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MONOMORPHISMS IN THE CATEGORIES OF CONNECTED ALGEBRAIC SYSTEMS WITH STRONG HOMOMORPHISMS

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INTRODUCTION

The actual motive to this work was given by the effort to describe monomorphisms in the category of all connected partially ordered sets with strong surjective homomorphisms.

In contrast to many categories of other algebraic structures where monomorphisms are always identical with injective morphisms, in this category we can find a number of examples where this is not the case (for, see [3]). But the complete characterization of monomorphisms in this category was not known.

The aim of this work is not only to solve this special problem but also to extend the obtained results to categories of more complicated algebraic structures of similar type as connected partially ordered semigroups, groups, rings, etc.

To this purpose we consider a certain formal type of a category of algebraic systems (see Definition 1.2.) involving among others also the above mentioned categories.

Theorem 2.3. gives a complete description of monomorphisms in the categories of that type. It is also proved that each category of this type is locally small (Theorem 2.6.).

The 3rd chapter contains formulations of the main results for several special categories.

In the end we would like to thank doc. RNDr. Milan Sekanina, CSc. We are obliged to him for many valuable suggestions.

1. INTRODUCTORY DEFINITIONS AND REMARKS

Notation and all used terms of the theory of categories are taken over from [1]. Similarly all closer undefined terms concerning the theory of algebraic systems are to be comprehended in the sense of [2].

We shall further agree on the following notation:

(1) If ϱ is a binary relation on a set **A**, then we denote as ϱ^{-1} the binary relation defined like this: $\varrho^{-1}(x, y)$ if and only if $\varrho(y, x)$ $(x, y \in \mathbf{A})$. Here $\varrho(y, x)$ means that ϱ is a true relation for y and x. Instead of $\varrho(y, x)$ we shall also write $y\varrho x$. In other words, we make no difference between the relations and the predicates corresponding to them. The relation ϱ^{-1} is called the inverse of ϱ. Instead of $a_0\varrho_1 a_1$, $a_1\varrho_2 a_2$, ..., ..., $a_{n-1}\varrho_n a_n$ we shall write shorter $a_0\varrho_1 a_1 \varrho_2 a_2, \ldots, a_{n-1}\varrho_n a_n$.

(2) Let **f** be a mapping of **A** into **B** (**A**, **B** are sets). Then $\mathbf{f}^{-1}(b)$ denotes the complete pre-image of $b \in \mathbf{B}$ under the mapping **f**.

(3) The symbol $|\mathbf{A}|$ stands for the cardinality of the set **A**.

(4) $\langle \mathbf{A}, \mathbf{\Omega}_F, \mathbf{\Omega}_P \rangle$ will denote an algebraic system with an underlying set **A**, an operator domain $\mathbf{\Omega}_F = \{F_i^{(n_i)} \mid i < k\}$ and a nonempty predicate domain $\mathbf{\Omega}_P = \{\varrho_j \mid j < l\}$, all relations of which are binary (i, j, k, l are ordinal numbers). The index n_i associated with the symbol of the basic operation represents its arity. As far as a misunderstanding is excluded, we shall use also any of the symbols $F_i^{(n)}$, $F^{(n)}$, F_i or only F for the basic operations. Further on we shall deal only with algebraic systems of the above introduced signature (ordinals k, l and n_i for each $i < k$ are assumed fixed). Let $\mathbf{\Sigma}$ denote the class of all these algebraic systems. For operator and predicate domains we shall always use the same symbols $\mathbf{\Omega}_F$ and $\mathbf{\Omega}_P$, respectively without distinguishing between separate algebraic systems of $\mathbf{\Sigma}$. That is why we shall write instead of $\langle \mathbf{A}, \mathbf{\Omega}_F, \mathbf{\Omega}_P \rangle$ only briefly **A**, whenever it is clear from the context that **A** stands for an algebraic system. For each $\mathbf{A} \in \mathbf{\Sigma}$ we denote $\mathbf{\Omega} = \{\varrho_j \mid j < 2l + 1$ where $\varrho_{l+j'} = \varrho_{j'}^{-1}$ for every $j' < l$ and $\varrho_{2l} = \Delta\}$ where Δ is the equality relation, i.e. for $x, y \in \mathbf{A}$ Δ is a true relation if and only if $x = y$.

(5) Let T be a term that has the signature of $\mathbf{\Sigma}$ and $x_1, \ldots, x_n, x'_1, \ldots, x'_n, v_1, \ldots, v_q$ object variables (n, q are natural numbers). If we write the term T in the form $T(x_i, x'_j, v_s)$, it means that some of the object variables are contained in T and at the same time they are the only ones occurring there. Thus it can even happen that T has not any object variable at all. If $a_1, \ldots, a_n, a'_1, \ldots, a'_n, e_1, \ldots, e_q$ are some elements of **A**, then $T(a_i, a'_j, e_s)$ is the value of the term T where each of the object variables x_i, x'_j, v_s occurring in T is replaced by respective element a_i, a'_j, e_s.

(6) Let **A** be an algebraic system of $\mathbf{\Sigma}$. We denote $\mathbf{M}_A(a_1\varrho_1 a'_1, \ldots, a_n\varrho_n a'_n)$ a family of elements $a_1, \ldots, a_n, a'_1, \ldots, a'_n$ in **A** with the property $a_1\varrho_1 a'_1, \ldots, a_n\varrho_n a'_n$ where $\varrho_1, \ldots, \varrho_n \in \mathbf{\Omega}$.

Definition 1.1. An algebraic system **A** is called directed if to each nonempty finite subset **M** of **A** we can find a $\varrho \in \mathbf{\Omega}$ and an $a_0 \in \mathbf{A}$ with the property $a\varrho\, a_0$ for every $a \in \mathbf{M}$.

Definition 1.2. Let us denote as $\mathbf{U}_{\mathbf{\Sigma}}(\mathfrak{A})$ the category defined in the following way:

a) Objects of the category $\mathbf{U}_{\mathbf{\Sigma}}(\mathfrak{A})$ are exactly those algebraic systems $\mathbf{A} \in \mathbf{\Sigma}$ for which the following three conditions are true.

I. The algebraic system **A** satisfies a prescribed collection $\mathfrak{A}$ of axioms for operations and relations. Each of these axioms must be heritable to subsystems and direct products of systems.

II. The algebraic system **A** is connected, which means that for each pair a, a' of elements in **A** is $a\varrho_1 \dots \varrho_n a'$. Here $\varrho_1 \dots \varrho_n$ is the composition of suitable relations $\varrho_1, \dots, \varrho_n \in \mathbf{\Omega}$.

III. Let us have given $F^{(n)} \in \mathbf{\Omega}_F$ $(n \geqq 1)$, $\varrho_1, \dots, \varrho_n \in \mathbf{\Omega}$ and a finite set family $\mathcal{M} = \{\mathbf{M}_A^p(a_1^p \varrho_1 a_1'^p, \dots, a_n^p \varrho_n a_n'^p| \, p \in \{1, \dots, v\}, \ v \text{ natural number}\}$ quite arbitrary. Then there exist terms $T_0(x_i, x_j', v_s)$, $T_1(x_i, x_j', v_s)$, $\dots$, $T_r(x_i, x_j', v_s)$,[1] relations $\sigma_1, \dots, \sigma_r \in \mathbf{\Omega}$ and, if necessary, elements $e_1, \dots, e_q \in \mathbf{A}$ with the following properties:

(1) $T_0 = F^{(n)}(x_1, \dots, x_n)$ and $T_r = F^{(n)}(x_1', \dots, x_n')$.

(2) If **A** is not a directed system and $\mathbf{\Omega}_F$ does not contain any 0-ary operation, then the object variables $v_1, \dots, v_q$ do not occur in the terms $T_0, T_1, \dots, T_r$.

(3) For every $p = 1, \dots, v$ is true

$$F^{(n)}(a_1^p, \dots, a_n^p) =$$
$$= T_0(a_i^p, a_j'^p, e_s)\, \sigma_1 T_1(a_i^p, a_j'^p, e_s)\, \sigma_2, \dots, \sigma_{r-1} T_{r-1}(a_i^p, a_j'^p, e_s)\, \sigma_r T_r(a_i^p, a_j'^p, e_s) =$$
$$= F^{(n)}(a_1'^p, \dots, a_n^p).$$

b) Morphisms in the category $\mathbf{U}_\Sigma(\mathfrak{A})$ are exactly all strong[1] surjective homomorphisms of algebraic systems. Condition III. must be omitted whenever $\mathbf{\Omega}_F = \emptyset$ or $\mathbf{\Omega}_F$ contains only 0-ary operations.

Let further $\mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$ denote the full subcategory of all finite systems in $\mathbf{U}_\Sigma(\mathfrak{A})$.

Lemma 1.1. *If $\mathbf{\Omega}_F$ contains at least one n-ary operation where $n \geqq 1$, then condition III. is true for each directed algebraic system in $\mathbf{\Sigma}$.*

Proof. Let **A** be a directed algebraic system in $\mathbf{\Sigma}$. Let us have an operation $F^{(n)} \in \mathbf{\Omega}_F$ $(n \geqq 1)$, relations $\varrho_1, \dots, \varrho_n \in \mathbf{\Omega}$ and a set $\mathcal{M} = \{\mathbf{M}_A^p(a_1^p \varrho_1 a_1'^p, \dots, a_n^p \varrho_n a_n'^p) \,|\, p \in \in \{1, \dots, v\},\ v \text{ natural number}\}$. Since **A** is directed, we can find $\varrho \in \mathbf{\Omega}$, $e_1 \in \mathbf{A}$ with the property $F^{(n)}(a_1^p, \dots, a_n^p)\, \varrho e_1, e_1 \varrho^{-1} F^{(n)}(a_1'^p, \dots, a_n'^p)$ for every $p = 1, \dots, v$. Then $F^{(n)}(x_1, \dots, x_n)$, v_1 and $F^{(n)}(x_1', \dots, x_n')$ are the terms, $\sigma_1 = \varrho$, $\sigma_2 = \varrho^{-1}$ the relations and $e_1 \in \mathbf{A}$ the element as required.

