Archivum Mathematicum

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Archivum Mathematicum, Vol. 13 (1977), No. 2, 105--109

Persistent URL: `http://dml.cz/dmlcz/106964`

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ARCH. MATH. 2, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XIII: 105—110, 1977

TOLERANCE RELATIONS ON SIMPLE TERNARY ALGEBRAS

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(Received September 6, 1976)

In the paper [7] Zelinka determines some basic properties of tolerance relations on finite tree structures. The purpose of this note is to determine the structure of the lattice $T(\mathfrak{A})$ of all compatible tolerance relations on a ternary algebra $\mathfrak{A}$ determining a tree structure.

Let $\mathfrak{B} = (B, \mathscr{F})$ be an algebra with the support B and with the set $\mathscr{F}$ of fundamental operations. A *tolerance relation* T on the set B is a reflexive and symmetric binary relation on B. T is called *compatible* with $\mathfrak{B}$, if and only if for each n-ary operation $f \in \mathscr{F}$ (where n is a positive integer) and for any $2n$ elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ of B which satisfy $x_i T y_i$ for $i = 1, \ldots, n$, we have $f(x_1, \ldots, x_n)\, Tf(y_1, \ldots, y_n)$.

In [2] Chajda, Niederle and Zelinka introduced the concept of *τ-covering*, which is an analogy for the class partition given by an equivalence relation on a set. Let M be a non-empty set. The family $\mathfrak{M} = \{M_\gamma, \gamma \in \Gamma,\}$ where Γ is a subscript set, is called a covering of M by subsets if and only if each M_γ for $\gamma \in \Gamma$ is a subset of M and $\cup_\gamma\{M_\gamma \mid \gamma \in \Gamma\} = M$, and $M_\gamma \neq M_\beta$ for $\gamma, \beta \in \Gamma$ and $\gamma \neq \beta$. A covering $\mathfrak{M} = \{M_\gamma, \gamma \in \Gamma\}$ of M by subsets is called a τ-covering of M, if and only if $\mathfrak{M}$ satisfies the following two conditions:

(1) if $\gamma_0 \in \Gamma$ and $\Gamma_0 \subseteq \Gamma$, then $M_{\gamma 0} \subseteq \cup_\gamma\{M_\gamma \mid \gamma \in \Gamma_0\} \Rightarrow \cap_\gamma\{M_\gamma \mid \gamma \in \Gamma_0\} \subseteq M_{\gamma 0}$:
(2) if $N \subseteq M$ and N is not contained in any set from $\mathfrak{M}$, then N contains a two-element subsets of the same property.

The following lemma shows the connection between tolerance relations on M and the τ-coverings of M [2, Thm. 1]:

Lemma 1. *Let M be a non-empty set. Then there exists a one-to-one correspondence between tolerance relations on M and τ-coverings of $\mathfrak{M}$ such that if T is a tolerance relation on M and $\mathfrak{M}_T$ is the τ-covering of M corresponding to T, then any two elements of M are in the relation T if and only if there exists a set from $\mathfrak{M}_T$ which contains both of them.*

Let V be a non-empty set and Q a ternary operation defined on V. The pair $(V, Q) = \mathfrak{A}$ is called a simple ternary algebra $\mathfrak{A}$, if Q satisfies the following demands:
(3) $Q(a, a, b) = a, \;\; a, b \in V$;

(4) $Q(a, b, c)$ is invariant under all 6 permutations of $a, b, c \in V$;

(5) $Q(Q(a, b, c), d, e) = (Q(a, d, e), Q(b, d, e), c)$, $a, b, c, d, e \in V$.

Let U and W be two non-empty subsets of V and s an element of V, then $Q(U, W, s) = \{Q(u, w, s) \mid u \in U \text{ and } w \in W\}$. A non-empty set $W \subseteq V$ is an ideal of $\mathfrak{A}$, whenever $Q(W, W, s) \subseteq W$ for each $s \in V$. According to (2), W is an ideal whenever $Q(W, W, s) = W$ for each $s \in V$. Let $\mathscr{W}$ be the family of all ideals of $\mathfrak{A}$. As shown in [5], $\mathscr{W}(\mathfrak{A}) = (\mathscr{W}, Q)$ is a simple ternary algebra over the ideals of $\mathfrak{A}$, where $Q(U, W, K) = \{Q(u, w, k) \mid u \in U, \; w \in W, \; k \in K \text{ and } U, W, K \in \mathscr{W}\}$. We denote by $I[x, z]$ the ideal $\{t \mid t = Q(x, z, t), \; x, z, t \in V\}$ of $\mathfrak{A}$. The ideal concept of simple ternary algebras is based on the definition of Nebeský given in [4].

