsilon)$ *and if* $(a,0) \in S(\epsilon)$ *then* $a = \epsilon$.

PROOF : We claim that

$$(5) \qquad ind(\Phi(\epsilon,a,.),0) = 1\ \forall\ a > \epsilon$$

$$(6) \qquad ind(\Phi(\epsilon,a,.),0) = 0\ \forall\ a < \epsilon$$

Here *ind* denotes the Leray-Schauder index.

To see (5) we take $0 < \epsilon < a$ and consider the homotopy $H(t,u) = u - tGf_\epsilon(u-a)$, $t \in [0,1]$. H is admissible on $V(r)$, $r > 0$ small enough, because, otherwise there are sequences $|u_n|_\infty \to 0$ and $t_n \in [0,1]$ such that $H(t_n,u_n) = 0$. Letting $v_n = \dfrac{u_n}{|u_n|_\infty}$ it follows

$$(7) \qquad -\Delta v_n = t_n \frac{f_\epsilon(u_n - a)}{u_n} v_n$$

Then (7) and $|v_n|_\infty = 1$ imply $v_n \to \overline{v}$ in E and $|\overline{v}|_\infty = 1$. But for n large $u_n(x) < a - \epsilon$, hence $f_\epsilon(u_n - a) \equiv 0$; therefore, passing to the limit into (7) one finds $-\Delta\overline{v} = 0$, a contradiction, and (5) follows.

Next let $a < \epsilon$. We prove (6) showing that there is $b > 0$ such that $\Phi(\epsilon, a, u) \neq 0$ for all $u \in V(b)$. In fact, otherwise, there is a sequence $u_n \in E$, $u_n > 0$, such that $u_n \to 0$ and satisfies $-\Delta u_n = f_\epsilon(u_n - a)$. Letting $u_n = t_n\phi + w_n$ with $\int_\Omega \phi w_n = 0$, it follows, for n large

$$\lambda(R)t_n\phi - \Delta w_n = f_\epsilon(u_n - a) = f(0+)(\frac{u_n - a}{\epsilon} + 1).$$

Multiplying by ϕ, integrating and letting $t_n \to 0$ one finds

$$0 = f(0+)(1 - \frac{a}{\epsilon}) \int_\Omega \phi,$$

a contradiction, proving (6).

By standard arguments in global bifurcation theory (see, e.g. [**3**]), (5) and (6) yield the existence of a global branch of positive solutions of ($4a$–ϵ) emanating from $(\epsilon, 0)$. ∎

Remarks 3. (a) As remarked above, $S(\epsilon) \subset [0, a^*] \times V(b^*)$. Then, by regularity, it follows readily that $(a, u) \in S(\epsilon)$ are bounded in $C^{1,\alpha}$, uniformly in ϵ.

(b) By the well known result of [**10**], the positive solutions of ($4a$–ϵ) are radial: $u = u(r)$ and $u'(r) < 0, \forall\, r > 0$.

Step 2. (Limit as $\epsilon \to 0$). To obtain the branch S of solutions of ($3a$–R) we shall let $\epsilon \to 0$ and show that $S(\epsilon)$ converges (in a suitable sense) to S. This will be obtained, as in [**6**], by means of the following topological lemma:

Lemma 4 ([**14**, Thm.9.1]). *Let X be a metric space and let S_n be a sequence of connected subsets of X. Let*

(i) $\liminf(S_n) \neq \emptyset$;
(ii) $\cup S_n$ *is precompact.*

Then $S =: \limsup(S_n)$ is (non empty) compact and connected.

In our case we take $X = \mathbf{R}^+ \times E$, $\epsilon = \frac{1}{n}$ and $S_n = S(\frac{1}{n})$. Lemma 2 implies that S_n is connected and $(0,0) \in \liminf(S_n)$. Moreover, using Remark 3-(a) it is easy to see that (ii) holds. By Lemma 4, it follows that $S = S(R) \neq \emptyset$ is compact and connected. In addition there results $S(R) \subset [0, a^*] \times V(b^*)$.

Using (f1), Remark 3-(a) and regularity, it follows readily that for any $u \in S$ there results $u \in C^2(B(R) - T(a)) \cap C^1(B(R))$.