Definition 1.3. Let $\mathbf{A} \in \mathbf{\Sigma}$ and $a_0, \dots, a_n \in \mathbf{A}$ (n is a nonnegative integer). The sequence $\alpha = (a_0, \dots, a_n)$ is called the path in **A** from a_0 to a_n (we write $\alpha\colon a_0 \to a_n$) if to every $j = 1, \dots, n$ we can find a $\varrho_j \in \mathbf{\Omega}$ with $a_{j-1} \varrho_j a_j$. The number n is the length of the path α (we write $L(\alpha) = n$). Each path in **A** with length 0 is said to be an empty path. If $\alpha = (a_0, \dots, a_n)$ is a path in **A**, then $\alpha^{-1} = (a_n, \dots, a_0)$ will be called the path inverse to α (the existence of α^{-1} is always guaranteed because $a_{j-1} \varrho_j a_j$ for some $j = 1, \dots, n$ implies $a_j \varrho_j^{-1} a_{j-1}$).

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[1] Notation for terms coincides with that in (5) on page 54. But here not only n but also q depends on the choice of $F^{(n)}$, $\varrho_1, \dots, \varrho_n$ and $\mathcal{M}$.

[1] Let us note that these homomorphisms are strong even with respect to all relations of $\mathbf{\Omega}$.

Definition 1.4. Let us have two paths $\alpha = (a_0, \dots, a_n)$ and $\beta = (b_0, \dots, b_m)$ with $a_n = b_0$ in $\mathbf{A} \in \mathbf{\Sigma}$. Then the path $\alpha\beta = (a_0, \dots, a_n, b_1, \dots, b_m)$ is called a composition of paths α and β. Evidently $L(\alpha\beta) = L(\alpha) + L(\beta)$.

Remark 1.1. If $\mathbf{A} \in \mathbf{U}_\Sigma(\mathfrak{A})$, then by condition II., to each pair $a, a' \in \mathbf{A}$ there exists a path $\alpha\colon a \to a'$. Let $\mathbf{W}(\mathbf{A})$ denote the set of all paths in $\mathbf{A}$ together with the partial operation of composition. When identifying all empty paths with the corresponding elements, so we can suppose that the underlying set $\mathbf{A}$ is a subset in $\mathbf{W}(\mathbf{A})$. Thus a morphism $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ in $\mathbf{U}_\Sigma(\mathfrak{A})$ can be extended in a natural way to a mapping of $\mathbf{W}(\mathbf{A})$ into $\mathbf{W}(\mathbf{B})$. For the mapping obtained in this way we shall use the same symbol $\mathbf{f}$ as for the corresponding morphism. Therefore we can write $\mathbf{f}(\alpha) = (\mathbf{f}(a_0), \dots, \mathbf{f}(a_n))$ for each $\alpha = (a_0, \dots, a_n) \in \mathbf{W}(\mathbf{A})$. Also the equation $L(\mathbf{f}(\alpha)) = L(\alpha)$ is true. We can see, too that the composition of paths is respected by $\mathbf{f} : \mathbf{W}(\mathbf{A}) \to \mathbf{W}(\mathbf{B})$, i.e. $\mathbf{f}(\alpha\beta) = \mathbf{f}(\alpha)\,\mathbf{f}(\beta)$. Further it is evident that the equation $(\mathbf{gf})(\alpha) = \mathbf{g}(\mathbf{f}(\alpha))$ is true whenever $\mathbf{gf}$ is defined in $\mathbf{U}_\Sigma(\mathfrak{A})$.

Definition 1.5. Let $\mathbf{A} \in \mathbf{U}_\Sigma(\mathfrak{A})$. The paths $\alpha = (a_0, \dots, a_n)$ and $\beta = (b_0, \dots, b_n)$ in $\mathbf{A}$ of the same length are said to be parallel if to every $j = 1, \dots, n$ there exists a $\varrho_j \in \mathbf{\Omega}$ with $a_{j-1}\varrho_j a_j$ and $b_{j-1}\varrho_j b_j$ (we write $\alpha // \beta$).

Definition 1.6. Let $\alpha = (a_0, \dots, a_n)$ and $\beta = (b_0, \dots, b_m)$ be two paths in $\mathbf{A}$ where $\mathbf{A} \in \mathbf{U}_\Sigma(\mathfrak{A})$. If the path α can be written in the form $\alpha = \alpha_0(b_0, b_1)\,\alpha_1(b_1, b_2) \dots \alpha_{m-1} \times (b_{m-1}, b_m)\,\alpha'_m$ for $m > 1$ or in the form $\alpha = \alpha_0(b_0)\,\alpha'_0$ in case $\beta = (b_0)$ is empty ($\alpha'_0, \alpha'_m, \alpha_i \in \mathbf{W}(\mathbf{A})$ with $i = 0, 1, \dots, m - 1$), then we say that β is a subpath of α (we write $\beta \subseteq \alpha$). If $\beta \subseteq \alpha$, then $L(\beta) \leqq L(\alpha)$ holds true.

Definition 1.7. Let $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ be a morphism in the category $\mathbf{U}_\Sigma(\mathfrak{A})$. The elements $a, a' \in \mathbf{A}$ with $a \neq a'$ and $\mathbf{f}(a) = \mathbf{f}(a')$ are called f-symmetric if the following two conditions are true.

(i) To each pair of elements $x, y \in \mathbf{A}$ where x stands in some relation $\varrho \in \mathbf{\Omega}$ to y (it is sufficient to suppose $\varrho \in \mathbf{\Omega}_P$) there exist parallel paths α_a and $\alpha_{a'}$ starting from a and a', respectively and satisfying $\mathbf{f}(\alpha_a) = \mathbf{f}(\alpha_{a'})$, $(x, y) \subseteq \alpha_a$, $(x, y) \subseteq \alpha_{a'}$ (parallelness is with $x\varrho y$ in α_a and $\alpha_{a'}$).

(ii) To each basic operation $F \in \mathbf{\Omega}_F$ there exist parallel paths $\beta_a : \to F(a, \dots, a)$ and $\beta_{a'} : a' \to F(a', \dots, a')$ with $\mathbf{f}(\beta_a) = \mathbf{f}(\beta_{a'})$.[1]

Remark 1.2. As far as the objects of $\mathbf{U}_\Sigma(\mathfrak{A})$ are models ($\mathbf{\Omega}_F = \emptyset$), then condition (ii) in the preceding definition is to be left out.

Definition 1.8. The category $\mathbf{U}_\Sigma(\mathfrak{A})$ or $\mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$ is said to be of type (i) (or alternatively of type (ii)) if with an arbitrarily chosen object $\mathbf{A} \in \mathbf{U}_\Sigma(\mathfrak{A})$ or $\mathbf{A} \in \mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$ and a morphism $\mathbf{f}$ starting from $\mathbf{A}$ each elements $a, a' \in \mathbf{A}$, $a \neq a'$, $\mathbf{f}(a) = \mathbf{f}(a')$ satisfying condition (i) (or alternatively condition (ii)) are already f-symmetric.

[1] If F is an 0-ary operation, then $F(a, \dots, a)$ or $F(a', \dots, a')$ denotes the element selected in A by F. In particular, $F(a, \dots, a) = F(a', \dots, a')$.

Lemma 1.2. *Let* $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ *be a morphism in* $\mathbf{U}_\Sigma(\mathfrak{A})$. *If there are* $\beta_a : a \to a_0$ *and* $\beta_{a'} : a' \to a_0$ *parallel paths with* $a \neq a'$ *and* $\mathbf{f}(\beta_a) = \mathbf{f}(\beta_{a'})$, *then* a, a' *satisfy condition* (*i*).

Proof. Let $x, y \in \mathbf{A}$ be elements where $x\varrho y$ for some relation $\varrho \in \mathbf{\Omega}$. Because of the connectedness of **A**, there exists a path $\beta : a_0 \to x$. Hence it is clear that the paths $\alpha_a = \beta_a\beta(x, y)$, $\alpha_{a'} = \beta_{a'}\beta(x, y)$ are parallel and satisfy $\mathbf{f}(\alpha_a) = \mathbf{f}(\alpha_{a'})$, $(x, y) \subseteq \subseteq \alpha_a$, $(x, y) \subseteq \alpha_{a'}$. But this is the assertion.

Corollary 1.1. *If there is at least one* 0*-ary operation in* $\mathbf{\Omega}_F$, *then* $\mathbf{U}_\Sigma(\mathfrak{A})$ *and* $\mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$ *are categories of type* (*ii*).

2. MONOMORPHISMS IN THE CATEGORIES $\mathbf{U}_\Sigma(\mathfrak{A})$ AND $\mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$

Theorem 2.1. *Let* $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ *be a morphism in* $\mathbf{U}_\Sigma(\mathfrak{A})$ (*or in* $\mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$). *Suppose that at least one of the following assumptions is true.* (1) *No pair of different elements in* **A** *satisfies condition* (*i*). (2) *The category* $\mathbf{U}_\Sigma(\mathfrak{A})$ (*or* $\mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$) *is of type* (*i*) *and there do not exist f-symmetric elements in* **A**. *Then each two parallel paths* $\alpha = (a_0, \ldots, a_n)$, $\beta = (b_0, \ldots, b_n) \in \mathbf{W}(\mathbf{A})$ *with* $\mathbf{f}(\alpha) = \mathbf{f}(\beta)$ *coincide whenever* $a_i = b_i$ *for some index* $i \in \{0, 1, \ldots, n\}$.

Proof. In case $n = 0$ the statement of the theorem is trivial. Suppose $n \geqq 1$ and let us have two different parallel paths $\alpha = (a_0, \ldots, a_n)$, $\beta = (b_0, \ldots, b_n) \in \mathbf{W}(\mathbf{A})$ such that $\mathbf{f}(\alpha) = \mathbf{f}(\beta)$ and $a_i = b_i$ for some index $i \in \{0, 1, \ldots, n\}$. In particular, there exists a $j \in \{0, 1, \ldots, n\}$, $j \neq i$ with $a_j \neq b_j$. We can confine ourselves to the case $j < i$. Then the paths $\alpha_0 = (a_j, a_{j+1}, \ldots, a_i$:, $\beta_0 = (b_j, b_{j+1}, \ldots, b_i)$ are parallel and $\mathbf{f}(\alpha_0) = \mathbf{f}(\beta_0)$. Thus, by lemma 1.2., the elements a_j and b_j satisfy condition (*i*), which is in contradiction with either of the assumptions (1) and (2).

