Stanislav Říha; Milan Sekanina
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SQUARES OF TRIANGULAR CACTI

S. ŘÍHA, M. SEKANINA, Brno

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Theorem 2 in [2] contains a necessary and sufficient condition for the hamiltonicity of the square of a cactus. In our paper triangular cacti are considered*) and the corresponding condition is deduced in the terms of forbidden subgraphs. Our condition seems to be more effective than that from [2].

If $G = (V, E)$ is a simple connected graph, $x, y \in V$, $d_G(x, y)$ denotes the distance of x and y in G, i.e. the number of edges in a shortest way connecting the vertices x and y. For positive integer n let $G^n = (V, E^n)$, where $E^n = \{xy : 1 \leqq d_G(x, y) \leqq n\}$. G^n is called the n-th power of G, for $n = 2$ we speak about the square of G.

A triangular cactus (briefly t-cactus) is a finite simple connected graph G, in which every cycle is a triangle and each edge is contained just in one triangle. For a t-cactus G $T(G)$ is the set of all triangles of G. A vertex of degree n is called an n-vertex in G. Notice, a vertex of a t-cactus G is a 2-vertex iff it is not a cut-point in G. $T \in T(G)$ containing at least two 2-vertices in a t-cactus G, is called an end-triangle, a triangle, which is not an end-triangle, is called an inner triangle. A triangle $T \in T(G)$ containing k 2-vertices, is called a triangle of genus k.

If G is a t-cactus and $M \subset T(G)$, $\cup M$ denotes the complete subgraph in G spanned by the set of the vertices of triangles from the system M. If $M \subset T(G)$, $T \in M$ and N is the set of all triangles of $T(G) - M$, which have at least one vertex with T in common and this vertex is a 2-vertex in $\cup M$, then N is called the growth of the graph $\cup M$ from the triangle T in G. If $m_1 \geqq m_2 \geqq m_3$ are the numbers of triangles having in a given growth N a given vertex with T in common, the growth N is said to be of the type (m_1, m_2, m_3). If $M, N \subset T(G)$, and $\cup M \cap \cup N$ consists of one vertex x, then $\cup N(\cup M)$ is said to be attached to $\cup M(\cup N)$ in the vertex x.

A generating sequence of a t-cactus G is a sequence σ $G_1, \ldots, G_s = G$ of its subgraphs, in which

1. Every G_i, $i = 1, \ldots, s$, is a t-cactus.

*) The case of the general cacti is considered by the first author in a paper which is under preparation.

2. G_1 is a triangle.

3. G_{i-1} is a subgraph of G_i and $G_{i-1} \neq G_i$.

4. $T(G_i) - T(G_{i-1})$ is the growth (so called i-th growth) of G_{i-1} from a certain $T_{i-1} \in T(G_{i-1})$ in the graph G.

If G_1 is an end-triangle, σ is called a prime generating sequence.

It is easily seen that there exists a prime generating sequence for every t-cactus.

Final growth in σ is every such growth $T(G_i) - T(G_{i-1})$ in σ, for which each $T \in T(G_i) - T(G_{i-1})$ is an end-triangle in G.

Let G be a t-cactus with a generating sequence σ having the following properties.

D1 G_1 is of genus 1 or 2.

D2 Every growth of σ is of the type (2, 0, 0) or of the type (1, 1, 0).

D3 Every final growth is of the type (1, 1, 0).

D4 Every growth of the type (1, 1, 0) is final.

D5 Every end-triangle from G different from G_1 is in a final growth.

Then G is called a diad and G_1 is a base of this diad.

It is not difficult to see that every diad possesses only one base. A 2-vertex in G of G_1 is called a base vertex of G.

If G', G'', G''' are diads having one vertex of their bases in common and this vertex is a base vertex in each of them (otherwise these diads are mutually disjoint), then the union $G' \cup G'' \cup G'''$ is called a 3-diad (an example of a 3-diad is in fig. 1).

