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ON THE INTEGRAL GEOMETRY OF THE LINEAR SUBSPACES IN A BIPLANAR SPACE

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Let J_p and K_q be fixed skew linear subspaces of the n-dimensional real projective space P_n, such that $\dim J_p = p$, $\dim K_q = q$, $p \leqq q$ and $p + q = n - 1$. The biplanar Kleinian space $B_n^{p,q}$ has P_n as the underlying space and its fundamental group G_L consists of all collineations of P_n preserving J_p and K_q. The subspaces J_p and K_q together form the absolute of $B_n^{p,q}$. The dimension of G_L is equal to $L = p^2 + q^2 + 2(p + q) + 1$.

We use a family of biplanar frames $R_L = (A_1 A_2 \dots A_{p+1} B_1 B_2 \dots B_{q+1})$ having the following property: the vertices $A_1, A_2, \dots, A_{p+1}$ lie in J_p and $B_1, B_2, \dots, B_{q+1}$ lie in K_q. An arbitrary element of G_L has the form

$$x'_i = \sum_{j=1}^{p+1} a_{ij} x_j, \qquad i = 1, \dots, p + 1,$$

$$y'_u = \sum_{v=1}^{q+1} b_{uv} y_v, \qquad u = 1, \dots, q + 1,$$

where $A = \det(a_{ij}) \neq 0$ and $B = \det(b_{uv}) \neq 0$. Without loss of generality we can suppose that

(1) $$A \, . \, B = 1.$$

Let

$$dA_i = \sum_{j=1}^{p+1} \omega_i^j A_j, \qquad i = 1, \dots, p + 1,$$

$$dB_u = \sum_{v=1}^{q+1} \psi_u^v B_v, \qquad u = 1, \dots, q + 1$$

be the infinitesimal transformations of R_L, where the 1-forms

(2)
$$\omega_i^j = \frac{1}{A} | A_1 \dots A_{j-1} dA_i A_{j+1} \dots A_{p+1} |,$$

$$\psi_u^v = \frac{1}{B} | B_1 \dots B_{v-1} dB_u B_{v+1} \dots B_{q+1} |$$

satisfy the structure equations

$$D\omega_i^j = \sum_{k=1}^{p+1} \omega_i^k \wedge \omega_k^j, \qquad i, j = 1, \ldots, p+1,$$

$$D\psi_u^v = \sum_{w=1}^{q+1} \psi_u^w \wedge \psi_w^v, \qquad u, v = 1, \ldots, q+1$$

and the unique non-differential condition

$$\sum_{i=1}^{p+1} \omega_i^i + \sum_{u=1}^{q+1} \psi_u^u = 0, \tag{3}$$

which is a consequence of (1). If we consider the homogeneous coordinates

$$\begin{aligned} &A_i(a_{1i}, a_{2i}, \ldots, a_{p+1.i}, 0, 0, \ldots, 0), \qquad i = 1, \ldots, p+1, \\ &(B_u(0, 0, \ldots, 0, b_{1u}, b_{2u}, \ldots, b_{q+1.u}), \qquad u = 1, \ldots, q+1 \end{aligned} \tag{4}$$

of the vertices of R_L, then from (2) we get

$$\begin{aligned} \omega_i^j &= \frac{1}{A} \sum_{k=1}^{p+1} A_{kj}\, \mathrm{d}a_{ki} = -\frac{1}{A} \sum_{k=1}^{p+1} a_{ki}\, \mathrm{d}A_{kj}, \\ \psi_u^v &= \frac{1}{B} \sum_{w=1}^{q+1} B_{wv}\, \mathrm{d}b_{wu} = -\frac{1}{B} \sum_{w=1}^{q+1} b_{wu}\, \mathrm{d}B_{wv}, \end{aligned} \tag{5}$$

where A_{kj} and B_{wv} are the cofactors of the elements a_{kj} and b_{wv}, respectively.