Corollary 2.1. *Supposing the same as in theorem* 2.1. *each two morphisms* $\mathbf{g}, \mathbf{h} : \mathbf{C} \to \to \mathbf{A}$ *in* $\mathbf{U}_\Sigma(\mathfrak{A})$ *with* $\mathbf{fg} = \mathbf{fh}$ *coincide whenever* $\mathbf{g}(c) = \mathbf{h}(c)$ *for some* $c \in \mathbf{C}$.

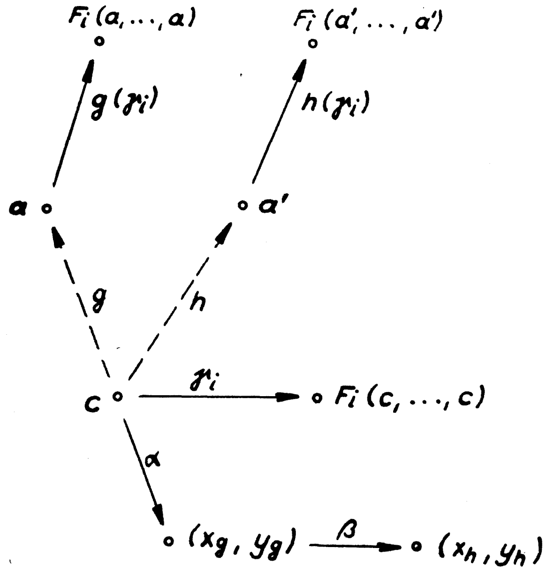
Proof. Let $\mathbf{fg} = \mathbf{fh}$ for some $\mathbf{g}, \mathbf{h} : \mathbf{C} \to \mathbf{A}$ and let further $c \in \mathbf{C}$ be that element for which $\mathbf{g}(c) = \mathbf{h}(c)$. Take an arbitrary element $x \in \mathbf{C}$. Because of the connectedness of **C**, there exists a path $\gamma : c \to x$. In consequence of this $\mathbf{g}(\gamma)$ and $\mathbf{h}(\gamma)$ are parallel paths in **A** starting from the same element $\mathbf{g}(c) = \mathbf{h}(c)$. By $\mathbf{fg} = \mathbf{fh}$, we get $\mathbf{f}(\mathbf{g}(\gamma)) = = \mathbf{f}(\mathbf{h}(\gamma))$ and hence we conclude $\mathbf{g}(\gamma) = \mathbf{h}(\gamma)$ using theorem 2.1. Thus in particular, $\mathbf{g}(x) = \mathbf{h}(x)$. Because of the free choice of x, we have got $\mathbf{g} = \mathbf{h}$.

Theorem 2.2. *Let* $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ *be a morphism in the category* $\mathbf{U}_\Sigma(\mathfrak{A})$. *If* **A** *is a directed algebraic system, then each two different elements* $a, a' \in \mathbf{A}$ *with* $\mathbf{f}(a) = \mathbf{f}(a')$ *are f-symmetric.*

Proof. Let **A** be a directed algebraic system of $\mathbf{U}_\Sigma(\mathfrak{A})$ and $a, a' \in \mathbf{A}$, $a \neq a'$ elements satisfying $\mathbf{f}(a) = \mathbf{f}(a')$. We can find $\varrho \in \mathbf{\Omega}$ and $a_0 \in \mathbf{A}$ with $a\varrho a_0$, $a'\varrho a_0$. The paths (a, a_0), (a', a_0) are parallel and have the same image under **f**. By lemma 1.2., we conclude a, a' satisfy condition (*i*). If $\mathbf{\Omega}_F \neq \emptyset$, then a, a' satisfy also condition (*ii*). For, let $F \in \mathbf{\Omega}_F$ be an arbitrary basic operation and $\sigma \in \mathbf{\Omega}$, $b_0 \in \mathbf{A}$ such that $a\varrho b_0$, $b_0\varrho^{-1}F(a, \ldots, a)$, $a'\varrho b_0$, $b_0\varrho^{-1}F(a', \ldots, a')$. Then, of course, the paths $\beta_a = (a, b_0, F(a, \ldots, a))$ and $\beta_{a'} = (a', b_0, F(a', \ldots, a'))$ are parallel and satisfy $\mathbf{f}(\beta_a) = \mathbf{f}(\beta_{a'})$.

Theorem 2.3. *Let* $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ *be a morphism in the category* $\mathbf{U}_\Sigma(\mathfrak{A})$ (*or in* $\mathbf{U}_\Sigma^{\text{fin}}(\mathbf{U})$). *Then* **f** *is a monomorphism in* $\mathbf{U}_\Sigma(\mathfrak{A})$ (*or in* $\mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$) *if and only if there are not any f-symmetric elements in* **A**.

.Proof. (1) Suppose first that **f** is not a monomorphism in $\mathbf{U}_\Sigma(\mathfrak{A})$ (or in $\mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$). Then, of course, there exist morphisms $\mathbf{g}, \mathbf{h} : \mathbf{C} \to \mathbf{A}$ in $\mathbf{U}_\Sigma(\mathfrak{A})$ (or in $\mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$) so that $\mathbf{g} \neq \mathbf{h}$ and $\mathbf{fg} = \mathbf{fh}$. In particular, we can choose a $c \in \mathbf{C}$ with $\mathbf{g}(c) \neq \mathbf{h}(c)$. Let us set $\mathbf{g}(c) = a$, $\mathbf{h}(c) = a'$. We are going to show that $a, a'$are f-symmetric. For, let $x, y \in \mathbf{A}$ be arbitrary elements such that $x\varrho y$ for some relation $\varrho \in \mathbf{\Omega}$. Since $\mathbf{g}, \mathbf{h}$ are strong surjective homomorphisms, there exist x_g, y_g, $x_h, y_h \in \mathbf{C}$ satisfying $x_g \varrho y_g$, $x_h \varrho y_h$ and $\mathbf{g}(x_g) = \mathbf{h}(x_h) = x$, $\mathbf{g}(y_g) = \mathbf{h}(y_h) = y$. According to connectedness of **C**,



we may choose paths $\alpha : c \to x_g$, $\beta : y_g \to x_h$. Thus, by composition, we get a path $\delta = \alpha(x_g, y_g)\, \beta(x_h, y_h)$. For the same reason in case $\mathbf{\Omega}_F \neq \emptyset$ a path $\gamma_i : c \to F_i(c, \ldots, c)$ may be associated with every basic operation $F_i \in \mathbf{\Omega}_F$. It holds true $\mathbf{g}(\delta)//\mathbf{h}(\delta)$ and eventually also $\mathbf{g}(\gamma_i)//\mathbf{h}(\gamma_i)$ for every $i < k$. The paths $\mathbf{g}(\delta)$ and $\mathbf{h}(\delta)$ are starting from a and a', respectively, and besides $(\mathbf{g}(x_g), \mathbf{g}(y_g)) = (x, y) \subseteq \mathbf{g}(\delta)$, $(\mathbf{h}(x_h), \mathbf{h}(y_h)) =$

$= (x, y) \subseteq \mathbf{h}(\delta)$. Because of $\mathbf{fg} = \mathbf{fh}$, we have also $\mathbf{f}(\mathbf{g}(\delta)) = \mathbf{f}(\mathbf{h}(\delta))$. Thus, in view of the free choice of x, y with $x \varrho y$, the elements a, a' satisfy condition (i). In case $\mathbf{\Omega}_F \neq \emptyset$, in the same way we prove the truth of condition (ii) using the paths

$$\mathbf{g}(\gamma_i) : a \to \mathbf{g}(F_i(c. \ldots, c)) = F_i(\mathbf{g}(c), \ldots, \mathbf{g}(c)) = F_i(a, \ldots, a)$$

and

$$\mathbf{h}(\gamma_i) : a' \to \mathbf{h}(F_i(c, \ldots, c)) = F_i(\mathbf{h}(c), \ldots, \mathbf{h}(c)) = F_i(a', \ldots, a').$$

Altogether we have showed that a, a' are f-symmetric elements.

(2) Conversely, suppose there exist f-symmetric elements $a, a' \in \mathbf{A}$. Let us define a set $\mathbf{C}$ like this: $\mathbf{C} = \{\langle x, x' \rangle \mid x, x' \in \mathbf{A}$ such that there exist parallel paths $\beta : a \to x$, $\beta' : a' \to x'$ with $\mathbf{f}(\beta) = \mathbf{f}(\beta')\}$. It is $\mathbf{C} \neq \emptyset$ because for example $\langle a, a' \rangle \in \mathbf{C}$. As far as $\mathbf{A}$ is a directed algebraic system or there is at least one 0-ary operation in $\mathbf{\Omega}_F$, then $\mathbf{C}$ contains even each pair $\langle x, x \rangle$ where $x \in \mathbf{A}$ (to show that, see proof of lemma 1.2. and condition *(ii)* especially with respect to 0-ary basic operations).