As a consequence of Remark 3-(b), one has that $u = u(r)$ and $u'(r) \leq 0$, $\forall\, r > 0$, for any $u \in S(R)$. To show that $u'(r) < 0$, $\forall\, r > 0$ one uses the maximum principle applied to $u'(r)$, similarly than in [**6**]. As a consequence $|T(a)| = 0$ and (iv) follows.

The same argument used for (6) allows us to prove (iii). Roughly, if $-\Delta u_n = f(u_n)$ with $u_n \in S$ and $|u_n|_\infty \to 0$, it follows

$$\lambda(R) \int_{B(R)} u_n \phi = \int_{B(R)} f(u_n)\phi$$

Passing to the limit one finds $0 = \int_{B(R)} f(0^+)\phi$, a contradiction.

By (iii) and $S(R) \subset [0, a^*] \times V(b^*)$ it follows that $S \cap \Sigma_0 \neq \emptyset$, yielding (ii).

This completes the proof of Theorem 1. ∎

As anticipated before, Theorem 1 holds in greater generality.

Let Ω be a bounded domain in $\mathbf{R}^N (N \geq 2)$, with smooth boundary $\partial\Omega$. We suppose that Ω is Steiner symmetric with respect to the hyperplane $\{x_1 = 0\}$,say. For shortness, we will simply say that Ω is Steiner symmetric. For $u \in H_0^1(\Omega)$, $u(x) > 0$, we denote by $u^*(x)$ the Steiner symmetrization (with respect to $\{x_1 = 0\}$) of u. For definitions concerning symmetrization, see, for ex., [**12**, § II.1-f].

Let $\lambda(\Omega)$ be the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$, and suppose that f satisfies (f1-2) and (f3) with Ω (resp.$\lambda(\Omega)$) instead of $B(R)$ (resp. $\lambda(R)$). Moreover, let $S(\Omega)$, $\Sigma(\Omega)$ and $\Sigma_0(\Omega)$ denote the sets $S(R), \Sigma(R)$ and $\Sigma_0(R)$, respectively, with Ω instead of $B(R)$.

Theorem 1'. *Let Ω be Steiner symmetric and suppose* (f1-2-3) *hold. Then there exists a global branch $S(\Omega) \subset cl(\Sigma(\Omega))$ (i.e. a closed, connected component of $\Sigma(\Omega)$) such that:*

(i) $(0,0) \in S(\Omega)$ *and if* $(a, 0) \in S(\Omega)$ *then* $a = 0$;

(ii) $S(\Omega)$ *is bounded in* $\mathbf{R}^+ \times E$ *and* $S(\Omega) \cap \Sigma_0(\Omega) \neq \emptyset$;

(iii) *if* $(a, u) \in S(\Omega)$, *with* $0 < |u|_\infty$ *small, then* $a > 0$. *As a consequence, there is* $a_0 > 0$ *such that for all* $a \in]0, a_0[$ $(1a)$ *has at least two distinct solutions with* $(a, u) \in S(\Omega)$.

(iv) *any* $u \in S(\Omega)$ *is Steiner symmetric and* $\partial u/\partial x_1 < 0\ \forall x_1 > 0$ *and* $|T(a)| = meas[T(a)] = 0$.

The proof of Theorem 1' is the same as that of Theorem 1 (with obvious changes) and is left to the reader.

Remarks 5. (i) Theorem 1' should be compared with the multiplicity results of [**5**], which are variational in nature.

(ii) The preceding arguments provide a complete proof of some results stated in [**2**], where we refer for applications to the Grad-Shafranov equation.

(iii) For a global result concerning $(1a)$ in a multivalued sense, see [**1**]1.

Example 6. Following [**8**] we consider a ionized gas confined in an electrically insulated cylinder having as cross section a bounded domain $\Omega \subset \mathbf{R}^2$.