Let $\mathfrak{A} = (V, Q)$ be a simple ternary algebra and $x \in V$ an arbitrary element. As shown by Avann [1, Lemma 3], one can associate with $\mathfrak{A}$ a *partial lattice* $L(\mathfrak{A}, x)$ having the following properties: (i) The order relation is given in $L(\mathfrak{A}, x)$ by $b \leqq c \Leftrightarrow Q(x, b, c) = b$. (ii) The zero element of $L(\mathfrak{A}, x)$ is x. (iii) $L(\mathfrak{A}, x)$ is closed with respect to the meet given by $b \wedge c = Q(x, b, c)$. (iv) The existence of an element m, $b, c \leqq m$, implies the existence of the join $b \vee c = Q(m, b, c)$. (v) If $b \vee c$ exists, then $d \wedge (b \vee c) = (d \wedge b) \vee (d \wedge c)$. (vi) For all triples $b, c, d \in V$ there exists $(b \wedge c) \vee (b \wedge d) \vee (c \wedge d) = Q(b, c, d)$.

Lemma 2. *Let T be a compatible tolerance relation on a simple ternary algebra $\mathfrak{A} = (V, Q)$. Then, if aTb, T collapses any two elements of the ideal $I[a, b]$.*

Proof. Let $t, x \in I[a, b]$, i.e. $t = Q(t, a, b)$ and $x = Q(x, a, b)$. As aTb, bTb, xTx and T is compatible, we obtain xTb. Similarly, tTb, too. The relations xTb, bTt and aTa imply now $x = Q(x, b, a) \; TQ(y, b, a) = t$.

Lemma 3. *Let $\mathfrak{A} = (V, Q)$ be a simple ternary algebra and $\mathscr{D}(\mathfrak{A}) = (\mathscr{D}, Q)$ a subalgebra of the simple ternary algebra $\mathscr{W}(\mathfrak{A}) = (\mathscr{W}, Q)$ closed with respect to the ternary operation Q. If for each $x \in V$, x belongs to at least one of the ideals of $\mathscr{D}$, the subsets from $\mathscr{D}$ constitute a τ-covering of $\mathfrak{A}$ determining a compatible tolerance relation on $\mathfrak{A}$, and conversely, the ideals of the τ-covering $\mathfrak{M}_T = \{M_\gamma, \gamma \in \Gamma\}$ of a compatible tolerance relation T on $\mathfrak{A}$ constitute a subalgebra of $\mathscr{W}(\mathfrak{A})$ closed with respect to Q.*

Proof. Let $\mathscr{D}(\mathfrak{A})$ be the subalgebra of the lemma; we show that a compatible tolerance relation $T_{\mathscr{D}}$ can be associated with $\mathscr{D}(\mathfrak{A})$, and this shows that the ideals from $\mathscr{D}$ constitute a τ-covering of V. We define the relation $T_{\mathscr{D}}$ as follows: $aT_{\mathscr{D}}b \Leftrightarrow$ there is an ideal $I \in \mathscr{D}$ such that $a, b \in I$. As each $x \in V$ belongs to at least one of the ideals from $\mathscr{D}$, $T_{\mathscr{D}}$ is reflexive. Obviously $T_{\mathscr{D}}$ is symmetric. If $X, Y, Z \in \mathscr{D}$, then $Q(X, Y, Z) \in \mathscr{D}$, and so $x_1 T_{\mathscr{D}} x_2$, $y_1 T_{\mathscr{D}} y_2$ and $z_1 T_{\mathscr{D}} z_2$ imply that $Q(x_1, y_1, z_1) \; T_{\mathscr{D}} \times \times Q(x_2, y_2, z_2)$, where $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $z_1, z_2 \in Z$.

Conversely, we show at first that each member of the τ-covering $\mathfrak{M}_T$ is an ideal of $\mathfrak{A}$. Let $x, y \in M_\gamma \in \mathfrak{M}_T$, and let $s \in V$ be an arbitrary element. As T is a compatible tolerance relation, xTx, xTy and sTs imply that $Q(x, y, s) \; TQ(x, x, s) = x$. Similarly, $Q(x, y, s) \; Ty$, and so $Q(x, y, s) \in \mathfrak{M}_\gamma$, i.e. to the same subset of V from the τ-covering

$\mathfrak{M}_T$ as x and y. Therefore M_y, and consequently each subset from $\mathfrak{M}_T$, is an ideal of $\mathfrak{A}$. As T is compatible, the relations x_1Tx_2, y_1Ty_2 and z_1Tz_2 imply that $Q(x_1, y_1, z_1)\, TQ(x_2, y_2, z_2)$. So the elements in the set $\{Q(x, y, z) \mid x \in X,\ y \in Y,\ z \in Z,\ X, Y, Z \in \mathfrak{M}_T\}$ constitute a class of elements each two of which are in the relation T, whence this class belongs to the τ-covering $\mathfrak{M}_T$. Thus the ideals from $\mathfrak{M}_T$ are closed with respect to Q in $\mathscr{W}(\mathfrak{A})$, and the lemma follows.