At this time we are going to show that $\mathbf{C}$ is a subsystem of $\mathbf{A} \otimes \mathbf{A}$ (here the sign $\otimes$ stands for the direct product of algebraic systems). If $\mathbf{\Omega}_F = \emptyset$ there is nothing to prove. So let $\mathbf{\Omega}_F$ be nonempty. To prove the statement, we must show that $\mathbf{C}$ is closed with respect to all basic operations of $\mathbf{\Omega}_F$. If $\langle x_0, x_0 \rangle$ is an element in $\mathbf{A} \otimes \mathbf{A}$ appointed by a 0-ary operation of $\mathbf{\Omega}_F$, then $\langle x_0, x_0 \rangle \in \mathbf{C}$ because even each pair $\langle x, x \rangle \in \mathbf{C}$. Therefore we may confine ourselves only to basic operations with nonzero arity. Let $F \in \mathbf{\Omega}_F$ be such a one (arbitrary) operation and $\langle x_1, x'_1 \rangle, \ldots, \langle x_n, x'_n \rangle \in \mathbf{C}$ some elements (n is the arity of F). By definition of $\mathbf{C}$, there exist paths $\beta_j = (b^j_0, \ldots, b^j_m) : a \to x_j$, $\beta'_j = (b'^j_0, \ldots, b'^j_m) : a' \to x'_j$ for every $j = 1, \ldots, n$ such that $\beta_j // \beta'_j$ and $\mathbf{f}(\beta_j) = \mathbf{f}(\beta'_j)$. We have supposed here that all the paths β_j (and β'_j) are of the same length. But it does not matter because if this was not the case, each path that is shorter than that one of maximal length can be prolonged by means of the relation $\mathbf{\Delta}$. Now let $i \in \{1, \ldots, m\}$ be an arbitrary but fixed index. Owing to $\beta_j // \beta'_j$ for every $j = 1, \ldots, n$, there exist relations $\varrho_1, \ldots, \varrho_n \in \mathbf{\Omega}$ such that $b^1_{i-1} \varrho_1 b^1_i, \ldots, b^n_{i-1} \times \varrho_n b^n_i$ and $b'^1_{i-1} \varrho_1 b'^1_i, \ldots, b'^n_{i-1} \varrho_n b'^n_i$. By condition III., we can find terms $T_0, T_1, \ldots, T_r$, relations $\sigma_1, \ldots, \sigma_r \in \mathbf{\Omega}$ and, if necessary, also elements $e_1, \ldots, e_q \in \mathbf{A}$ such that

$$F(b^1_{i-1}, \ldots, b^n_{i-1}) \, \sigma_1 T_1(b^j_{i-1}, b^j_i, e_s) \, \sigma_2, \ldots, \sigma_r F(b^1_i, \ldots, b^n_i)$$

and

$$F(b'^1_{i-1}, \ldots, b'^n_{i-1}) \, \sigma_1 T_1(b'^j_{i-1}, b'^j_i, e_s) \, \sigma_2, \ldots, \sigma_r F(b'^1_i, \ldots, b'^n_i).$$

As a result we obtain parallel paths

$$\alpha_i : F(b^1_{i-1}, \ldots, b^n_{i-1}) \to F(b^1_i, \ldots, b^n_i)$$

and

$$\alpha'_i : F(b'^1_{i-1}, \ldots, b'^n_{i-1}) \to F(b'^n_i, \ldots, b'^n_i).$$

Using the result from [2] (6.1., theorem 1) we can write for every $p = 0, 1, \ldots, r$ $\mathbf{f}(T_p(b_{i-1}^j, b_i^j, e_s)) = T_p(\mathbf{f}(b_{i-1}^j), \mathbf{f}(b_i^j), \mathbf{f}(e_s)) = \mathrm{T}_p(\mathbf{f}(b_{i-1}'^j), \mathbf{f}(b_i'^j), \mathbf{f}(e_s)) = \mathbf{f}(T_p(b_{i-1}'^j, b_i'^j, e_s))$. But this means that $\mathbf{f}(\alpha_i) = \mathbf{f}(\alpha_i')$. As the elements a, a' are f-symmetric, they satisfy condition (*ii*). Let $\beta_a : a \to F(a, \ldots, a)$ and $\beta_{a'} : a' \to F \times \times (a', \ldots, a')$ be some parallel paths with $\mathbf{f}(\beta_a) = \mathbf{f}(\beta_{a'})$. By composition, we obtain paths $\beta = \beta_a \alpha_1 \ldots \alpha_m : a \to F(x_1 \ldots x_n)$ and $\beta' = \beta_{a'} \alpha_1 \ldots \alpha_m' : a' \to F \times \times (x_1', \ldots, x_n')$. These paths are parallel and satisfy $\mathbf{f}(\beta) = \mathbf{f}(\beta')$, which means that

$$\langle F(x_1, \ldots, x_n), F(x_1', \ldots, x_n') \rangle = F(\langle x_1, x_1' \rangle, \ldots, \langle x_n, x_n' \rangle) \in \mathbf{C}.$$

Thus $\mathbf{C}$ is an algebraic system of Σ. Now it is necessary to show that $\mathbf{C} \in \mathbf{U}_\Sigma(\mathfrak{A})$.

The truth of condition I. is obvious because the axioms of $\mathfrak{A}$ are supposed to be heritable to direct products and subsystems.

We now verify condition II. Let $\langle x, x' \rangle, \langle y, y' \rangle \in \mathbf{C}$ be arbitrary. By definition of $\mathbf{C}$, let $\beta_x : a \to x$, $\beta_{x'} : a' \to x'$, $\beta_y : a \to y$, $\beta_{y'} : a' \to y'$ be those paths satisfying $\beta_x // \beta_{x'}$, $\beta_y // \beta_{y'}$ and $\mathbf{f}(\beta_x) = \mathbf{f}(\beta_{x'})$, $\mathbf{f}(\beta_y) = \mathbf{f}(\beta_{y'})$. Observe that also the paths $\beta_x^{-1}, \beta_{x'}^{-1}$ are parallel and $\mathbf{f}(\beta_x^{-1}) = \mathbf{f}(\beta_{x'}^{-1})$. We set $\beta = (b_0, \ldots, b_n) = \beta_x^{-1}\beta_y : x \to y$ and $\beta' = (b_0', \ldots, b_n') = \beta_{x'}^{-1}\beta_{y'} : x' \to y'$. Clearly $\beta // \beta'$ and $\mathbf{f}(\beta) = \mathbf{f}(\beta')$ as well. It is easy to see that $\langle b_i, b_i' \rangle \in \mathbf{C}$ $(i = 0, 1, \ldots, n)$ and that $\langle b_{i-1}, b_{i-1}' \rangle \, \varrho_i \langle b_i, b_i' \rangle$ for every $i = 1, \ldots, n$ $(\varrho_i \in \mathbf{\Omega})$. But this says that $(\langle b_0, b_0' \rangle, \ldots, \langle b_n, b_n' \rangle)$ is a path in $\mathbf{C}$ from $\langle x, x' \rangle$ to $\langle y, y' \rangle$. In case $\mathbf{\Omega}_F \neq \emptyset$ it remains to check condition III. for $\mathbf{C}$.

For, we arbitrarily choose an n-ary operation $F \in \mathbf{\Omega}_F$ $(n \geqq 1)$, relations $\varrho_1, \ldots, \ldots, \varrho_n \in \mathbf{\Omega}$ and for every $p \in \{1, \ldots, v\}$ a set $\mathbf{M}_C^p(\langle a_1^p, b_1^p \rangle \, \varrho_1 \langle a_1'^p, b_1'^p \rangle, \ldots, \langle a_n^p, b_n^p \rangle \times \times \varrho_n \langle a_n'^p, b_n'^p \rangle)$. In particular, for every $p \in \{1, \ldots, v\}$ we have in $\mathbf{A}$ $\mathbf{M}_A^{p,a}(a_1^p \varrho_1 a_1'^p, \ldots, \ldots, a_n^p \varrho_n a_n'^p)$, $\mathbf{M}_A^{p,b}(b_1^p \varrho_1 b_1'^p, \ldots, b_n^p \varrho_n b_n'^p)$. All these sets create a family with $2v$ elements. Thus, by condition III. for $\mathbf{A}$, there exist terms $T_0, T_1, \ldots, T_r$, relations $\sigma_1, \ldots, \sigma_r \in \mathbf{\Omega}$ and, if necessary, also elements $e_1, \ldots, e_q \in \mathbf{A}$ with the corresponding properties (1), (2), (3). In particular owing to (3), it holds true for every $p = 1, \ldots, v$

$$F(a_1^p, \ldots, a_n^p) \, \sigma_1 T_1(a_i^p, a_j'^p, e_s) \, \sigma_2, \ldots, \sigma_{r-1} T_{r-1}(a_i^p, a_j'^p, e_s) \, \sigma_r F(a_1'^p, \ldots, a_n'^p)$$

$$F(b_1^p, \ldots, b_n^p) \, \sigma_1 T_1(b_i^p, b_j'^p, e_s) \, \sigma_2, \ldots, \sigma_{r-1} T_{r-1}(b_i^p, b_j'^p, e_s) \, \sigma_r F(b_1'^p, \ldots, b_n'^p).$$

Instead of this we can write

$$\langle F(a_1^p, \ldots, a_n^p), F(b_1^p, \ldots, b_n^p) \rangle \, \sigma_1 \langle T_1(a_i^p, a_j'^p, e_s), T_1(b_i^p, b_j'^p, e_s) \rangle \, \sigma_2, \ldots,$$
$$\ldots, \sigma_{r-1} \langle T_{r-1}(a_i^p, a_j'^p, e_s), T_{r-1}(b_i^p, b_j'^p, e_s) \rangle \, \sigma_r \langle F(a_1'^p, \ldots, a_n'^p), F(b_1'^p, \ldots, b_n'^p) \rangle.$$

Hence, by definition of operations on $\mathbf{A} \times \mathbf{A}$, we easily get

$$F(\langle a_1^p, b_1^p \rangle, \ldots, \langle a_n^p, b_n^p \rangle) \, \sigma_1 T_1(\langle a_i^p, b_i^p \rangle, \langle a_j'^p, b_j'^p \rangle, \langle e_s, e_s \rangle) \, \sigma_2, \ldots$$
$$\ldots, \sigma_{r-1} T_{r-1}(\langle a_i^p, b_i^p \rangle, \langle a_j'^p, b_j'^p \rangle, \langle e_s, e_s \rangle) \, \sigma_r F(\langle a_1'^p, b_1'^p \rangle, \ldots, \langle a_n'^p, b_n'^p \rangle).$$

But this is already property (3) for $\mathbf{C}$. Besides (1) the terms $T_0, \ldots, T_1, T_r$ have also the property (2). But this is clear if we consider that $\mathbf{A}$ directed implies $\mathbf{C}$ directed (indeed,

$\mathbf{A}$ directed implies $\mathbf{A} \times \mathbf{A}$ directed and hence $\mathbf{C}$ is also directed in view of $\langle x, x\rangle$ being in $\mathbf{C}$ for each $x \in \mathbf{A}$). The elements $\langle e_s, e_s\rangle$ are considered only when at least one object variable of type v_s is in $T_0, T_1, \ldots, T_r$. According to (2) this may be the case exactly when $\mathbf{A}$ (and hence $\mathbf{C}$) is directed or $\mathbf{\Omega}_F$ contains at least one 0-ary operation. In both cases $\langle e_s, e_s\rangle \in \mathbf{C}$, which means that the terms $T_0, T_1, \ldots, T_r$ are really operating on $\mathbf{C}$. So condition III. is also true and we get $\mathbf{C} \in \mathbf{U}_\Sigma(\mathfrak{A})$. As far as $\mathbf{A} \in \mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$, then $\mathbf{A} \times \mathbf{A}$ and $\mathbf{C}$ are finite, which means that also $\mathbf{C} \in \mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$.