2. Let L_t $(0 \leqq t \leqq p)$ be an arbitrary linear subspace of $B_n^{p,q}$, without common points with the absolute. Using the method of L. A. Santalo in [6], we find conditions for the density of L_t to be invariant with respect to G_L. Assume that L_t is determined by the linearly independent points $M_1 = A_1 + B_1, M_2 = A_2 + B_2, \ldots, M_{t+1} = A_{t+1} + B_{t+1}$. The isotropy group of L_t is given by the following completely integrable Pfaffian system

$$\omega_\lambda^\mu - \psi_\lambda^\mu = 0, \qquad \omega_\lambda^\alpha = 0, \qquad \psi_\lambda^\beta = 0,$$

where

$$\lambda, \mu = 1, \ldots, t+1; \qquad \alpha = t+2, \ldots, p+1;$$
$$\beta = t+2, \ldots, q+1.$$

We now consider the $(t+1)(n-t)$-form

$$\mathrm{d}L_t = \bigwedge_{\lambda=1}^{t+1} [\bigwedge_{\alpha=t+2}^{p+1} \omega_\lambda^\alpha \wedge \bigwedge_{\beta=t+2}^{q+1} \psi_\lambda^\beta \wedge \bigwedge_{\mu=1}^{t+1} (\omega_\lambda^\mu - \psi_\lambda^\mu)]. \tag{6}$$

By the well-known criterion of S. S. Chern [4], there exists a unique biplanar density (6) of L_t iff $D(\mathrm{d}L_t) = 0$. But

$$D(\mathrm{d}L_t) = \sum_{\lambda=1}^{t+1} [(p+1)\,\omega_\lambda^\lambda + (q+1)\,\psi_\lambda^\lambda] \wedge \mathrm{d}L_t$$

and $D(\mathrm{d}L_t) = 0$ iff

(7) $$\sum_{\lambda=1}^{t+1} [(p+1)\,\omega_\lambda^\lambda + (q+1)\,\psi_\lambda^\lambda] = 0.$$

From (3) we see that (7) holds iff $p = q$ and $t = p$. Thus obtain the following theorem:

Theorem 1. *The subspaces L_t have the density* (6) *invariant with respect to G_L iff $p = q$ and $t = p$.*

The theorem shows that the only case in which the linear subspaces L_t without common points with the absolute have a biplanar density is one of maximal dimension $t = p$ and $B_n^{p,q} = B_{2p+1}^{pp}$. The latter means that the biplanar space is the generalized biaxial space. Thus we obtain a new characterization of the generalized biaxial space.

Remark. One can analogously show that the linear subspaces which have common points with J_p and K_q have no density invariant under G_L.

Let us now consider the set of m-tuples of skew linear subspaces $\Sigma_1 = (L_{t_1}, L_{t_2}, \ldots, L_{t_m})$ without common points with J_p and K_q and whose dimensions t_i satisfy the condition

$$t_1 + t_2 + \ldots + t_m + m \leqq p + 1.$$

Assume that L_{t_1} is determined by the linearly independent points $M_1, M_2, \ldots, M_{t_1+1}$ and L_{t_2} by the linearly independent points $M_{t_1+2}, M_{t_1+3}, \ldots, M_{t_1+t_2+2}$ and so on, where $M_i = A_i + B_i$, $i = 1, \ldots, t_1 + t_2 + \ldots + t_m + m$. In the exterior product

(8) $$\mathrm{d}\Sigma_1 = \mathrm{d}L_{t_1} \wedge \mathrm{d}L_{t_2} \wedge \ldots \wedge \mathrm{d}L_{t_m}$$

the factor (for $1 \leqq s \leqq m$)

$$\mathrm{d}L_{t_s} = \bigwedge_{\lambda_s} [\bigwedge_{\alpha'_s} (\omega_{\lambda_s}^{\alpha'_s} \wedge \psi_{\lambda_s}^{\alpha'_s}) \wedge \bigwedge_{\alpha''_s} \omega_{\lambda_s}^{\alpha''_s} \wedge \bigwedge_{\beta''_s} \psi_{\lambda_s}^{\beta''_s} \wedge \bigwedge_{\mu_s} (\omega_{\lambda_s}^{\mu_s} - \psi_{\lambda_s}^{\mu_s})],$$

$$t_1 + \ldots + t_{s-1} + s \leqq \lambda_s, \quad \mu_s \leqq t_1 + \ldots + t_s + s,$$