Denote by v the temperature of the gas, by $\sigma = \sigma(v)$ its electrical conductivity and by δ its temperature of discharge. Since the gas is ionized, σ has a simple discontinuity at $v = \delta$: precisely we suppose that

$$\sigma(v) = 0 \quad \text{for} \quad v \leq \delta,$$

$$\sigma(v) > 0 \quad \text{and continuous for} \quad v > \delta.$$

Assuming that the electric field $\mathcal{E}$ is constant and directed along the axis of the cylinder, and up to a normalization of the constants, such as the thermal conductivity and $|\mathcal{E}|$, one is led to the equations

$$-\Delta v = 0 \quad \text{in} \quad \{x \in \Omega : v(x) \le \delta\} \tag{8}$$

$$-\Delta v = \sigma(v) \quad \text{in} \quad \{x \in \Omega : v(x) > \delta\} \tag{9}$$

together with the boundary condition

$$v = v_0 \quad \text{on} \quad \partial\Omega \tag{10}$$

where $v_0 \in \mathbf{R}$. Let us assume that $v_0 \le \delta$, otherwise the problem is unaffected by the discontinuity.

In order to evidence the dependence on δ and v_0, it is convenient to set $u = v - v_0$ and $a = \delta - v_0 (\ge 0)$.

Introducing the Heaviside function $H : \mathbf{R} \longrightarrow \mathbf{R}$ defined by

$$H(s) = 0 \ \forall\, s \le 0, \ H(s) = 1 \ \forall\, s > 0,$$

we set

$$f(s) = H(s)\sigma(s+\delta). \tag{11}$$

With these notations, (8-9-10) become:

$$-\Delta u = f(u-a) \quad \text{in} \quad \Omega, \ u = 0 \quad \text{on} \quad \partial\Omega.$$

This is exactly problem ($1a$) and the preceding bifurcation results apply.

3. Global Bifurcation for Problems in $\mathbf{R}^N$. In this section we deal with global branching for problem ($2a$).

As usual, ($2a$) is approximated by Dirichlet problems ($3a$–R). The existence of a global branch of positive solutions of ($2a$) will be estabilished with another application of Lemma 4, letting $R \to \infty$. The meaning of positive solution of ($2a$) is the same given for those of ($3a$–R).

We note explicitely that in all this section $N \ge 3$.

Here we suppose that f satisfies (f1-2) and

(f4) There exist $c \ge d > 0$ such that $d \le f(s) \le c$ for all $s > 0$.

Fixed $a, R > 0$, let $(a, u_R) \in S(R)$. Set $T_R(a) = \{x \in \mathbf{R}^N : u_R(r) = a\}$ and denote by $\rho = \rho(R, a)$ the radius of the sphere $T_R(a)$. For $v \in H_0^1(B(R))$, respectively $u \in \mathcal{D}^{1,2}(\mathbf{R}^N)$, we set

$$\|v\|_R^2 = \int_{B(R)} |\nabla v|^2 \quad \text{and} \ \|u\|^2 = \int_{\mathbf{R}^N} |\nabla u|^2.$$

In order to pass to the limit as $R \to \infty$ we need to estimate $\rho(R, a)$.

Lemma 7. *If* (f1-2-4) *hold then* $\rho = \rho(R, a)$ *satisfies:*

$$kd^2\rho^{N+2} \le \int_{\{u_R>a\}} |\nabla u|^2 \le kc^2\rho^{N+2}, \tag{12}$$

where $k = \dfrac{\omega_{N-1}}{N^2(N+2)}$ *and* ω_{N-1} *denotes the measure of the unit sphere* $\partial B(1)$.

PROOF : Being a radial solution of ($3a$–R) $u = u_R(r)$ satisfies

$$\frac{d^2u}{dr^2} + \frac{N-1}{r}\frac{du}{dr} + f(u-a) = 0,$$

namely

$$(r^{N-1}u')' = -r^{N-1}f(u-a).$$

By this and the fact that $f = 0$ for $s \le 0$, there results

$$u_R(r) = a + \int_r^{\rho} s^{1-N}ds \int_0^s t^{N-1}f(u(t)-a)dt, \quad \text{for} \quad 0 \le r \le r; \tag{13}$$

$$u_R(r) = \frac{a}{1-(\frac{\rho}{R})^{N-2}}[(\frac{\rho}{r})^{N-2} - (\frac{\rho}{R})^{N-2}], \quad \text{for} \quad \rho \le r \le R. \tag{13'}$$

with $\rho = \rho(R, a)$ determined by

$$u'(\rho^-) = u'(\rho^+). \tag{13"}$$

By (13) we deduce:

$$\int_{\{u_R>a\}} |\nabla u_R|^2 = \omega_{N-1}\int_0^{\rho} s^{1-N}ds[\int_0^s t^{N-1}f(u_R(t)-a)dt]^2 \tag{14}$$

Using (14) and (f4) the lemma follows. ∎

We denote by Σ the set of pairs $(a, u) \in \mathbf{R}^+ \times \mathcal{D}^{1,2}(\mathbf{R}^N)$ such that u is a positive solution of ($2a$) and by $T(a)$ the set $\{x \in \mathbf{R}^N \ : \ u(x) = a\}$.