Now we define mappings $\mathbf{g}, \mathbf{h} : \mathbf{C} \to \mathbf{A}$ as projections, i.e. $\mathbf{g}(\langle x, x'\rangle) = x$, $\mathbf{h}(\langle x, x'\rangle) = x'$ for each $\langle x, x'\rangle \in \mathbf{C}$. Since $\mathbf{g}(\langle a, a'\rangle) = a \neq a' = \mathbf{h}(\langle a, a'\rangle)$, we have $\mathbf{g} \neq \mathbf{h}$. It is also $\mathbf{fg} = \mathbf{fh}$ because $\mathbf{f}(x) = \mathbf{f}(x')$ for each $\langle x, x'\rangle \in \mathbf{C}$ (cf. definition of $\mathbf{C}$). Clearly $\mathbf{g}, \mathbf{h}$ are homomorphisms of algebraic systems. In order to show that $\mathbf{g}, \mathbf{h}$ are morphisms in $\mathbf{U}_\Sigma(\mathfrak{A})$, we must prove that they are strong (then, by connectedness of $\mathbf{A}$, they are also surjective). For, let $x, y \in \mathbf{A}$ be arbitrary elements with $x \varrho y$ for some $\varrho \in \mathbf{\Omega}_P$. Using property (*i*) of f-symmetric elements a, a' we can choose parallel paths $\alpha_a, \alpha_{a'}$ starting from a, a', respectively and satisfying $\mathbf{f}(\alpha_a) = \mathbf{f}(\alpha_{a'})$, $(x, y) \subseteq \alpha_a$, $(x, y) \subseteq \alpha_{a'}$. Thus ,we may write $\alpha_a, \alpha_{a'}$ in the form $\alpha_a = \beta_a(x, y)\gamma_1$, $\alpha_{a'} = \beta_{a'}(x', y')\gamma_2$ where $\mathbf{f}(\beta_a) = \mathbf{f}(\beta_{a'})$, $\beta_a // \beta_{a'}$ and $x' \varrho y'$. Then, of course, $\langle x, x'\rangle$, $\langle y, y'\rangle \in \mathbf{C}$, $\langle x, x'\rangle \varrho \langle y, y'\rangle$ and $\mathbf{g}(\langle x, x'\rangle) = x$, $\mathbf{g}(\langle y, y'\rangle) = y$, which means that $\mathbf{g}$ is strong. In order to show the same for $\mathbf{h}$, we proceed in an analogous way.

Altogether we have proved that $\mathbf{f}$ is not a monomorphism.

Corollary 2.2. *A morphism of* $\mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$ *is a monomorphism in this category if and only if it is a monomorphism* in $\mathbf{U}_\Sigma(\mathfrak{A})$.

Corollary 2.3. *Let* $\mathbf{U}_\Sigma(\mathfrak{A})$ (*or* $\mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$) *be a category of type* (*i*) *and* $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ *a monomorphism in* $\mathbf{U}_\Sigma(\mathfrak{A})$ (*or* $\mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$). *Then each two parallel paths* $\alpha = (a_0, \ldots, a_n)$, $\beta = (b_0, \ldots, b_n) \in \mathbf{W}(\mathbf{A})$ *with* $\mathbf{f}(\alpha) = \mathbf{f}(\beta)$ *coincide whenever* $a_i = b_i$ *for some index* $i \in \{0, 1, \ldots, n\}$.

Proof. The statement is a consequence of theorems 2.1. and 2.3.

Theorem 2.4. *Let* $\mathbf{A}$ *be a directed algebraic system and* $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ *a monomorphism in the category* $\mathbf{U}_\Sigma(\mathfrak{A})$ (*or in* $\mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$). *Then* $\mathbf{f}$ *is an isomorphism.*

Proof. By theorems 2.2. and 2.3., $\mathbf{f}$ is injective. But that means already that $\mathbf{f}$ is an isomorphism.

Corollary 2.4. *Suppose that all algebraic systems in* $\mathbf{U}_\Sigma(\mathfrak{A})$ (*or in* $\mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$) *are directed. Then monomorphisms in* $\mathbf{U}_\Sigma(\mathfrak{A})$ (*or in* $\mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$) *are exactly all isomorphisms (in other words, besides isomorphisms there do not exist any other monomorphisms in* $\mathbf{U}_\Sigma(\mathfrak{A})$ *or* $\mathbf{U}_\Sigma^{\text{fin}}(\mathfrak{A})$).

Remark 2.1. Consider that the main features of the method used in the proof of the theorem 2.3. could be applied also to categories where one (or some) of the conditions mentioned in the definition of $\mathbf{U}_{\Sigma}(\mathfrak{A})$ is (are) omitted or weakened. For example, we need not require the connectedness, or as morphisms we can take only the surjective homomorphisms not strong. Up to now we have supposed $\mathbf{\Omega}_P \neq \emptyset$. But in case $\mathbf{\Omega}_P = \emptyset$ we could proceed in a quite analogous way. Each object of $\mathbf{U}_{\Sigma}(\mathfrak{A})$ would be an algebra satisfying only condition I. (conditions II. and III. are in this case groundless). As morphisms we would consider homomorphisms or surjective homomorphisms of algebras. Using the method of the proof in the theorem 2.3. it is easy to see that the monomorphisms in this category are exactly all injective morphisms.

Lemma 2.1. *Let* $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ *be a morphism in the category* $\mathbf{U}_{\Sigma}^{\text{fin}}(\mathfrak{A})$ *and* $\alpha : a \to b$, $\alpha' : a' \to b'$ *two parallel paths in* $\mathbf{A}$ *with* $\mathbf{f}(\alpha) = \mathbf{f}(\alpha')$. *Then there exist parallel paths* $\beta : a \to b$, $\beta' : a' \to b'$ such *that* $\beta \subseteq \alpha$, $\beta' \subseteq \alpha'$, $\mathbf{f}(\beta) = \mathbf{f}(\beta')$ *and* $L(\beta) = L(\beta') \leqq \leqq \sum_{b \in \mathbf{B}} |\mathbf{f}^{-1}(b)|^2 - 1$.

Proof. When excluding the trivial case $\mathbf{A} = \{a\}$, $\mathbf{B} = \{b\}$, we can assume that $\sum_{b \in \mathbf{B}} |\mathbf{f}^{-1}(b)|^2 - 1$ is a natural number. If $L(\alpha) = L(\alpha') \leqq \sum_{b \in \mathbf{B}} |\mathbf{f}^{-1}(b)|^2 - 1$, there is nothing to prove. So let $L(\alpha) = L(\alpha') > \sum_{b \in \mathbf{B}} |\mathbf{f}^{-1}(b)|^2 - 1$. We denote $\alpha = (a_0, \ldots, a_n)$, $\alpha' = (a'_0, \ldots, a'_n)$ and set $\mathbf{f}(a_i) = \mathbf{f}(a'_i) = b_i$ for every $i = 0, 1, \ldots, n$. By hypothesis, $n + 1 > \sum_{b \in \mathbf{B}} |\mathbf{f}^{-1}(b)|^2$. Then we can find an element $b \in \mathbf{B}$ such that $b = b_{i_1} = \ldots = b_{i_r}$ where $0 \leqq i_1 < i_2 < \ldots < i_r \leqq n$ and $r > |\mathbf{f}^{-1}(b)|^2$. This means that a_{i_j}, $a'_{i_j} \in \mathbf{f}^{-1}(b)$ and thus $[a_{i_j}, a'_{i_j}] \in \mathbf{f}^{-1}(b) \mathbf{x} \mathbf{f}^{-1}(b)$ for every $j = 1, \ldots, r$ (here the sign $\mathbf{x}$ stands for a cartesian product of sets). Because of $r > |\mathbf{f}^{-1}(b) \mathbf{x} \mathbf{f}^{-1}(b)|$, there exist indices $p, q \in \{i_1, \ldots, i_r\}$, $p < q$ with $[a_p, a'_p] = [a_q, a'_q]$, i.e. $a_p = a_q$ and $a'_p = a'_q$.

Therefore we can write $\alpha = \beta_1\beta_0\beta_2$, $\alpha' = \beta'_1\beta'_0\beta'_2$ where $\beta_1 = (a_0, \ldots, a_p)$, $\beta_0 = (a_p, \ldots, a_q)$, $\beta_2 = (a_q, \ldots, a_n)$ and $\beta'_1 = (a'_0, \ldots, a'_p)$, $\beta'_0 = (a'_p, \ldots, a'_q)$, $\beta'_2 = (a'_q, \ldots, a'_n)$. We can set $\alpha_1 = \beta_1\beta_2$, $\alpha'_1 = \beta'_1\beta'_2$. It is easy to see that $\alpha_1 : a \to b$, $\alpha'_1 : a' \to b'$, $\alpha_1 \subseteq \alpha$, $a'_1 \subseteq \alpha'$, $\alpha_1 // \alpha'_1$, $\mathbf{f}(\alpha_1) = \mathbf{f}(\alpha'_1)$ and at the same time $n_1 = L(\alpha_1) = L(\alpha'_1) = L(\beta_1) + L(\beta_2) = p + n - q = n - (q - p) < n$. As far as $n_1 \leqq \leqq \sum_{b \in \mathbf{B}} |\mathbf{f}^{-1}(b)|^2 - 1$, it suffices to set $\beta = \alpha_1$, $\beta' = \alpha'_1$. In the opposite case we revise the whole construction replacing α, α' by α_1, α'_1, etc. After a finite number of steps we obtain the required paths.

