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A LINEAR INEQUALITY OF GRONWALL'S TYPE CONTAINING MULTIPLE INTEGRAL

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Let $C(J, R^+)$ be the class of continuous functions $J \to R^+$, where $R^+ = [0, \infty)$ and $J = [a, b)$, $-\infty < a < b \leqq \infty$. For $u_1, \ldots, u_m \in C(J, R^+)$ let us define

$$(1) \qquad K[u_1, \ldots, u_m](\alpha, \beta) = \int_\alpha^\beta u_1(t_1) \int_\alpha^{t_1} u_2(t_2) \ldots \int_\alpha^{t_{m-1}} u_m(t_m)\,\mathrm{d}t_m \ldots \mathrm{d}t_1,$$
$$a \leqq \alpha \leqq \beta < b.$$

If the functions $u_1, \ldots, u_m$ are fixed, then $K(\alpha, \beta)$ is nonnegative and continuously differentiable for every $a \leqq \alpha \leqq \beta < b$, and

$$(2) \qquad \frac{-\partial K}{\partial \alpha}[u_1, \ldots, u_m](\alpha, \beta) = u_m(\alpha)\,K[u_1, \ldots, u_{m-1}](\alpha, \beta),$$

$$(3) \qquad \frac{\partial K}{\partial \beta}[u_1, \ldots, u_m](\alpha, \beta) = u_1(\beta)\,K[u_2, \ldots, u_m](\alpha, \beta).$$

(For $m = 1$ the right side of (2), (3) equals $u_1(\alpha)$, $u_1(\beta)$, respectively.)

In this paper we shall obtain some upper bounds of functions $x \in C(J, R^+)$ satisfying on J the inequality

$$(4) \qquad x(t) \leqq f(t) + g(t)\,K[p_1, \ldots, p_{n-1}, p_n x](a, t),$$

where $f, g, p_1, \ldots, p_n$ are fixed elements of $C(J, R^+)$. More general inequality containing multiple integral

$$(5) \qquad x(t) \leqq f(t) + g(t) \int_a^t \int_a^{t_1} \ldots \int_a^{t_{n-1}} p(t, t_1, \ldots, t_n)\,x(t_n)\,\mathrm{d}t_n \ldots \mathrm{d}t_1$$

has been investigated by M. Ráb in [1]. Applying the general result proved in [1] to the special inequality (4), we receive the following

Lemma 1. *Let $x, f, g, p_1, \ldots, p_n \in C(J, R^+)$ and let* (4) *be valid for $t \in J$. Then*

$$(6) \qquad x(t) \leqq f(t) + g(t) \int_a^t \frac{\partial K}{\partial \beta}[p_1, \ldots, p_{n-1}, p_n f](a, s) \times$$
$$\times \exp \int_s^t \frac{\partial K}{\partial \beta}[p_1, \ldots, p_{n-1}, p_n g](a, r)\,\mathrm{d}r\,\mathrm{d}s, \qquad t \in J.$$

If f and g are nondecreasing on J, then (6) *implies that*

$$x(t) \leqq f(t) \exp\{g(t)\, K[p_1, \ldots, p_n](a, t)\}, \qquad t \in J. \tag{7}$$

This result has been proved in [1] by the method of comparison of the integral inequality (5) with certain linear scalar differential inequality of the first order. In our paper we realize analogous comparison with a system of m linear scalar differential inequalities, $1 \leqq m \leqq n$; we get some upper bounds, similar to (6). It is interesting that the functional argument of K in the obtained bounds permits some of cyclical permutations.