Lemma 3 shows that the structure of the lattice $T(\mathfrak{A})$ of all compatible tolerance relations on $\mathfrak{A}$ is equivalent with the structure of the complete lattice of all closed subalgebras of $\mathscr{W}(\mathfrak{A})$ containing every element $x \in V$ in at least one ideal of the subalgebra. In the following we shall consider the structure of $T(\mathfrak{A})$ in the case where the simple ternary algebra $\mathfrak{A} = (V, Q)$ determines a tree. A simple ternary algebra $\mathfrak{A} = (V, Q)$ is a tree, if for any $x \in V$ the partial lattice $L(\mathfrak{A}, x)$ is a tree, i.e. no two non-comparable elements a and b of $L(\mathfrak{A}, x)$ have a common upper bound in $L(\mathfrak{A}, x)$. Simple ternary algebras determining a tree are called *tree algebras.*

Lemma 4. *Let $\mathfrak{A} = (V, Q)$ be a tree algebra, and let $M, K, J \in \mathscr{W}$. $Q(M, K, J)$ is either an element of V, or $Q(M, K, J) \subseteq (K \cap M) \cup (K \cap J) \cup (M \cap J)$.*

Proof. At first we show that if $Q(M, K, J)$ contains an element $x \in V$ not belonging to M, K, or to J, then $Q(M, K, J)$ is the one element set $\{x\}$. Assume that $Q(m, k, j) = x \notin M \cup K \cup J$; we consider the situation in the partial lattice $L(\mathfrak{A}, x)$.

In $L(\mathfrak{A}, x)$, $x = (k \wedge m) \vee (k \wedge j) \vee (j \wedge m)$, and as x is the least element of $L(\mathfrak{A}, x)$, $k \wedge m = k \wedge j = m \wedge j = x$. Assume that there are two elements $k' \in K$ and $j' \in J$ such that $k' \wedge j' > x$, whence $k \wedge k' \wedge j' \geqq x$. As $L(\mathfrak{A}, x)$ is a tree and $k' \geqq k \wedge k'$, $k' \wedge j'$, the elements $k \wedge k'$ and $k' \wedge j'$ are comparable in $L(\mathfrak{A}, x)$. If $k \wedge k' \geqq k' \wedge j'$, then $k \wedge k' \wedge j' = k' \wedge j' > x$. If $k \wedge k' < k' \wedge j'$, then $k \wedge k' \wedge j' = k \wedge k'$, and as $x \notin K \cup J \cup M$, $k \wedge k' > x$, and consequently in any case $k \wedge k' \wedge j' > x$. Similarly we see that $j \wedge j' \wedge k' \to x$. As $j' \wedge k' \geqq j' \wedge k' \wedge k, j \wedge j' \wedge k'$, the elements $j \wedge j' \wedge k$ and $j' \wedge k' \wedge k$ are comparable in $L(\mathfrak{A}, x)$. Hence the meet $k \wedge j \wedge k' \wedge j'$ is equal to $k \wedge k' \wedge j'$ or to $j \wedge j' \wedge k'$, and so greater than x. But then $k \wedge j > x$ as well, which is a contradiction. Thus the meet of any two elements from K and J is equal to x; this holds also for all meets $m' \wedge j'$ and $m' \wedge k'$, where $m' \in M$, $k' \in K$ and $j' \in J$. Consequently, $Q(k', m', j') = x$ for all triples k', j', m'.

According to the proof above, we can assume in the following that $Q(M, K, J) \subseteq M \cup K \cup J$ without loosing generality. Let $k, j' \in Q(M, K, J)$, $k \in K, j' \in J$ but $k, j' \notin K \cap J$; we shall show that then $k, j' \in M$, which proves the assertion that $Q(M, K, J) \subseteq (M \cap J) \cup (M \cap K) \cup (K \cap J)$. According to the definitions of k and j', $k = Q(k, m, j)$ and $j' = Q(k', m', j')$. We consider the partial lattice $L(\mathfrak{A}, k)$, where $0 = k = (k \wedge m) \vee (k \wedge j) \vee (m \wedge j)$ and $j' = (j' \wedge m') \vee (k' \wedge m') \vee (j' \wedge k')$. So $m \wedge j = k$, and as $L(\mathfrak{A}, k)$ is a tree, j' is equal to at least one of the elements $(j' \wedge m')$, $(k' \wedge m')$, $(k' \wedge j')$. If $(k' \wedge m')$ or $(k' \wedge j')$ were equal to j', then $k' \geqq j'$, and as $j' \in I[k', k]$ and $I[k', k] \subseteq K$, also $j' \in K$, which is a contradiction. Hence $j' \wedge m' =$