We are now in position to state:

Theorem 8. *Let* $N \ge 3$ *and* (f1-2-4) *hold. Then there is a global, unbounded branch* $\mathbf{S} \subset cl(\Sigma)$ *such that:*

(i) $(0, 0) \in \mathbf{S}$;

(ii) *if* $(a, u) \in \mathbf{S}$ *then* u *is a radial, positive solution of* ($1a$), $u'(r) < 0\ \forall\ r > 0$ *and* $|T(a)| = 0$.

PROOF : Since the proof is very similar to that of [6], based again on Lemma 4, we will be sketchy.

First, any solution $u_R \in S(R)$ can be extended on all $\mathbf{R}^N$ setting $u_R = 0$ for $r > R$. Fixed an integer $j \gg 1$, let $X_j = \{(a, u) \in \mathbf{R}^+ \times \mathcal{D}^{1,2}(\mathbf{R}^N) \ : \ a^2 + \|u\|^2 \le j^2\}$. Taken a sequence $R_n \uparrow \infty$, we set $S_{n,j} = S(R_n) \cap X_j$. We claim that (i) and (ii) of Lemma 4 hold true for $S_n = S_{n,j}$. In fact, (i) is trivially verified. As for (ii), let us take a sequence $(a_h, u_h) \in \cup_{n\in\mathbf{N}} S_{n,j}$. This means that u_h is a solution of $(3a_h - R_{n(h)})$ for some $R_{n(h)}$. Set $\rho(h) = \rho(R_{n(h)}, a_h)$. Since $a_h \le j$, and $\|u_h\| \le j$, the left hand side of (12) implies:

$$kd^2 \rho(h)^{N+2} \le \int_{\{u_h > a_h\}} |\nabla u_h|^2 \le \|u_h\|^2$$

Hence there exists $\rho^* > 0$ such that $\rho(h) \le \rho^*$ for all h, and $\{u_h > a_h\} \subset \{r < \rho^*\}$. From this and since $-\Delta u = f(u-a) = 0$ on $\{u < a\}$, it follows as in [**6**] that u_h converges, up to a subsequence, in $\mathcal{D}^{1,2}(\mathbf{R}^N)$. This shows that (ii) holds. Applying Lemma 4 we find a non-empty, closed, connected set $S_j = \limsup(S_{n,j})$.

Next, note that for $a = 0$ problem ($3a$–R) has positive solutions $u \in \Sigma_0(R)$ and obviously one has $\rho(R, 0) = R$. Hence, using the left hand side of (12) one finds $\|u\|^2 \ge kd^2 R^{N+2}$ for all $u \in \Sigma_0(R)$. Since each $S(R)$ is connected, it follows that $\forall\ j$ there is $n(j)$ such that, for all $n \ge n(j)$ there exists $(a_n, u_n) \in S(R_n)$ such that $a_n^2 + \|u_n\|^2 = j^2$. The preceding compactness argument shows that, up to a subsequence, (a_n, u_n) converges to some $(a, u) \in S_j$ with $a^2 + \|u\|^2 = j^2$. Therefore set $\mathbf{S} = \cup_{j\in\mathbf{N}} S_j$ is unbounded, yielding the searched global branch.

The required properties of the solutions listed in (ii) follow by standard arguments as in [**6**]. ∎

In order to control the behaviour of the branch **S** we state:

Lemma. *Let u be any radial solution of* (2a) *and let $\rho(a) = \{r : u(r) = a\}$. Then there results*

$$\sqrt{\frac{N(N-2)a}{c}} \le \rho(a) \le \sqrt{\frac{N(N-2)a}{d}}. \tag{15}$$

PROOF : First of all we remark that formulas (13'-13") hold true for $u(r)$ with 0 instead of ρ/R.