$$1 \leqq \alpha'_a \leqq t_1 + \ldots + t_{s-1} + s - 1, \qquad t_1 + \ldots + t_s + s + 1 \leqq \alpha''_s \leqq p + 1,$$

$$t_1 + \ldots + t_s + s + 1 \leqq \beta''_s \leqq q + 1$$

is the exterior product of all left sides of the equations of the completely integrable Pfaffian system, which determines the isotropy group of the element L_{t_s}. It follows from

$$D(\mathrm{d}\Sigma_1) = \sum_{\lambda=1}^{t_1+t_2+\ldots+t_m+m} [(p+1)\,\omega_\lambda^\lambda + (q+1)\,\psi_\lambda^\lambda] \wedge \mathrm{d}\Sigma_1$$

and the criterion of S. S. Chern that a necessary and sufficient condition for existence of the biplanar density (8) of the m-tuples Σ_1 is

$$\sum_{\lambda=1}^{t_1+t_2+\ldots+t_m+m} [(p+1)\omega_\lambda^\lambda + (q+1)\psi_\lambda^\lambda] = 0.$$

From (3) we conclude that the above equality has place iff $p = q$ and $t_1 + t_2 + \ldots + t_m + m = p + 1$.

Hence we have

Theorem 2. *The m-tuples Σ_1 have the invariant density* (8) *under G_L iff $p = q$ and $t_1 + t_2 + \ldots + t_m + m = p + 1$.*

The Theorem 1 is the special case $m = 1$ of the Theorem 2. Remark that the existence of invariant density for Σ_1 in $B_{2p+1}^{p;p}$ is established by G. Stanilov [3].

Let Σ_2 be an arbitrary system of $h + s + m$ skew linear subspaces $L_{p_1}, L_{p_2}, \ldots, L_{p_h}, L_{q_1}, L_{q_2}, \ldots, L_{q_s}, L_{r_1}, L_{r_2}, \ldots, L_{r_m}$ which satisfy the conditions

$$\bigcup_{\lambda=1}^{h} L_{p_\lambda} \subset J_p, \quad \bigcup_{\sigma=1}^{s} L_{q_\sigma} \subset K_q, \quad (\bigcup_{a=1}^{m} L_{r_a}) \cap (J_p \cup K_q) = \emptyset,$$

$$P + R + h + m \leqq p + 1, \qquad Q + R + s + m \leqq q + 1,$$

where

$$P = \sum_{\lambda=1}^{h} p_\lambda, \qquad Q = \sum_{\sigma=1}^{s} q_\sigma, \qquad R = \sum_{a=1}^{m} r_a$$

and

$$p_\lambda = \dim L_{p_\lambda}, \qquad \lambda = 1, \ldots, h,$$
$$q_\sigma = \dim L_{q_\sigma}, \qquad \sigma = 1, \ldots, s,$$
$$r_a = \dim L_{r_a}, \qquad a = 1, \ldots, m.$$

Suppose that the linear subspaces L_{p_λ} ($\lambda = 1, \ldots, h$), L_{q_σ} ($\sigma = 1, \ldots, s$), L_{r_a} ($a = 1, \ldots, m$) are determined by the linearly independent points

$$L_{p_\lambda} = (A_{P-p_\lambda+\lambda}, A_{P-p_\lambda+\lambda+1}, \ldots, A_{P+\lambda}),$$
$$L_{q_\sigma} = (B_{Q-q_\sigma+\sigma}, B_{Q-q_\sigma+\sigma+1}, \ldots, B_{Q+\sigma}),$$
$$L_{r_a} = (M_{R-r_a+a}, M_{R-r_a+a+1}, \ldots, M_{R+a}),$$

where $M_{R-r_a+a+\varepsilon} = A_{E+a+\varepsilon} + B_{F+a+\varepsilon}$, $E = P + R + h - r_a$ and $F = Q + R + s - r_a$ for $\varepsilon = 0, 1, \ldots, r_a$. The completely integrable Pfaffian system, which determines the isotropy group of Σ_2 has the form