$= j'$, and thus $m' \geq j'$. Now, $m \wedge j'$ and $j \wedge j'$ are comparable, since $j' \geq m \wedge j'$, $j' \wedge j$. Thus $m \wedge j' \wedge j \wedge j' = m \wedge j \wedge j' = k$ is equal to $m \wedge j$ or to $j \wedge j'$. If $j \wedge j' = m \wedge j \wedge j'$, then $j \wedge j' = k$, and as $j \wedge j' \in J$, also $k \in J$, which is a contradiction. Hence $m \wedge j' = k$. As $m' \geq m \wedge m', j'$, they are comparable, whence $m \vee m' \wedge j' = m \wedge j'$ is equal to j' or to $m \wedge m'$. As $m \wedge j' = k$, $m \wedge j' \neq j'$, since in the other case $j' \in K$, which is a contradiction. Hence $m \wedge j' = m \wedge m' = k$, and as $m \wedge m' \in M$, also $k \in M$. On the other hand $j' \in I[m', m \wedge m']$, and $I[m', m \wedge m'] \subseteq M$, whence also $j' \in M$. This completes the proof.

Lemma 5. *Let $\mathfrak{A} = (V, Q)$ be a tree algebra and $T, R \in T(\mathfrak{A})$. Then $T \vee R = T \cup R$, i.e. $T(\mathfrak{A})$ is a sublattice of the lattice of all binary relations on the set V.*

Proof. Obviously the relation $T \cup R$ is reflexive and symmetric; we must only show that the relation $S = T \cup R$ is compatible. The definition of S implies then that $S = T \vee R$ in $T(\mathfrak{A})$.

Let x_1Sx_2, y_1Sy_2 and z_1Sz_2. The ideals $I[x_1, x_2]$, $I[y_1, y_2]$, $I[z_1, z_2]$ belong to the τ-coverings $\mathfrak{M}_T$ and $\mathfrak{M}_R$. According to Lemma 4, $Q(I[x_1, x_2], I[y_1, y_2], I[z_1, z_2])$ is contained in $(I[x_1, x_2] \cap I[y_1, y_2]) \cup (I[x_1, x_2] \cap I[z_1, z_2]) \cup (I[y_1, y_2] \cap I[z_1, z_2])$ or is equal to an element of V. But in both of these two cases, any two of the elements in $Q(I[x_1, x_2], I[y_1, y_2], I[z_1, z_2])$ are collapsed by T or R, whence $Q(x_1, y_1, z_1)\, SQ(x_2, y_2, z_2)$, too. Thus S is compatible. This completes the proof.

As the join operation in $T(\mathfrak{A})$ is equivalent with the set union, we can write as a direct corollary to Lemma 5.

Theorem 1. *Let $\mathfrak{A} = (V, Q)$ be a tree algebra, then $T(\mathfrak{A})$ is a distributive lattice.*

The following theorem illuminates the Boolean property of $T(\mathfrak{A})$.

Theorem 2. *Let $\mathfrak{A} = (V, Q)$ be a tree algebra.* $T(\mathfrak{A})$ is a boolean lattice if and only if V contains at most two elements.

Proof. Let V contain at least three elements x, y, z. We can always find a partial lattice where x, y and z constitute a chain, and let it be $L(\mathfrak{A}, x)$ and the chain $x < y < z$. $T[x, y] \vee T[y, z] = R$ is a tolerance relation on $\mathfrak{A}$. Let R' be the complement of R in $T(\mathfrak{A})$; so $x(R \vee R')\, z$. According to Lemma 5, xRz or $xR'z$. The definition of R shows that xRz does not hold, whence $xR'z$. According to Lemma 2, $xR'y$, too, and so $x(R \wedge R')\, y$, whence R' is not a complement of R; this a contradiction. Obviously $T(\mathfrak{A})$ is Boolean when V contains at most two elements, and the theorem follows.

For further information about tolerance relations on lattices and other algebraic structures the reader is referred to [3], [8], [9] and to [10]. Congruence relations on simple ternary algebras are considered in the paper [6].

Acknowledgement

I would like to express my sincere thanks to Dr. Chajda for his kind comments and suggestions for preparing this paper.

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