Theorem. *Let $x, f, g, p_1, \ldots, p_n \in C(J, R^+)$ and let* (4) *be valid on J. Put $f_n = f$, $g_n = g$ and*

$$f_m(t) = K[p_{m+1}, \ldots, p_{n-1}, p_n f](a, t),$$
$$g_m(t) = K[p_{m+1}, \ldots, p_{n-1}, p_n g](a, t), \qquad t \in J, m = 1, \ldots, n-1.$$

Then

$$x(t) \leqq f(t) + g(t) \int_a^t \frac{-\partial K}{\partial \alpha}[p_1, \ldots, p_{m-1}, p_m f_m](s, t) \times \tag{8}$$
$$\times \exp K[q_1, \ldots, q_m](s, t)\, \mathrm{d}s, \qquad t \in J,$$

where $q_1, \ldots, q_m$ is an arbitrary cyclical permutation of the system $p_1, \ldots, p_{m-1}, p_m g_m$, for all $m = 1, 2, \ldots, n$.

The proof of Theorem is based on the following

Lemma 2. *Let the real function c be continuous on J and let $p_1, \ldots, p_m \in C(J, R^+)$ Denote*

$$c^+ = \frac{1}{2}(c + |c|).$$

If the system of functions $u_1, \ldots, u_m$ is the solution of the initial value problem

$$u_k' = p_k(t)\, u_{k+1}, \qquad k = 1, 2, \ldots, m-1,$$
$$u_m' = p_m(t)\, u_1 + c(t),$$
$$u_1(a) = \ldots = u_m(a) = 0,$$

then

$$u_1(t) \leqq \int_a^t \frac{-\partial K}{\alpha}[p_1, \ldots, p_{m-1}, c^+](s, t) \times$$
$$\times \exp K[q_1, \ldots, q_m](s, t)\, \mathrm{d}s, \qquad t \in J,$$

where $q_1, \ldots, q_m$ is an arbitrary cyclical permutation of the system $p_1, \ldots, p_m$.

Proof of Lemma 2. Using the method of variation of constants, we receive

$$u_k(t) = \int_a^t c(s)\, v_k(t, s)\, \mathrm{d}s, \qquad t \in J, k = 1, \ldots, m, \tag{9}$$

where the functions $v_k(t) = v_k(t, s)$, $k = 1, \ldots, m$, satisfy the system of equations

$$(10) \qquad v_k' = p_k(t)\, v_{k+1}, \qquad k = 1, 2, \ldots, m-1,$$
$$v_m' = p_m(t)\, v_1,$$

and

$$(11) \qquad v_k(s) = 0, \qquad k = 1, 2, \ldots, m-1,$$
$$v_m(s) = 1,$$

for all fixed $s \in J$.

It is easily seen that the functions $v_k(t)$ are nonnegative on $[s, b)$. Then the equations (9) imply that

$$(12) \qquad u_k(t) \leqq \int_a^t c^+(s)\, v_k(t, s)\, \mathrm{d}s, \qquad t \in J,\ k = 1, 2, \ldots, m.$$

Integrating the equations (10) from s to t, we receive (with respect to (3) and (11))

$$(13) \qquad v_k(t) = K[p_k, \ldots, p_{m-1}](s, t) + K[q_1, \ldots, q_{m-1}, q_m v_k](s, t), \qquad t \in [s, b),$$

where $q_1, \ldots, q_m$ is the cyclical permutation of the system $p_1, \ldots, p_m$ such that $q_1 = p_k$; $k = 1, 2, \ldots, m$.

Now we apply Lemma 1 to the integral equations (13); since the functions $K[p_k, \ldots, p_{m-1}](s, t)$ are nondecreasing in t on $[s, b)$, we can write (see (7))

$$v_k(s, t) \leqq K[p_k, \ldots, p_{m-1}](s, t) \exp K[q_1, \ldots, q_m](s, t), \qquad t \in [s, b),\ k = 1, 2, \ldots, m.$$

Consequently, in view of the equations

$$v_1(s, t) = K[p_1, \ldots, p_{k-2}, p_{k-1} v_k(s, .)](s, t), \qquad t \in [s, b),\ k = 2, \ldots, m$$