Theorem 2.5. *Let* $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ *be a morphism in the category* $\mathbf{U}_{\Sigma}^{\text{fin}}(\mathfrak{A})$. *The elements* $a, a' \in \mathbf{A}$ *with* $a \neq a'$ *and* $\mathbf{f}(a) = \mathbf{f}(a')$ *are f-symmetric if and only if the following two conditions are true.*

(i') *To each pair of elements* $x, y \in \mathbf{A}$ *where* x *stands in some relation* $\varrho \in \mathbf{\Omega}$ *to* y *there exist paths* $\alpha, \alpha', \bar{\alpha}, \bar{\alpha}'$ *(*α, $\bar{\alpha}$ *starting from* a *and* α', $\bar{\alpha}'$ *starting from* a'*) such that* $\alpha // \alpha'$, $\bar{\alpha} // \bar{\alpha}'$, $\mathbf{f}(\alpha) = \mathbf{f}(\alpha')$, $\mathbf{f}(\bar{\alpha}) = \mathbf{f}(\bar{\alpha}')$, $(x, y) \subseteq \alpha$, $(x, y) \subseteq \bar{\alpha}'$ *and at the same time the length of each of the paths is not greater than* $\sum_{b \in \mathbf{B}} |\mathbf{f}^{-1}(b)|^2$ (parallelness is with $x\varrho y$ in α and $\bar{\alpha}'$).

(ii')[1] *To each basic operation* $F \in \mathbf{\Omega}_F$ *there exist parallel paths* β_a: $a \to F(a, \ldots, a)$, $\beta_{a'}$: $a' \to F(a', \ldots, a')$ *with* $\mathbf{f}(\beta_a) = \mathbf{f}(\beta_{a'})$ *such that their length is not greater than* $\sum_{b \in \mathbf{B}} |\mathbf{f}^{-1}(b)|^2 - 1$.

Proof. (1) Let $a, a' \in \mathbf{A}$, $a \neq a'$, $\mathbf{f}(a) = \mathbf{f}(a')$ be elements satisfying conditions (i') and (ii'). If $x, y \in \mathbf{A}$ are arbitrary elements with $x\varrho y$ for some $\varrho \in \mathbf{\Omega}$ and $\alpha, \alpha', \bar{\alpha}, \bar{\alpha}'$ paths with the properties mentioned in (i'), then paths $\alpha_a = \alpha\alpha^{-1}\bar{\alpha}$, $\alpha_{a'} = \alpha'\alpha'^{-1}\bar{\alpha}'$ are parallel and satisfy $\mathbf{f}(\alpha_a) = \mathbf{f}(\alpha_{a'})$, $(x, y) \subseteq \alpha_a$, $(x, y) \subseteq \alpha_{a'}$. Thus, elements a, a' satisfy condition (*i*). The second condition is an immediate consequence of (*ii'*). Altogether we have showed that a, a' are f-symmetric.

(2) Conversely, let a, a' be f-symmetric elements. Choose $x, y \in \mathbf{A}$ with $x\varrho y$ for some $\varrho \in \mathbf{\Omega}$ quite arbitrarily. In particular, by condition (*i*), we can find paths $\alpha = \gamma_a(x, y) : a \to y, \alpha' = \gamma_{a'}(x', y') : a' \to y', \bar{\alpha} = \bar{\gamma}_a(\bar{x}, \bar{y}) : a \to \bar{y}, \bar{\alpha}' = \bar{\gamma}_{a'}(x, y) : a' \to y$ that have the properties required in (*i'*) (α, $\bar{\alpha}$ are suitable subpaths in α_a and so are $\bar{\alpha}$, $\bar{\alpha}'$ in $\alpha_{a'}$). By lemma 2.1., we can assume $L(\gamma_a), L(\bar{\gamma}_a) \leqq \sum_{b \in \mathbf{B}} |\mathbf{f}^{-1}(b)|^2 - 1$. Hence $L(\alpha), L(\bar{\alpha}) \leqq \sum_{b \in \mathbf{B}} |\mathbf{f}^{-1}(b)|^2$. Because of equations $L(\alpha) = L(\alpha')$, $L(\bar{\alpha}) = L(\bar{\alpha}')$ the same holds true also for $L(\alpha')$ and $L(\bar{\alpha}')$. Thus, condition (*i'*) is true. The truth of (*ii'*) follows immediately from (*ii*) and lemma 2.1.

Remark 2.2. If there is given a morphism $\mathbf{f}: \mathbf{A} \to \mathbf{B}$ in $\mathbf{U}_\Sigma^{\mathrm{fin}}(\mathfrak{A})$, then going over all paths in $\mathbf{A}$ that are not longer[1] than $\sum_{b \in \mathbf{B}} |\mathbf{f}^{-1}(b)|^2$ (there is only a finite number of them), we can decide by means of the preceding theorem whether there are any f-symmetric elements in $\mathbf{A}$ or not. In this way, and considering theorem 2.3., it is possible to decide after a finite number of steps whether $\mathbf{f}$ is a monomorphism or not.

Remark 2.3. Let $\mathbf{B} \in \mathbf{U}_\Sigma(\mathfrak{A})$. A pair $(\mathbf{A}, \mathbf{f})$ where $\mathbf{f}: \mathbf{A} \to \mathbf{B}$ is a monomorphism is called a subobject of the object $\mathbf{B}$. Let us denote by $\mathrm{S}(\mathbf{B})$ the class of all subobjects of $\mathbf{B}$. We now define an equivalence relation $\sim$ on $\mathrm{S}(\mathbf{B})$ like this: Subobjects $(\mathbf{A}, \mathbf{f})$, $(\mathbf{A}', \mathbf{f}') \in \mathrm{S}(\mathbf{B})$ are said to be equivalent (we write) $(\mathbf{A}, \mathbf{f}) \sim (\mathbf{A}', \mathbf{f}')$) if there exist morphisms $\mathbf{g}: \mathbf{A} \to \mathbf{A}'$, $\mathbf{g}': \mathbf{A}' \to \mathbf{A}$ such that $\mathbf{f} = \mathbf{f}'\mathbf{g}$ and $\mathbf{f}\mathbf{g}' = \mathbf{f}'$ (obviously $\mathbf{g}, \mathbf{g}'$ are also monomorphisms). Let $\mathrm{S}(\mathbf{B})/\!\sim$ denote the quotient class of $\mathrm{S}(\mathbf{B})$ by $\sim$. We further denote by $[(\mathbf{A}, \mathbf{f})]$ the equivalence class of $\mathrm{S}(\mathbf{B})/\!\sim$ containing a subobject $(\mathbf{A}, \mathbf{f}) \in \mathrm{S}(\mathbf{B})$. For each pair of objects $\mathbf{A}, \mathbf{B} \in \mathbf{U}_\Sigma(\mathfrak{A})$ we shall set $\mathscr{F}(\mathbf{A}, \mathbf{B}) = \{[\mathbf{A}, \mathbf{f})] \mid \mathbf{f}: \mathbf{A} \to \mathbf{B} \text{ is monomorphism}\}$. Now we are going to show that the category $\mathbf{U}_\Sigma(\mathfrak{A})$ is locally small, which means that $\mathrm{S}(\mathbf{B})/\!\sim$ is a set for each object $\mathbf{B} \in \mathbf{U}_\Sigma(\mathfrak{A})$.

At first we prove the following lemma.

[1] Condition (ii') is considered only when $\mathbf{\Omega}_F \neq 0$ (cf. remark 1.2.).

[1] According to the relation $\Delta \in \mathbf{\Omega}$ we can confine ourselves only to the investigation of paths with the length exactly equal to $\sum_{b \in \mathbf{B}} |\mathbf{f}^{-1}(b)|^2$.

Lemma 2.2. *Let* $\mathbf{B} \in \mathbf{U}_\Sigma(\mathfrak{A})$ *be an arbitrary object. If* $\mathbf{A}_1, \mathbf{A}_2 \in \mathbf{U}_\Sigma(\mathfrak{A})$ *are isomorphic objects, then* $\mathscr{F}(\mathbf{A}_1, \mathbf{B}) = \mathscr{F}(\mathbf{A}_2, \mathbf{B})$.

Proof. Let i: $\mathbf{A}_1 \to \mathbf{A}_2$ be an isomorphism. Assume at first $\mathscr{F}(\mathbf{A}_1, \mathbf{B}) \neq \emptyset$ and choose $[(\mathbf{A}_1, \mathbf{f})] \in \mathscr{F}(\mathbf{A}_1, \mathbf{B})$ quite arbitrarily. Then, of course $(\mathbf{A}_1, \mathbf{f}) \sim (\mathbf{A}_2, \mathbf{f}i^{-1})$, which means that $\mathscr{F}(\mathbf{A}_2, \mathbf{B})$ is also nonempty and $[(\mathbf{A}_1, \mathbf{f})] = [(\mathbf{A}_2, \mathbf{f}i^{-1})] \in \mathscr{F}(\mathbf{A}_2, \mathbf{B})$. Thus we have got $\mathscr{F}(\mathbf{A}_1, \mathbf{B}) \subseteq \mathscr{F}(\mathbf{A}_2, \mathbf{B})$. In case $\mathscr{F}(\mathbf{A}_1, \mathbf{B}) = \emptyset$ this inclusion is trivial. By exchange of roles of $\mathbf{A}_1$ and $\mathbf{A}_2$ we get also $\mathscr{F}(\mathbf{A}_2, \mathbf{B}) \subseteq \mathscr{F}(\mathbf{A}_1, \mathbf{B})$ and finally $\mathscr{F}(\mathbf{A}_1, \mathbf{B}) = \mathscr{F}(\mathbf{A}_2, \mathbf{B})$.

Remark 2.4. Under the hypothesis $\mathscr{F}(\mathbf{A}_1, \mathbf{B}) \neq \emptyset, \mathscr{F}(\mathbf{A}_2, \mathbf{B}) \neq \emptyset$ also the converse of the preceding lemma is true. To prove that, it suffices to realize that $(\mathbf{A}_1, \mathbf{f}_1) \sim$ $\sim (\mathbf{A}_2, \mathbf{f}_2)$ necessitates an isomorphism between $\mathbf{A}_1$ and $\mathbf{A}_2$.

