ARCH. MATH. 2, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XVII: 87—90, 1981

BEST APPROXIMATION AND STRICT CONVEXITY OF METRIC SPACES

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(Received November 8, 1979)

The notion of strict convexity in metric spaces was introduced in [1] and certain existence and uniqueness theorems on best approximation in such a space were proved in [1] and [2]. In this note we take a stronger version of the notion of strict convexity and characterize such metric spaces. As a result we get the unicity theorem of best approximation 'Every convex proximinal set in a strictly convex metric space is chebyshev' and its converse.

Before proceeding to our main results, we recall few definitions:

Let (X, d) be a metric space and $x, y, z \in X$. We say that the point z is *between* x and y (writing xzy) if

$$d(x, z) + d(z, y) = d(x, y).$$

For any two points x, y of X, the set

$$\{z \in X : d(x, z) + d(z, y) = d(x, y)\},$$

i.e. the set of all those points which lie between x and y, is called the *segment* $[x, y]$.

A metric space (X, d) is said to be *convex* [4] if for every two points x and $y \in X$, there exists $z \in X$ such that $x \neq y \neq z$ and xzy i.e. if for every x, y in X and for every $t, 0 \leqq t \leqq 1$ there exists at least one point z such that

$$d(x, z) = (1 - t)\, d(x, y) \qquad \text{and} \qquad d(z, y) = td(x, y).$$

The space is said to be *strongly convex* [4] if such a z exists and is unique for each pair x and y of X.

Thus for strongly convex metric spaces each t, $0 \leqq t \leqq 1$, determines a unique point of the segment $[x, y]$.

A strongly convex metric space (X, d) is said to be *strictly convex* if for every x, y of X and $r > 0$,

$d(x, p) \leqq r$, $d(y, p) \leqq r$ imply $d(z, p) < r$ unless $x = y$, where p is arbitrary but fixed point of X and z is any point in the open segment $]x, y[$.

Therefore, in a strictly convex metric space if x and y are any two points on the boundary of a sphere then $]x, y[$ lies strictly inside the sphere.

A subset K of a metric space (X, d) is said to be *convex* [1] if for every $x, y \in K$, any point between x and y is also in K i.e. for each x, y in K, the segment $[x, y]$ lies in K.

Let S be a subset of a metric space (X, d) and z be a point of S. Let

$$S_z = \{x \in X : d(x, z) = d(x, S)\},$$

i.e. S_z is the set of all those points of X having z as a nearest point in S.

S is said to be *proximinal* if for each point x in X there is a point of S nearest to x i.e. for each x in X there exists at least one point $z \in S$ such that $x \in S_z$. If there is a unique such point z for each x in X then S is said to be *uniquely proximinal* or *Chebyshev*.

In [1] and [2] the conditions under which S is uniquely proximinal have been studied. We have the following unicity theorem of best approximation, the proof of which is contained in Theorem 2 of [1].

Theorem 1. *In a strictly convex metric space whenever a convex set is proximinal, it is uniquely proximinal.*

In order to show that the converse of the above theorem also holds, we establish a lemma.

Lemma. *For any two points x, y in a strongly convex metric space (X, d) the function*

$$\Phi = \Phi_{x,y} : [x, y] \to [0, d(x, y)] \subseteq R,$$

taking $z \in [x, y]$ to the real number $d(x, z)$ is an isometry.

Proof. We can assume $x \neq y$. Let $z \in [x, y]$ and $z' \in [z, y]$. Then

$$d(x, y) = d(x, z) + d(z, y) =$$
$$= d(x, z) + d(z, z') + d(z', y) \geqq d(x, z') + d(z', y) \geqq d(x, y).$$

Hence

(1) $$d(x, y) = d(x, z') + d(z', y)$$

and

$$\Phi(z') = d(x, z') = d(x, z) + d(z, z') = \Phi(z) + d(z, z')$$

implying

(2) $$| \Phi(z') - \Phi(z) | = d(z, z').$$

The equality (1) shows that $z' \in [x, y]$ and implies that $[x, y]$ is convex, and the equality (2) shows that Φ is an isometry.

Corollary. *For any two points x, y in a strongly convex metric space (X, d) the segment $[x, y]$ is a compact set.*

The following theorem shows that the converse of the unicity Theorem (Theorem 1) is also true.

Theorem 2. *Let (X, d) be a strongly convex metric space. Then the following statements are equivalent:*

(*i*) *X is strictly convex.*

(*ii*) *For each convex set S and distinct points x and y of S, $S_x \cap S_y = \emptyset$.*

(*iii*) *Whenever a convex set is proximinal, it is uniquely proximinal.*

Proof. (i) $\Rightarrow$ (ii).

Let, if possible, $S_x \cap S_y \neq \emptyset$ and let $z \in S_x \cap S_y$. This implies

$$d(z, x) = d(z, y) = d(z, S).$$

Now $x, y \in X$ and X is a convex space, therefore there exists $q \in X$ such that xqy. $q \in [x, y]$ and S is a convex set, therefore $q \in S$.

Strict convexity of the space implies $d(z, q) < d(z, S)$, which is a contradiction.

(ii) $\Rightarrow$ (iii).

Let a convex set S be proximinal. Let $p \in X$. Since S is proximinal, there exists $x \in S$ such that $p \in S_x$.

Let if possible, $y \neq x$ be also nearest to p, then $p \in S_y$. Thus $p \in S_x \cap S_y$, $x \neq y$, which is a contradiction.

(iii) $\Rightarrow$ (i).

Let $x \neq y$, p be points of (X, d) with $d(x, p) = d(y, p) = r$ (say). Define

$$f : I : = [0, d(x, y)] \rightarrow \mathbf{R},$$

as

$$f(t) = d(p, \Phi_{x,y}^{-1}(t)).$$

Then f is continuous. Moreover, since $[x, y]$ is a compact, convex subset, the hypothesis (iii) implies that there exists no subinterval $[t_1, t_2] \subseteq I$, $t_1 < t_2$, such that

$$f(t_1) = f(t_2) = \min\{f(t) : t_1 \leqq t \leqq t_2\}.$$

We affirm that all interior points $t \in]0, d(x, y)[$ satisfy

$$f(t) < \max f = f(0) = f(d(x, y)). \tag{3}$$

Let, if possible, $f(t_0) \geqq \max f$ for some interior point. Set

$$m' = \min\{f(t) : t \leqq t_0\},$$
$$m'' = \min\{f(t) : t \geqq t_0\}.$$

Suppose $m' \geqq m''$. Define

$$t_0' = \inf\{t : t \leqq t_0, \min\{f(t_1) : t \leqq t_1 \leqq t_0\} \geqq m'\},$$
$$t_0'' = \sup\{t : t \geqq t_0, \min\{f(t_2) : t_0 \leqq t_2 \leqq t\} \geqq m'\}.$$

Since f is continuous it follows that

$$f(t_0') = f(t_0'') = \min\{f(t) : t_0' \leqq t \leqq t_0''\}.$$

If $t_0' < t_0''$ then $[t_0', t_0'']$ is the subinterval leading to a contradiction, if $t_0' = t_0''$ then $I = [0, d(x, y)]$ is the subinterval leading to a contradiction. If $m' \leqq m''$, a contradiction can be reached in an analogous fashion.

Since Φ is an isometry, (3) implies $d(z, p) < r$ for any point z in the open segment $]x, y[$. Hence the space is strictly convex.

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