(see (10) and (11))
we have

$$(14) \qquad v_1(s, t) \leqq K[p_1, \ldots, p_{k-2}, p_{k-1} K[p_k, \ldots, p_{m-1}](s, .) \times$$
$$\times \exp K[q_1, \ldots, q_m](s, .)](s, t), \qquad t \in [s, b)$$

for all $k = 1, 2, \ldots, m$.
Since the function $\exp K[q_1, \ldots, q_m](s, t_1)$ is nondecreasing in $t_1 \in [s, t)$, the inequality (14) can be simplified to the following one

$$(15) \qquad v_1(s, t) \leqq K[p_1, \ldots, p_{m-1}](s, t) \exp K[q_1, \ldots, q_m](s, t), \qquad t \in [s, b).$$

From (12) (with $k = 1$) and (15) we obtain

$$(16) \qquad u_1(t) \leqq \int_a^t c^+(s)\, K[p_1, \ldots, p_{m-1}](s, t) \times$$
$$\times \exp K[q_1, \ldots, q_m](s, t)\, \mathrm{d}s.$$

By (2) the right side of (16) equals

$$\int_a^t \frac{-\partial K}{\partial \alpha} [p_1, \ldots, p_{m-1}, c^+] (s, t) \exp K[q_1, \ldots, q_m] (s, t) \, ds.$$

The proof of Lemma 2 is complete.

Proof of Theorem. The functions

$$u_k(t) = K[p_k, \ldots, p_{n-1}, p_n x] (a, t), \qquad t \in J, k = 1, 2, \ldots, n,$$

satisfy the following system of equations

$$(17) \qquad u_k' = p_k(t) u_{k+1}, \qquad k = 1, 2, \ldots, n-1,$$
$$u_n' = p_n(t) g(t) u_1 + p_n(t) h(t),$$

where

$$(18) \qquad h(t) = x(t) - g(t) u_1(t), \qquad t \in J.$$

The inequality (4) can be written in the form

$$(19) \qquad x(t) \leqq f(t) + g(t) u_1(t), \qquad t \in J.$$

From (18) and (19) it follows that

$$(20) \qquad h(t) \leqq f(t), \qquad t \in J.$$

Now, let m be a fixed integer, $1 \leqq m \leqq n$. Using (17) and (20) we receive, with respect to $u_{m+1}(a) = \ldots = u_n(a) = 0$,

$$(21) \qquad u_m'(t) \leqq p_m(t) K[p_{m+1}, \ldots, p_{n-1}, p_n g u_1] (a, t) +$$
$$+ p_m(t) K[p_{m+1}, \ldots, p_{n-1}, p_n f] (a, t), \qquad t \in J.$$

Since u_1 is nondecreasing on J, it holds

$$(22) \qquad K[p_{m+1}, \ldots, p_{n-1}, p_n g u_1] (a, t) \leqq$$
$$\leqq u_1(t) K[p_{m+1}, \ldots, p_{n-1}, p_n g] (a, t), \qquad t \in J.$$

The inequalities (21) and (22) imply that

$$(23) \qquad u_m'(t) \leqq p_m(t) g_m(t) u_1(t) + p_m(t) f_m(t), \qquad t \in J.$$

(The functions f_m and g_m are defined in Theorem.)

Let us consider the system of m scalar equations

$$u_k' = p_k(t) u_{k+1}, \qquad k = 1, 2, \ldots, m-1,$$
$$u_m' = p_m(t) g_m(t) u_1 + c(t),$$

where

$$c(t) \leqq p_m(t) f_m(t), \qquad t \in J$$

(see (17) and (23)).

Taking in account that $u_1(t) = \dots = u_m(t) = 0$, we can apply Lemma 2:

$$u_1(t) \leqq \int_a^t \frac{-\partial K}{\partial \alpha} [p_1, \dots, p_{m-1}, p_m f_m](s, t) \times \\ \times \exp K[q_1, \dots, q_m](s, t) \, ds, \qquad t \in J, \tag{24}$$

where $q_1, \dots, q_m$ is an arbitrary cyclical permutation of $p_1, \dots, p_{m-1}, p_m g_m$.

Using (19) and (24) we obtain the desired inequality (8).

REFERENCE

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