Theorem 2.6. *The category* $\mathbf{U}_\Sigma(\mathfrak{A})$ *is locally small.*

Proof. For the purpose of this proof we introduce the following notation. For each $\mathbf{A} \in \mathbf{U}_\Sigma(\mathfrak{A})$ we set $\overline{\mathbf{W}}(\mathbf{A}) = \{[\alpha, (\varrho_1, \ldots, \varrho_n)] \mid \alpha = (a_0, \ldots, a_n) \in \mathbf{W}(\mathbf{A}), a_0\varrho_1 a_1, \ldots, \ldots, a_{n-1}\varrho_n a_n$ with $\varrho_1, \ldots, \varrho_n \in \mathbf{\Omega}\}$. We further denote by $\mathbf{Z}(\mathbf{A})$ the set of all mappings of the set of all ordinals that are strictly less than $k + 1$ into the set $\overline{\mathbf{W}}(\mathbf{A})$ (we recall that k is the ordinal number of the well-ordered set $\mathbf{\Omega}_F$). With each morphism $\mathbf{f}$: $\mathbf{A} \to \mathbf{B}$ in $\mathbf{U}_\Sigma(\mathfrak{A})$ we associate a mapping $\overline{\mathbf{f}}$: $\overline{\mathbf{W}}(\mathbf{A}) \to \overline{\mathbf{W}}(\mathbf{B})$ defined in this way: $\overline{\mathbf{f}}([\alpha, (\varrho_1, \ldots, \varrho_n)]) = [\mathbf{f}(\alpha), (\varrho_1, \ldots, \varrho_n)]$ for each $[\alpha, (\varrho_1, \ldots, \varrho_n)] \in \overline{\mathbf{W}}(\mathbf{A})$. We further denote by the same symbol $\overline{\mathbf{f}}$ a mapping of $\mathbf{Z}(\mathbf{A})$ into $\mathbf{Z}(\mathbf{B})$ where for $z \in \mathbf{Z}(\mathbf{A})$ is $\overline{\mathbf{f}}(z)(i) = \overline{\mathbf{f}}(z(i))$ with each $i < k + 1$.

Now we shall begin the actual proof. Let $\mathbf{f}$: $\mathbf{A} \to \mathbf{B}$ be a monomorphism in the category $\mathbf{U}_\Sigma(\mathfrak{A})$ and $a_0 \in \mathbf{A}$ an arbitrary but fixed element. By connectedness of $\mathbf{A}$, we can associate with each $a \in \mathbf{A}$ a mapping $z_a \in \mathbf{Z}(\mathbf{A})$ such that for every $i < k$ is $z_a(i) = [\alpha_i, (\varrho_1^i, \ldots, \varrho_{n_i}^i)]$ with α_i: $a_0 \to F_i(a, \ldots, a)$ and $z_a(k) = [\alpha, (\varrho_1, \ldots, \varrho_n)]$ with α: $a \to a_0$. Suppose the mappings z_a are chosen fixedly for all $a \in \mathbf{A}$. We denote $\mathbf{f}^*$: $\mathbf{A} \to \mathbf{Z}(\mathbf{B})$ a mapping defined as follows: foı $a \in \mathbf{A}$ is $\mathbf{f}^*(a) = \overline{\mathbf{f}}(z_a)$. We are going to show that $\mathbf{f}^*$ is injective. For, suppose $a, a' \in \mathbf{A}$ are elements with $\mathbf{f}^*(a) = \mathbf{f}^*(a')$. Thus $\overline{\mathbf{f}}(z_a) = \overline{\mathbf{f}}(z_{a'})$, which means $\overline{\mathbf{f}}(z_a(i)) = \overline{\mathbf{f}}(z_a)(i) = \overline{\mathbf{f}}(z_{a'})(i) = \overline{\mathbf{f}}(z_{a'}(i))$ for every ordinal $i < k + 1$. When denoting $z_{a'}(i) = [\alpha_i', (\varrho_1'^i, \ldots, \varrho_{n'_i}'^i)]$ for every $i < k$ and $z_{a'}(k) = [\alpha', (\varrho_1', \ldots, \varrho_{n'}')]$, we can write $\mathbf{f}(\alpha_i) = \mathbf{f}(\alpha_i')$ (hence $n_i = n_i'$), $\varrho_1^i = \varrho_1'^i, \ldots, \ldots, \varrho_{n_i}^i = \varrho_{n_i}'^i$ for every $i < k$ and $\mathbf{f}(\alpha) = \mathbf{f}(\alpha')$ (hence $n = n'$), $\varrho_1 = \varrho_1', \ldots, \varrho_n = \varrho_n'$. It results from the preceding that $\alpha_i // \alpha_i'$ for every $i < k$ and $\alpha // \alpha'$. Hence we conclude $a = a'$. For, let us suppose $a \neq a'$, then α: $a \to a_0$, α': $a' \to a_0$ satisfy suppositions ot lemma 1.2. and therefore a, a' satisfy condition (i). But condition (ii) is also true because for every $i < k$ $\alpha\alpha_i$: $a \to F_i(a, \ldots, a)$, $\alpha'\alpha_i'$: $a' \to F_i(a', \ldots, a')$ are parallel paths with $\mathbf{f}(\alpha\alpha_i) = \mathbf{f}(\alpha'\alpha_i')$. Altogether a, a' are f-symmetric. By theorem 2.3., this contradicts the hypothesis that $\mathbf{f}$ is a monomorphism.

Since **f** is injective, we have in particular $|\mathbf{A}| \leqq |\mathbf{Z(B)}|$. So, this unequality holds true whenever $(\mathbf{A}, \mathbf{f})$ is a subobject of **B**.

If there is a given $\mathbf{B} \in \mathbf{U}_\Sigma(\mathfrak{A})$, then, with respect to lemma 2.2., we need not distinguish between subobjects $(\mathbf{A}_1, \mathbf{f}_1)$, $(\mathbf{A}_2, \mathbf{f}_2) \in \mathbf{S(B)}$ as far as $\mathbf{A}_1$ is isomorphic to $\mathbf{A}_2$. Thus, we can confine ourselves only to suitable pairwise nonisomorphic undeılying sets with cardinality less or equal to $|\mathbf{Z(B)}|$. If we consider at the same time that there is only a set of all algebraic systems of $\mathbf{U}_\Sigma(\mathfrak{A})$ with the same underlying set, we see easily that also $\mathbf{S(B)}/\sim$ must be a set. Since **B** was an arbitrary object, we can state that the category $\mathbf{U}_\Sigma(\mathfrak{A})$ is locally small.

3. MONOMORPHISMS IN SOME SPECIAL CATEGORIES $\mathbf{U}_\Sigma(\mathfrak{A})$

In this chapter we shall use the following notation:

$\mathbf{O}(\leqq)$ ... the category of all connected partially ordered sets,
$\mathbf{L}(\wedge, \vee, \leqq)$ ... the category of all lattices (here $x \wedge y$ and $x \vee y$ stand for infimum and supremum of x, y, respectively),
$\mathbf{SG}(+, \leqq)$ ... the category of all connected partially ordered semigroups (grupoids[1]),
$\mathbf{G}(+, \leqq)$ ... the category of all connected partially ordered groups,
$\mathbf{R}(+, ., \leqq)$ ... the category of all connected partially ordered rings.

Morphisms in all above mentioned categories are strong surjective homomorphisms. We recall that for $a_1, a_2 \in \mathbf{A}$, $a_1 \leqq a_2$ where $\mathbf{A} \in \mathbf{SG}(+, \leqq)$ or $\mathbf{A} \in \mathbf{G}(+, ., \leqq)$ is $a_1 + b \leqq a_2 + b$ and $b + a_1 \leqq b + a_2$ for each $b \in \mathbf{A}$. Moreover, in case $\mathbf{A} \in \mathbf{R}(+, ., \leqq)$ it holds also true $a_1 . b \leqq a_2 . b$ and $b . a_1 \leqq b . a_2$ for each $b \geqq 0$ of **A** (here 0 is the zero element of the ring **A**). We are going to show that each of these categories is a category $\mathbf{U}_\Sigma(\mathfrak{A})$ (with appropriate choice of Σ and $(\mathfrak{A})$). For. $\mathbf{O}(\leqq)$ is this statement evident because objects of $\mathbf{O}(\leqq)$ satisfy conditions I. and II. Since in this case $\Omega_F = \emptyset$, we do not consider condition III. For the objects of the remaining categories $\mathbf{L}(\wedge, \vee, \leqq)$, $\mathbf{SG}(+, \leqq)$, $\mathbf{G}(+, \leqq)$ and $\mathbf{R}(+, ., \leqq)$ we must verify the only nontrivial condition, namely exactly condition III.

Theorem 3.1. *Objects of the category* $\mathbf{SG}(+, \leqq)$ *satisfy condition* III.

Proof. Let $\mathbf{A} \in \mathbf{SG}(+, \leqq)$ and $a_1, a_2, a_1', a_2' \in \mathbf{A}$. The statement follows easily from the following implications:

If $a_1 \leqq a_1'$ and $a_2 \leqq a_2'$, then $a_1 + a_2 \leqq a_1' + a_2 \leqq a_1' + a_2'$.
If $a_1 \leqq a_1'$ and $a_2 \geqq a_2'$, then $a_1 + a_2 \leqq a_1' + a_2 \geqq a_1' + a_2'$.
If $a_1 \geqq a_1'$ and $a_2 \leqq a_2'$, then $a_1 + a_2 \geqq a_1' + a_2 \leqq a_1' + a_2'$.

[1] See that all proved for $\mathbf{SG}(+, \leqq)$ in this chapter holds true also for the category of al connected partially ordered grupoids because associativity of addition does not play any role in the following considerations.

Lemma 3.1. *Each connected partially ordered group is directed.*

Proof. Let **A** be a connected partially ordered group. We show that there exists an upper bound for each pair of elements in **A**. Then, by induction on n, each subset in **A** with a finite number of n elements has an upper bound.

So, let $a_1, a_2 \in \mathbf{A}$ be two arbitrary elements. Since **A** is connected, we may suppose that there exist elements $b_1, b'_1, \ldots, b_m, b'_m \in \mathbf{A}$ (m is a natural number) such that $b'_m = a_2$ and $a_1 \leqq b_1 \geqq b'_1 \leqq b_2 \geqq \ldots \leqq b_m \geqq a_2$ (for, consider also the reflexivity and transitivity of the relation $\leqq$). We prove the statement by induction on m. If $m = 1$, then b_1 is the required upper bound for a_1, a_2. Assume now $m > 1$ and the truth of the statement in case $m - 1$. In particular, we can find a $c_{m-1} \in \mathbf{A}$ with $c_{m-1} \geqq a_1$ and $c_{m-1} \geqq b'_{m-1}$. At the same time we have also $b_m \geqq b'_{m-1}$. Adding element $b_m - b'_{m-1}$ from the left to the unequality $c_{m-1} \geqq b'_{m-1}$ we obtain $b_m - b'_{m-1} + c_{m-1} \geqq b_m$. Similarly adding $-b'_{m-1} + c_{m-1}$ from the right to the unequality $b_m \geqq b'_{m-1}$ we get $b_m - b'_{m-1} + c_{m-1} \geqq c_{m-1}$. If we set $a = b_m - b'_{m-1} + c_{m-1}$, we may write $a_1 \leqq c_{m-1} \leqq a$, $a \geqq b_m \geqq a_2$, which means that a is the searched upper bound.

Corollary 3.1. *Each connected partially ordered ring is directed.*

Theorem 3.2. *Objects of each of the categories* $\mathbf{L}(\wedge, \vee, \leqq)$, $\mathbf{G}(+, \leqq)$, $\mathbf{R}(+, ., \leqq)$ *satisfy condition* III.

Proof. The statement follows from lemma 1.1.

Thus, using theorem 2.3., we can formulate criteria for monomorphisms in the categories $\mathbf{O}(\leqq)$, $\mathbf{L}(\wedge, \vee, \leqq)$, $\mathbf{SG}(+, \leqq)$, $\mathbf{G}(+, \leqq)$, $\mathbf{R}(+, ., \leqq)$.

Theorem 3.3. *A morphism* $\mathbf{f}: \mathbf{A} \to \mathbf{B}$ *in the category* $\mathbf{O}(\leqq)$ *is a monomorphism if and only if there do not exist elements* $a, a' \in \mathbf{A}$, $a \neq a'$, $\mathbf{f}(a) = \mathbf{f}(a')$ *with this property: To each pair of comparable elements* $x, y \in \mathbf{A}$ *there exist parallel paths* α_a *and* $\alpha_{a'}$ *starting from* a *and* a', *respectively and satisfying* $\mathbf{f}(\alpha_a) = \mathbf{f}(\alpha_{a'})$, $(x, y) \subseteq \alpha_a$, $(x, y) \subseteq \alpha_{a'}$.

Example of a monomorphism in $\mathbf{O}(\leqq)$ that is not injective (see [3])

Let us define connected partially ordered sets **A**, **B** like this: $\mathbf{A} = \{a_1, a_2, a_3, a_4\}$ where $a_i \neq a_j$ for $i \neq j$ and $a_1 \leqq a_2, a_3 \leqq a_2, a_3 \leqq a_4$, $\mathbf{B} = \{b_1, b_2, b_3\}$ where $b_i \neq b_j$ for $i \neq j$ and $b_1 \leqq b_2$, $b_3 \leqq b_1$, $b_3 \leqq b_2$. We define $\mathbf{f}: \mathbf{A} \to \mathbf{B}$ in this way $\mathbf{f}(a_1) = \mathbf{f}(a_4) = b_1$, $\mathbf{f}(a_i) = b_i$ for $i = 2, 3$.

By theorem 3.3., **f** is clearly a monomorphism in $\mathbf{O}(\leqq)$ that is not injective.

Theorem 3.4. *A morphism* $\mathbf{f}: \mathbf{A} \to \mathbf{B}$ *in the category* $\mathbf{SG}(+, \leqq)$ *is a monomorphism if and only if there do not exist elements* $a, a' \in \mathbf{A}$, $a \neq a'$, $\mathbf{f}(a) = \mathbf{f}(a')$ *with the following properties:*

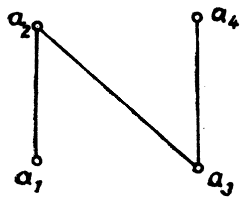
(1) *To each pair of comparable elements* $x, y \in \mathbf{A}$ *there exist parallel paths* α_a *and*

$\alpha'_{a'}$ *starting from* a *and* a', *respectively and satisfying* $\mathbf{f}(\alpha_a) = \mathbf{f}(\alpha_{a'})$, $(x, y) \subseteq \alpha_a$, $(x, y) \subseteq \alpha_{a'}$.

(2) *There exist parallel paths* β_a: $a \to a + a$, $\beta_{a'}$: $a' \to a' + a'$ *with* $\mathbf{f}(\beta_a) = \mathbf{f}(\beta_{a'})$.

B:

A:



Remark 3.1. Also in $\mathbf{SG}(+, \leqq)$ we can find monomorphisms that are not injective. For, see **A**, **B** and **f** from the preceding example with addition defined as follows $a_i + a_j = a_1$ for each $a_i, a_j \in \mathbf{A}$ and $b_i + b_j = b_1$ for each $b_i, b_j \in \mathbf{B}$.

Theorem 3.5. *Monomorphisms in the categories* $\mathbf{L}(\wedge, \vee, \leqq)$, $\mathbf{G}(+, \leqq)$ *and* $\mathbf{R}(+, ., \leqq)$ *are exactly all isomorphisms (in other words, besides isomorphisms there do not exist any other monomorphisms in these categories).*

Proof. For, see corollary 2.4.

We denote now by $\mathbf{O}(<)$ a subcategory of $\mathbf{O}(\leqq)$ defined like this

(1) Objects of $\mathbf{O}(<)$ are exactly all objects of $\mathbf{O}(\leqq)$,

(2) A morphism $\mathbf{f}$: $\mathbf{A} \to \mathbf{B}$ in $\mathbf{O}(\leqq)$ is also a morphism in $\mathbf{O}(<)$ exactly when $\mathbf{f}(a) \neq \mathbf{f}(a')$ whenever $a, a' \in \mathbf{A}$, $a \neq a'$ and $a \leqq a'$.

We may identify $\mathbf{O}(<)$ with (more exactly: $\mathbf{O}(<)$ is isomorphic to) the category of all connected "strictly ordered" sets where "strict ordering" (we use for this relation the sign $<$) is an areflexive ($< \cap \Delta = \emptyset$), asymmetric ($x < y$ implies $y \nless x$) and transitive relation; morphisms are again all strong surjective homomorphisms.

This identification is carried out by the following correspondence between the partial and "strict" ordering: $a < a'$ exactly when $a \leqq a'$ and $a \neq a'$.

We shall now be looking for the relationship between the monomorphisms in $\mathbf{O}(\leqq)$ and $\mathbf{O}(<)$.

Lemma 3.2. *Let* $\mathbf{f}: \mathbf{A} \to \mathbf{B}$ *be a morphism in* $\mathbf{O}(<)$. *Elements* $a, a' \in \mathbf{A}$, $a \neq a'$, $\mathbf{f}(a) = \mathbf{f}(a')$ *are f-symmetric in* $\mathbf{O}(<)$ *if and only if they are f-symmetric in* $\mathbf{O}(\leqq)$.

Proof. One implication is evident. So, let $a, a' \in \mathbf{A}$ be f-symmetric elements in $\mathbf{O}(\leqq)$. We want to show that they are f-symmetric also in $\mathbf{O}(<)$. But this is an easy consequence of the following consideration. If $\alpha = (a_0, \ldots, a_n)$, $\alpha' = (\alpha'_0, \ldots, a'_n)$ are two parallel paths in $\mathbf{A} \in \mathbf{O}(\leqq)$ with $\mathbf{f}(\alpha) = \mathbf{f}(\alpha')$, then they are parallel also as paths in $\mathbf{A} \in \mathbf{O}(<)$. Indeed, if this were not true, then we could find an index $i \in \{1, \ldots, n\}$ so that a_{i-1}, a_i would be "strictly" comparable and $a'_{i-1} = a'_i$ or conversely $a_{i-1} = a_i$ and a'_{i-1}, a'_i "strictly" comparable. But this is not possible because two different comparable elements would have the same image under $\mathbf{f}$, which is in contradiction with the definition of morphisms in $\mathbf{O}(<)$.

The preceding lemma and theorem 2.3. say that a morphism in $\mathbf{O}(<)$ is a monomorphism in $\mathbf{O}(<)$ if and only if it is a monomorphism in $\mathbf{O}(\leqq)$.

We can state more, namely that monomorphisms of $\mathbf{O}(<)$ exhaust all monomorphisms in $\mathbf{O}(\leqq)$ in this way because each monomorphism $\mathbf{f}: \mathbf{A} \to \mathbf{B}$ in $\mathbf{O}(\leqq)$ is already a morphism in $\mathbf{O}(<)$. Indeed, if this is not true, we can find elements a, $a' \in \mathbf{A}$, $a \neq a'$, $a \leqq a'$ (hence $a < a'$) with $\mathbf{f}(a) = \mathbf{f}(a')$. But then the paths $\alpha = (a, a')$ and $\alpha' = (a', a')$ are parallel in $\mathbf{O}(\leqq)$ and satisfy $\mathbf{f}(\alpha) = \mathbf{f}(\alpha')$, which means, by lemma 1.2., that a, a' are f-symmetric in $\mathbf{O}(\leqq)$. This contradicts the hypothesis that $\mathbf{f}$ is a monomorphism in $\mathbf{O}(\leqq)$.

Thus we have proved:

Theorem 3.6. *Each monomorphism in* $\mathbf{O}(<)$ *is also a monomorphism in* $\mathbf{O}(\leqq)$ *and conversely.*

In the end, using theorem 2.6., we can state:

Theorem 3.7. *The categories* $\mathbf{O}(\leqq)$, $\mathbf{O}(<)$, $\mathbf{L}(\wedge, \vee, \leqq)$, $\mathbf{SG}(+, \leqq)$, $\mathbf{G}(+, \leqq)$ *and* $\mathbf{R}(+, \cdot, \leqq)$ *are locally small.*

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