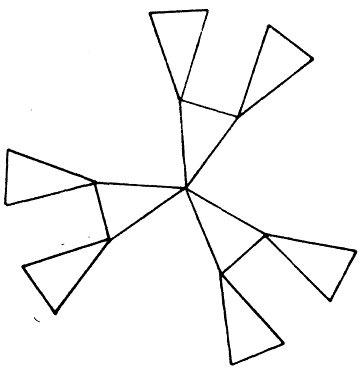

Fig. 1

Every Hamiltonian circle H in G^2 in some graph G gives a certain cyclical ordering χ of the set V of vertices of G. If $G' = (V', E')$ is a subgraph of G, the restriction χ/V' is a cyclical ordering of V' and we put $H/G' = \chi/V'$. If H/G' defines a Hamiltonian circle in G', we denote this Hamiltonian circle as H/G', too.

In the sequel G means a t-cactus if not stated explicitelly otherwise.

If H is a Hamiltonian circle in G^2 and $T \in T(G)$, T is called to be (at least) of the type (H, i), if T has (at least) i edges with H in common. If one of these edges is connecting two 2-vertices of T, T is called to be (at least) of the type $(H, \bar{i})$.

Lemma 1. *Let G, G' be t-cacti, G a subgraph in G' and $T(G') - T(G)$ the growth of G from some T in $T(G)$ of a type $(m, n, 0)$ in the graph G'. Let H be a Hamiltonian circle in G^2 and T is at least of the type $(H, \bar{1})$. If the growth $T(G') - T(G)$ is of the type $(m, n, 0)$ $m \geqq n \geqq 1$, T is at least of the type $(H, \bar{2})$. Then in the graph $(G')^2$ there exists a Hamiltonian circle H' with the following properties:*

a) *If T is at least of the type $(H, \bar{2})$, then*

*a*1. $H'/\cup(T(G) - \{T\}) = H/\cup(T(G) - \{T\})$.

a2. $T' \in T(G') - T(G) \Rightarrow T'$ *is at least of the type* $(H', \bar{1})$.