$$\text{(9)} \qquad \begin{aligned} &\omega_{i_\lambda}^{j_\lambda} = 0,\ \psi_{u_\sigma}^{v_\sigma} = 0,\ \omega_{\xi_a}^{\eta_a} = 0,\ \psi_{\varrho_a}^{\tau_a} = 0, \\ &\omega_{E+a+\varepsilon}^{E+a+\delta} - \psi_{F+a+\varepsilon}^{F+a+\delta} = 0, \end{aligned}$$

where

$$P - p_\lambda + \lambda \leqq i_\lambda \leqq P + \lambda, \qquad 1 \leqq \lambda \leqq h,$$
$$1 \leqq j_\lambda \leqq P - p_\lambda + \lambda - 1, \qquad P + \lambda + 1 \leqq j_\lambda \leqq p + 1,$$
$$Q - q_\sigma + \sigma \leqq u_\sigma \leqq Q + \sigma, \qquad 1 \leqq \sigma \leqq s,$$

$$
(10) \qquad \begin{array}{ll}
1 \leqq v_\sigma \leqq Q - q_\sigma + \sigma - 1, & Q + \sigma + 1 \leqq v_\sigma \leqq q + 1, \\
E + a \leqq \xi_a \leqq E + r_a + a, & 1 \leqq a \leqq m, \\
1 \leqq \eta_a \leqq E + a - 1, & E + r_a + a + 1 \leqq \eta_a \leqq p + 1, \\
F + a \leqq \varrho_a \leqq F + r_a + a, & 0 \leqq \varepsilon, \delta \leqq r_a, \\
1 \leqq \tau_a \leqq F + a - 1, & F + r_a + a + 1 \leqq \tau_a \leqq q + 1.
\end{array}
$$

After exterior differentiation of the form

$$
(11) \qquad d\Sigma_2 = \Lambda \omega_{i_\lambda}^{j_\lambda} \wedge \psi_{u_\sigma}^{v_\sigma} \wedge \omega_{\xi_a}^{\eta_a} \wedge \psi_{\varrho_a}^{\tau_a} \wedge (\omega_{E+a+\varepsilon}^{E+a+\delta} - \psi_{F+a+\varepsilon}^{F+a+\delta})
$$

which contains as factors all left sides of (9) (the indices have the ranges (10), we get

$$
D(d\Sigma_2) = [(p + 1) \sum_{i=1}^{P+R+h+m} \omega_i^i + (Q + s) \sum_{j=1}^{p+1} \omega_j^j + (q + 1) \sum_{u=1}^{Q+R+s+m} \psi_u^u +
$$

$$
+ (P + h) \sum_{v=1}^{q+1} \psi_v^v] \wedge d\Sigma_2 .
$$

It follows from (3) that the form $d\Sigma_2$ is closed iff

$$
(p + 1) \sum_{i=1}^{P+R+h+m} \omega_i^i + (Q + s) \sum_{j=1}^{p+1} \omega_j^j + (q + 1) \sum_{u=1}^{Q+R+s+m} \psi_u^u + (P + h) \sum_{v=1}^{q+1} \psi_v^v = 0.
$$

It is not hard to see that the above equality is possible exactly in one of the following three cases:

$$
(12) \qquad \begin{array}{lll}
\text{(i)} & P + R + h + m = p + 1, & Q + R + s + m = q + 1; \\
\text{(ii)} & P + h = p + 1, & s = 0, \ m = 0; \\
\text{(iii)} & Q + s = q + 1, & h = 0, \ m = 0.
\end{array}
$$

Therefore we get the following theorem:

Theorem 3. *The systems Σ_2 have the biplanar density* (11) *only in the cases* (12).

Theorem 3 allows us to determine in a simple manner the systems Σ_2 with invariant biplanar density. As an example, we consider the four-dimensional biplanar space $B_4^{1,2}$ with absolute of a line J_1 and a plane K_2. From Theorem 3 the possible cases are:

I. $h = 2, s = 3, m = 0, P = 0, Q = 0, R = 0;$
II. $h = 2, s = 2, m = 0, P = 0, Q = 1, R = 0;$
III. $h = 1, s = 2, m = 1, P = 0, Q = 0, R = 0;$
IV. $h = 1, s = 1, m = 1, P = 0, Q = 1, R = 0;$
V. $h = 0, s = 1, m = 2, P = 0, Q = 0, R = 0;$
VI. $h = 0, s = 1, m = 1, P = 0, Q = 0, R = 1;$
VII. $h = 2, s = 0, m = 0, P = 0, Q = 0, R = 0;$
VIII. $h = 0, s = 3, m = 0, P = 0, Q = 0, R = 0;$
IX. $h = 0, s = 2, m = 0, P = 0, Q = 1, R = 0.$

If $p_1 = p_2 = \dots = p_h = 0$, $q_1 = q_2 = \dots = q_s = 0$ and $r_1 = r_2 = \dots = r_m = 0$, then the systems Σ_2 consist of $h + s + m$ linearly independent points: h points in τ_p, s points in K_q and m points outside $J_p \cup K_q$. This special case is considered in [1]. An analogue of Theorem 3 for $B_{2p+1}^{p,p}$ is proved in [2].

3. Here we give a geometric interpretation of the biplanar density of some systems of special linear subspaces.

Example 1. Let $h = 2$, $s = 2$, $m = 0$ and choose $p_1 = 0$, $p_2 = p - 1$, $q_1 = 0$, $q_2 = q - 1$. The conditions (i) in (12) are satisfied and, from Theorem 3, the systems $\Sigma'_2 = \{$(a point in J_p) + (a $(p-1)$-dimensional plane in J_p) + (a point in K_q) + (a $(q-1)$-dimensional plane in K_q)$\}$ have an invariant biplanar density. The subspaces are chosen to be skew. Let

$$\Sigma'_2 = \{L_{p_1} = (A_1), L_{p_2} = (A_2, \dots, A_{p+1}), L_{q_1} = (B_1), L_{q_2} = (B_2, \dots, B_{q+1})\}.$$

Then the density of the systems Σ'_2 is given by the following $2(p+q)$-form

$$\mathrm{d}\Sigma'_2 = \bigwedge_{i=2}^{p+1} (\omega_1^i \wedge \omega_i^1) \wedge \bigwedge_{u=2}^{q+1} (\psi_1^u \wedge \psi_u^1). \tag{13}$$

In view of [5], we get an other expression for (13). At least one of the coordinates $(a_{11}, a_{21}, \dots, a_{p+1,1}, 0, 0, \dots, 0)$ of the point L_{p_1} and at least one of the coordinates $(0, 0, \dots, 0, b_{11}, b_{21}, \dots, b_{q+1,1})$ of the point L_{q_1} are non-zero. Let $a_{p+1,1} \neq 0$, $b_{q+1,1} \neq 0$. Setting

$$X_i = a_{i1}/a_{p+1,1}, \qquad Y_u = b_{u1}/b_{q+1,1}, \qquad i = 1, \dots, p+1;\ u = 1, \dots, q+1,$$

from (5) we obtain

$$\begin{aligned} \omega_1^i &= \frac{1}{A} a_{p+1,1} \sum_{k=1}^{p} A_{ki}\,\mathrm{d}X_k + \frac{\mathrm{d}a_{p+1,1}}{a_{p+1,1}} \delta_{1i}, \\ \psi_1^u &= \frac{1}{B} b_{q+1,1} \sum_{v=1}^{q} B_{vu}\,\mathrm{d}Y_v + \frac{\mathrm{d}b_{q+1,1}}{b_{q+1,1}} \delta_{1u}, \end{aligned} \tag{14}$$

where δ_{**} is the well-known symbol of Kronecker.

It follows from (4) that the $(p-1)$-plane L_{p_2} and the $(q-1)$-plane L_{q_2} have the have the equations

$$\begin{aligned} L_{p_2}\colon\quad & A_{11}x_1 + A_{21}x_2 + \dots + A_{p1}x_p + A_{p+1,1}x_{p+1} = 0, \\ & y_1 = 0,\ y_2 = 0, \dots, y_{q+1} = 0, \\ L_{q_2}\colon\quad & x_1 = 0,\ x_2 = 0, \dots, x_{p+1} = 0, \\ & B_{11}y_1 + B_{21}y_2 + \dots + B_{q1}y_q + B_{q+1,1}y_{q+1} = 0. \end{aligned}$$

At least one of the coefficients $A_{11}, A_{21}, \dots, A_{p+1,1}$ in the first equation of L_{p_2} and at least one of the coefficients $B_{11}, B_{21}, \dots, B_{q+1,1}$ in the latter equation of

L_{q_2} are non-zero. Suppose that $A_{p+1,1} \neq 0$, $B_{q+1,1} \neq 0$. Then, setting

$$\alpha_i = A_{i1}/A_{p+1,1}, \qquad i = 1, \ldots, p+1,$$
$$\beta_u = B_{u1}/B_{q+1,1}, \qquad u = 1, \ldots, q+1,$$

from (5) we get

$$\omega_i^1 = -\frac{1}{A} A_{p+1,1} \sum_{k=1}^{p} a_{ki}\, \mathrm{d}\alpha_k - \frac{\mathrm{d}A_{p+1,1}}{A_{p+1,1}} \delta_{1i},$$
(15)
$$\psi_u^1 = -\frac{1}{B} B_{q+r,1} \sum_{v=1}^{q} b_{vu}\, \mathrm{d}\beta_v - \frac{\mathrm{d}B_{q+1,1}}{B_{q+1,1}} \delta_{1u}.$$

Putting (14) and (15) in (13) we have

$$\mathrm{d}\Sigma_2' = \left(\frac{a_{p+1,1}A_{p+1,1}}{A}\right)^{p+1} \left(\frac{b_{q+1,1}B_{q+1,1}}{B}\right)^{q+1} \bigwedge_{i=1}^{p} (\mathrm{d}X_i \wedge \mathrm{d}\alpha_i) \wedge \bigwedge_{u=1}^{q} (\mathrm{d}Y_u \wedge \mathrm{d}\beta_u).$$

But

$$a_{p+1,1}A_{p+1,1} = A(\alpha_1 X_1 + \ldots + \alpha_p X_p + 1)^{-1},$$
$$b_{q+1,1}B_{q+1,1} = B(\beta_1 Y_1 + \ldots + \beta_q Y_q + 1)^{-1}$$

and therefore

$$\mathrm{d}\Sigma_2' = \frac{\bigwedge_{i=1}^{p} (\mathrm{d}X_i \wedge \mathrm{d}\alpha_i) \wedge \bigwedge_{u=1}^{q} (\mathrm{d}Y_u \wedge \mathrm{d}\beta_u)}{|\alpha_1 X_1 + \ldots + \alpha_p X_p + 1|^{p+1} |\beta_1 Y_1 + \ldots + \beta_q Y_q + 1|^{q+1}}.$$

Example 2. Let $h = 2$, $s = 0$, $m = 0$ and choose $p_1 = 0$, $p_2 = p - 1$. Then the conditions (ii) in Theorem 3 are satisfied. The systems Σ_2'', which consist of skew (point in J_p) + ($(p-1)$-dimensional plan in J_p) have the biplanar density

$$\mathrm{d}\Sigma_2'' = \bigwedge_{i=2}^{p+1} (\omega_1^i \wedge \omega_i^1).$$

As in Example 1, using the same notations, we obtain

$$\mathrm{d}\Sigma_2'' = \frac{\bigwedge_{i=1}^{p} (\mathrm{d}X_i \wedge \mathrm{d}\alpha_i)}{\underset{i=1}{}|\alpha_1 X_1 + \ldots + \alpha_p X_p + 1|^{p+1}}.$$

Example 3. Let $h = 0$, $s = 2$, $m = 0$ and choose $q_1 = 0$, $q_2 = q - 1$. Clearly, these values satisfy the conditions (iii) in (12) and hence the systems Σ_2''' of skew (point in K_q) + ($(q-1)$-dimensional plane in K_q) have the invariant biplanar density

$$\mathrm{d}\Sigma_2''' = \bigwedge_{u=2}^{q+1} (\psi_1^u \wedge \psi_u^1).$$

Using the notations of the Example 1, we have

$$d\Sigma_1''' = \frac{\bigwedge_{u=1}^{q} (dY_u \wedge d\beta_u)}{|\beta_1 Y_1 + \ldots + \beta_q Y_q + 1|^{q+1}}.$$

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