Using (13") one finds

$$\rho^{1-N} \int_0^\rho t^{N-1} f(u(t) - a)dt = (N-2)\frac{1}{\rho}a$$

and (15) follows. ∎

Theorem 10. *Let $N \ge 3$ and* (f1-2-4) *hold. Then $\forall a > 0$ any positive, radial solution u of* (1a) *satisfies*

$$\frac{a^{(N+2)/2}K}{c^{(N-2)/2}}[\frac{d^2}{c^2}\frac{N-2}{N+2} + 1] \le \|u\|^2 \le \frac{a^{(N+2)/2}K}{d^{(N-2)/2}}[\frac{c^2}{d^2}\frac{N-2}{N+2} + 1] \tag{16}$$

where

$$K = \frac{\omega_{N-1}}{N}[(N-2)N]^{N/2}.$$

Hence $\forall\, a > 0$, $(2a)$ *possesses a positive, radial solution* u, *with* $(a, u) \in \mathbf{S}$.

PROOF : As remarked in Lemma 9, $u(r)$ has the form (13'-13") with 0 instead of ρ/R. In particular: (i) Lemma 7 holds with u instead of u_R; and (ii) one has:

$$\int_{\{u<a\}} |\nabla u|^2 = \omega_{N-1} \int_\rho^\infty [(2-N)a\rho^{N-2}]^2 r^{1-N} dr = \omega_{N-1}(N-2)a^2\rho^{N-2}$$

Then:

$$\|u\|^2 = \int_{\{u>a\}} |\nabla u|^2 + \int_{\{u<a\}} |\nabla u|^2 = \int_{\{u>a\}} |\nabla u|^2 + \omega_{N-1}(N-2)a^2\rho^{N-2}.$$

By (12) it follows:

$$kd^2\rho^{N+2} + \omega_{N-1}(N-2)a^2\rho^{N-2} \le \|u\|^2 \le kc^2\rho^{N+2} + \omega_{N-1}(N-2)a^2\rho^{N-2}.$$

Using the estimates for ρ found in Lemma 9 we obtain (16).

Lastly, suppose that, for some $a > 0$, $(2a)$ has no solutions u, with $(a, u) \in \mathbf{S}$. Since, by Theorem 8, $\mathbf{S}$ is connected, unbounded and bifurcates from $(0,0)$, there would exist $A > 0$ and sequences $(a_n, u_n) \in \mathbf{S}$ with $a_n \uparrow A$ and $\|u_n\| \to \infty$. This contradicts the right hand side of (16). ∎

Remarks 11. (i) If $N = 2$ problem $(1a)$ can have no (radial) solutions. This is the case when $f(s) = c > 0\ \forall\, s > 0$.

(ii) Using the same arguments as above, one can find the following estimate for the L^∞ norm $|u|_\infty$ of any radial solution of $(1a)$:

$$a[1 + \frac{d}{c}\frac{N-2}{2}] \le |u|_\infty = u(0) \le [1 + \frac{c}{d}\frac{N-2}{2}]a.$$

(iii) A global bifurcation result for vortex rings in an ideal fluid is proved in [6]. Such a paper does not provide any bound of the type we prove in Theorem 10, and the behaviour of the branch cannot be controlled (see, [6, §4]). Unfortunately our approach relies on the special symmetry of the problem $(2a)$ and we are not able to extend the arguments to cover the case of vortex rings, where a cylindrical (rather than radial) symmetry arises. An extension of the Amick & Turner results describing the asymptotic behaviour of the bifurcating branch would be an interesting question to pursue.

Example 12. The application discussed in Example 6 extends to the case of an isotropic, ionized gas filling all of $\mathbf{R}^3$. One is led to the free boundary problem (8-9) with Ω substituted by $\mathbf{R}^3$. A natural boundary condition to be added is now

$$v \longrightarrow v_0 \quad \text{as} \quad |x| \longrightarrow \infty. \tag{17}$$

If f is given by (11) and keeping the same notations as in Example 6, (8-9-17) give rise to a problem of the form $(2a)$, and the preceding bifurcation results apply.

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