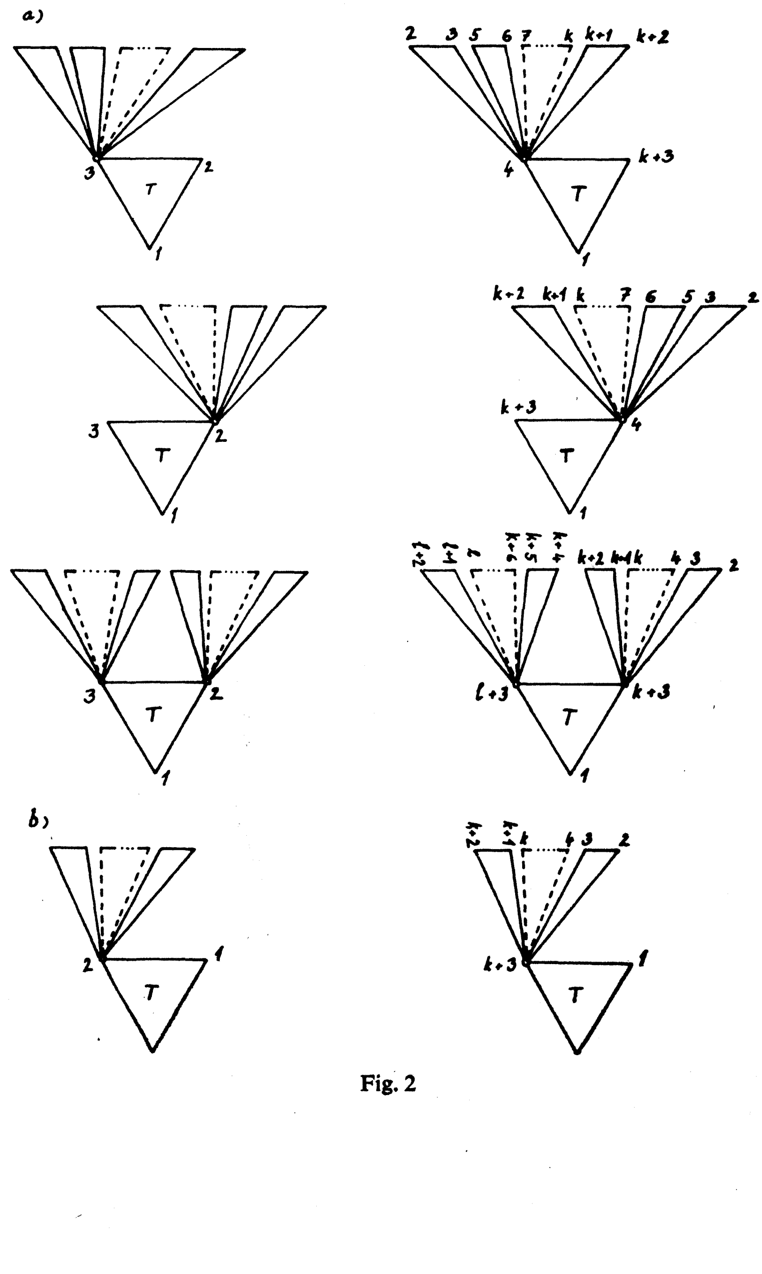

Fig. 2

a3. *If the growth* $T(G') - T(G)$ *is of the type* $(m, 0, 0)$, $m \geqq 1$, *and* $T_1, T_2 \in \in T(G') - T(G)$ *are arbitrary but fixed (chosen in advance), then* T_1 *and* T_2 *are of the type* $(H', \bar{2})$.

a4. *If the growth* $T(G') - T(G)$ *is of the type* $(m, n, 0)$, $m \geqq n \geqq 1$, *then every 2-vertex in G of T is contained in at least one triangle* T_3 *from* $T(G') - T(G)$ *of the type* $(H', \bar{2})$. T_3 *can be chosen in advance arbitralily but fixedly from* $T(G') - T(G)$.

b) *If T is of the type* $(H, \bar{1})$ *and* $T(G') - T(G)$ *is of the type* $(m, 0, 0)$, $m \geqq 1$, *then*

b1. $H'/\cup(T(G) - \{T\}) = H/\cup(T(G) - \{T\})$.

b2. *Every triangle from* $T(G') - T(G)$ *is at least of the type* $(H', \bar{1})$.

b3. *At least one triangle* T_4 *chosen in advance from* $T(G') - T(G)$ *is of the type* $(H', \bar{2})$.

Proof can be obtained via numbering given in fig. 2, where on the left hand side the relevant part of the ordering of the set of the vertices in H is considered, on the right hand side the ordering of the set of the vertices in H' is given. In the rest of G the orderings for H and H' coincide.

Let G be a t-cactus not containing any 3-diad as a subgraph. Let $T \in T(G)$ be an end-triangle. The triangle T has evidently a vertex in common with at most two diads lying in $\cup(T(G) - \{T\})$ as a base vertex (see fig. 3, B denotes the base of a diad). Denote the growth of T from T in G as M.
The set of the vertices of the graph $\cup(M \cup \{T\})$ will be ordered as follows

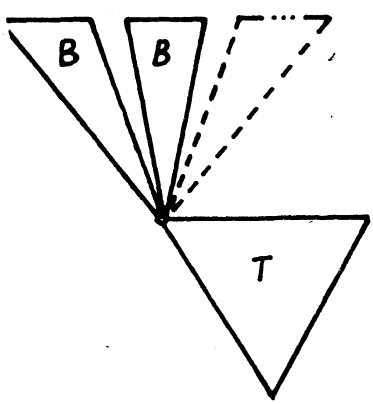

Fig. 3

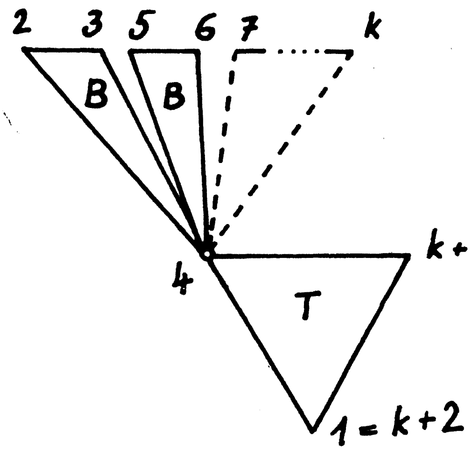

Fig. 4

Hence we get

Lemma 2. *The graph* $[\cup(M \cup \{T\})]^2$ *is Hamiltonian and in the Hamiltonian circle H given by numbering in fig. 4 the bases B are of the type* $(H, \bar{2})$.

Let $G = T, \ldots, G_i, \ldots, G$ be a prime generating sequence of a t-cactus G and let H_i be a Hamiltonian circle in G_i^2 such that

(P_i): the triangles S of the genus 2 in G_i (the genus taken in respect to G_i) which are the bases of diads lying in $G_S = S \cup \cup(T(G) - T(G_i))$ have at least the type $(H_i, \bar{2})$, the other triangles of the genus 2 in G_i different from T are at least of the type $(H_i, \bar{1})$.

Now, we construct H_{i+1} with property (P_{i+1}) (H_1, H_2 with properties (P_1), (P_2) evidently exist by Lemma 2). Suppose $i \geqq 2$.

Let the $(i + 1)$-th growth be from $S \in T(G_i)$. If the triangle S is not a base for a diad lying in G_S the growth is of the type $(m, 0, 0)$. If a is the vertex of S, which is 2-vertex in G_i, but not a 2-vertex in G_{i+1} which is a base vertex of this diad then at most one diad lying in $\cup(T(G) - T(G_i))$ is attached to G_i in the vertex a and the existence of H_{i+1} with property (P_{i+1}) follows from Lemma 1b, (the base of our diad, if it exists, chosen for T_4).

Let the triangle S be a base for a diad lying in G_S. Then S is at least of the type $(H_i, \bar{2})$ and let a, b be 2-vertices in S (in G_i). If a is contained in two bases of diads lying in $\cup(T(G) - T(G_i))$ as a base vertex and so exactly in two such bases, then is a 2-vertex in G (otherwise a 3-diad would exist in G) and the existence of H_{i+1} with (P_{i+1}) follows from Lemma 1, a1.–a3. (the bases of diads under consideration taken as T_1, T_2). If each of the vertices a and b is contained as a base vertex at most in one diad lying in $\cup(T(G) - T(G_i))$, the existence of H_{i+1} with (P_{i+1}) follows from Lemma 1, a1., a2., a4. (the bases of diads taken as triangles denoted as T_3).

Hence

Proposition 1. *If a t-cactus does not contain any 3-diad, it has the Hamiltonian square.*

Lemma 3. *Let G be a simple connected finite graph (not necessarily a t-cactus), for which G^2 is Hamiltonian. Let H be a Hamiltonian circle in G^2 and g a cut-vertex in G with $G - \{g\} = G_1 \cup \dots \cup G_s$ as the decomposition in components. Let G_1 have at most two vertices as neighbors to g in G. Then*

a) *$(G - G_1)^2$ is Hamiltonian.*

b) *If G_1 has at least three vertices and no neighbor in H of the vertex g lies in G_1, the vertices of G_1 form an interval in H with the ends in distance 1 from g in G.*

Proof. Let H be of the form

$$g, a_1, \dots, a_k, a_{k+1}, \dots, a_m, a_{m+1}, \dots, a_n, a_{n+1}, \dots, a_p, a_{p+1}, \dots, a_r, a_{r+1}, \dots$$

where

$$a_1, \dots, a_k \notin G_1, a_{k+1}, \dots, a_m \in G_1, a_{m+1}, \dots, a_n \notin G_1, a_{n+1}, \dots, a_p \in G_1,$$
$$a_{p+1}, \dots, a_r \notin G_1, a_{r+1} \in G_1.$$

For the case b) it is

$$d_G(a_k, g) = d_G(a_{k+1}, g) = d_G(a_m, g) = d_G(a_{m+1}, g) =$$

$$= d_G(a_n, g) = d_G(a_{n+1}, g) = d_G(a_p, g) = d_G(a_{p+1}, g) =$$
$$= d_G(a_r, g) = d_G(a_{r+1}, g) = 1.$$

Ad a. $g, a_1, \ldots, a_k, a_{m+1}, \ldots, a_n, a_{p+1}, \ldots, a_r, \ldots$ is a Hamiltonian circle in $(G - G_1)^2$.

Ad b. Admit there exists a_{n+1}. Then $a_{k+1} = a_m \neq a_{n+1} = a_p$ and there exists a_{r+1} different from a_m and a_p. So at least three vertices in G_1 are neighbors of g in G, a contradiction.

Remark. Compare Lemma 3 and Lemma 5 with the results of [1].

Lemma 4. *Let T be the base of a diad G. Then for no Hamiltonian circle H in G^2 T is of the type $(H, 2)$ in such a way that two edges of H are edges of T containing a base vertex in G.*

Proof. a). Let

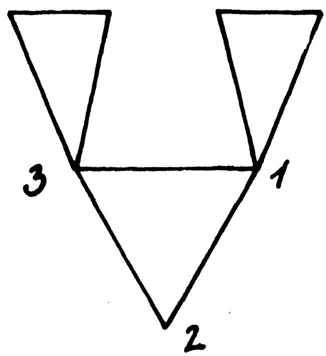

Fig. 5

One sees that G^2 does not contain any Hamiltonian circle with edges 12,23.

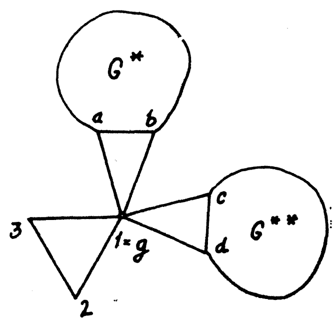

Fig. 6

b) Suppose Lemma 4 is true for all diads with fewer than n triangles and let G have n triangles. Let G be as on Fig. 6, where G^* and G^{**} are diads with fewer than n triangles. Suppose edges 12,23 are in a Hamiltonian circle H in G^2. By Lemma 4b. the set of vertices different from g of at least one of diads G^*, G^{**} form an interval in H. Let it be G^*. The ends of this interval are a and b and $(G^*)^2$ contains a Hamiltonian circle with edges $a1$, $1b$. This contradicts to the supposition of induction.

Lemma 5. *Let G be a simple connected finite graph. Let g be its cut-vertex and G_i, $i \in I$, the components of $G - \{g\}$. Let G^2 be Hamiltonian and H be a Hamiltonian circle in G^2 of the form $\ldots, a, g, b, \ldots$, where $a \notin G_i$, $b \notin G_i$ and the component G_i has at least two vertices. Then there exists a Hamiltonian circle H' in $(G_i \cup \{g\})^2$, in which two edges of G coincide to g.*

Proof. As $(G_i \cup \{g\})^2$ is a subgraph in G^2 it is sufficient to put $H' = H/(G_i \cup \{g\})^2$.

Corollary 1. Let G be a simple connected finite graph having at least three vertices, g a vertex of G which is not a cut-vertex and let no Hamiltonian circle H in G^2 contain two edges of G incident to g. Then for the graph G^*, which consists of three copies of G with amalgamated g,$(G^*)^2$ is not Hamiltonian.

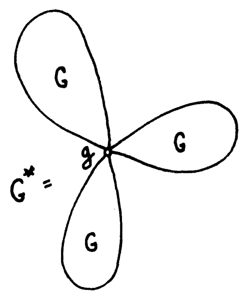

Fig. 7

Corollary 2. For a 3-diad G G^2 is not Hamiltonian.
It follows from Corollary 1 and Lemma 4.

Lemma 6. *Let G_1, G_2 be t-cacti, G_1 a subgraph in G_2. If G_2^2 is Hamiltonian, G_1^2 is Hamiltonian, too.*

Proof follows from Lemma 3a as G_1 can be obtained from G_2 by successive deleting suitable end-triangles.

Theorem. *If G is a t-cactus then G^2 is Hamiltonian iff G does not contain any 3-diad.*

Proof follows from Lemma 6, Corollary 2 and Proposition 1.
The least t-cactus not having the Hamiltonian square is in Fig. 1.

REFERENCES

[1] H. Fleischner, H. V. Kronk: *Hamiltonische Linien im Quadrat brückenloser Graphen mit Artikulationen,* Monatshefte für Math. 76 (1972), 112—117.

[2] A. M. Hobbs: *Hamiltonian squares of cacti,* Journal of Combinatorial Theory, Series B 26, (1979), 50—65.

S. Říha, M. Sekanina
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia