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ON COVARIETIES OF COALGEBRAS

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Abstract. Categories of algebras over the base category Set^{op} are studied under the influence of various general approaches known in categorical algebra. As results constructions of cofree coalgebras and various Birkhoff theorems are obtained. Several remarks concern distinguishing subsets.

Key words: coalgebra, cofree coalgebra, covariety, cogeneration, distinguishing subset

The coalgebras we speak about were studied by Drbohlav [3], [4]. Also Isbell [5] dealt with them and obtained different existence criterion for cofree coalgebras. Davis [1], [2] considered much more general dualizations of universal algebra, we concretize it here.

We denote by Set the category of sets and maps, by Card its representative subclass of cardinals; all in the Gödel–Bernays set theory. The value of a map $f : X \to Y$ in $x \in X$ is written as fx or f_x, the kernel equivalence relation of f is denoted by $\ker f \subseteq X \times X$. For $A \in Set$, $n \in$ Card, nA stands for the n-th copower of A i.e. the disjoint union of n copies of A. Clearly, giving a map $\bar{f} : A \to nB$ is the same thing as giving a couple of maps $(f : A \to B, f' : A \to n)$ with $f'a$ pointing the copy whose fa yields $\bar{f}a$, for all $a \in A$. Agreement· Throughout all the paper primes and bars indicate such a relation of maps and $\bar{f}$ and (f, f') define one another.

I. BASIC NOTIONS

1.1. Definition (see [3], [4])· A *type* is a couple (I, n) where I is a class and $n : I \to$ Card is an (*arity*) map. A *coalgebra* of the type (I, n) is a couple $\boldsymbol{A} = (A, (\bar{f}_i)_{i \in I})$ where $A \in Set$ and each $\bar{f}_i : A \to n_i A$ is a map, called an n_i-ary coalgebraic *operation*. The components $f_i : A \to A$ and $f'_i : A \to n_i$ we call *unary* and *labelling* operations, respectively.

A *homomorphism* of coalgebras $\boldsymbol{A} = (A, (\bar{f}_i))$, $\boldsymbol{B} = (B, (\bar{g}_i))$ is defined as any map $h : A \to B$ making

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f_i"'] & B \arrow[d, "g_i"] \\ n_i A \arrow[r, "n_i h"'] & n_i B \end{tikzcd}$$

commutative for all $i \in I$, i.e. as any label preserving homomorphism of the underlying unary algebras.

By a *subcoalgebra* of a coalgebra $A = (A, \bar{f}_i))$ we mean any coalgebra $(B, (\bar{g}_i))$ with $B \subseteq A$ and $\bar{g}_i = \bar{f}_i|_A$. Evidently, embeddings of subcoalgebras represent injective homomorphisms. We call a coalgebra A *single generated* if it contains an element a such that the smallest subcoalgebra containing a is A itself.

Congruence relations of a coalgebra $A = (A, (\bar{f}_i))$ are defined as equivalence relations $\varrho \subseteq A \times A$ such that $\varrho \subseteq (f_i \times f_i)^{-1} \varrho$ and $\varrho \subseteq \ker f_i'$ for each $i \in I$, in order to give natural representants of onto homomorphisms via the obvious quotient coalgebras.

Sums of coalgebras are constructed as disjoint unions, in the obvious way.

The duals of free algebras are cofree coalgebras:

1.2. Definition. Let K be any class of coalgebras of a type (I, n). A coalgebra $T_X = (T_X, (\bar{k}_i)) \in K$ is said to be cofree in K cogenerated by a map $\xi : T_X \to X$ if it has the following universal property: For every coalgebra $A = (A, (\bar{f}_i)) \in K$ and every map $h : A \to X$ there is a unique homomorphism $h^* : A \to T_X$ making

$$\begin{tikzcd} X & & T_X \arrow[ll, "\xi"'] \\ & A \arrow[ul, "h"] \arrow[ur, "h^*"'] & \end{tikzcd}$$

commutative. The class K is said to possess all cofree coalgebras if in K there exists the respective cofree coalgebra T_X for every $X \in Set$ — so that the right adjoint functor to the obvious underlying set functor is induced.

1.3. Definition. Denote by **H**, **S**, Σ the obvious closures (of a class of coalgebras) under all homomorphic images, all subcoalgebras and all sums, respectively. Any class K of coalgebras of the same type with $\mathbf{H}K = K$ and $\Sigma K = K$ is called a *quasicovariety* (see [4]). By a *covariety* we mean any quasicovariety with $\mathbf{S}K = K$ and possessing all cofree coalgebras (implicit at the restriction to bounded types as in [3], [4]).

1.4. Definition. Write I^* or I^+ for the class of all words over the alphabet I with the empty word ω included or excluded, respectively. Being given a coalgebra

$(A, (f_i))$ of the type (I, n) and $s = i_1 i_2 \dots i_r \in I^+$, denote by f_s the composition $f_{i_1} f_{i_2} \cdots f_{i_r}$, by f'_s the composition $f'_{i_1} f_{i_2} \dots f_{i_r}$. Put $f_\omega = \mathrm{id}$, f'_ω leaving undefined.

Let further I^* denote the class I^* endowed with "unary operations" $l_i : I \to I$ of $s \to is$. Call a map $h : I^* \to A$ admissible from I^* to a coalgebra $(A, (f_i))$ whenever

$$\begin{tikzcd} I^* \arrow[r, "h"] \arrow[d, "l_i"'] & A \arrow[d, "f_i"] \\ I^* \arrow[r, "h"'] & A \end{tikzcd}$$

commutes for all $i \in I$.

Clearly, I^* enjoys the same universal property as the free unary algebras:

1.5. Assertion. Admissible maps $h : I^* \to A$ are in one to one correspondence with all the elements of A (the images of the empty word ω). Explicitly $hs = f_s h\omega$.

Another auxiliary assertion we shall need is a slight modification of an earlier lemma of Słominski [11]:

1.6. Lemma. *Let $A = (A, (f_i))$ be a coalgebra and let $\sigma \subseteq A \times A$ be an equivalence relation. Then $\underline{\sigma} \subseteq A \times A$ defined by*

$$a\underline{\sigma}b \Leftrightarrow \underset{s \in I^*}{\forall} f_s a \sigma f_s b \qquad \text{and} \qquad \underset{s \in I^+}{\forall} f'_s a = f'_s b$$

is a congruence relation and it is just the greatest congruence relation of A contained in σ.

1.7. Remark. Equivalence relations σ such that $\underline{\sigma} = \mathrm{id}$ are of frequent use in computer science. They are generally called cogenerating relations. Special attention is paid to two-block equivalences with this property—see [6], where their blocks are called *distinguishing subsets*. We mention them in 2.7 and 4.8 of this paper.

II. COFREE COALGEBRAS

Existence criteria and corresponding constructions for cofree coalgebras are presented in this section.

2.1. Theorem. *Let K be a quasicovariety of coalgebras. Then the following statements are equivalent:*

1° *K possesses all cofree coalgebras*

2° *K possesses a cofree coalgebra cogenerated by a set with at least two elements*

3° *K has a representative set of single generated coalgebras.*

Proof: 1° implies 2° trivially, 3° implies 1° by SAFT (see [8]) since single-generated coalgebras act as cogenerators.

To prove 3° provided 2° let $\boldsymbol{T} = (T, (\bar{k}_i))$ be the cofree coalgebra of 2°, cogenerated by a map $\xi : \boldsymbol{T} \to X$ with card $X \geqq 2$. Any single-generated coalgebra $\boldsymbol{A} = (A, (\bar{f}_i))$ of $\boldsymbol{K}$ clearly has a unique representation by the admissible (cf. 1.4.) quotient map $h : \boldsymbol{I^*} \to \boldsymbol{A}$ together with the I-tuple $(f_i'h : I^* \to n_i)$. Let for arbitrary map $\chi : A \to X$ χ denote the pair $(\chi, (f_i'))$, then the coalgebra $\boldsymbol{A}$ induces a set of compositions $\chi h = (\chi h, (f_i'h))$. This set $\{\chi h; \chi : A \to X\}$ determines the surjective map h up to bijection so that it determines the coalgebra $\boldsymbol{A}$ up to isomorphism.

But there is only a set of the compositions $\boldsymbol{\chi} h$ for all $\boldsymbol{A} \in \boldsymbol{K}$ and all χ. The reason is that the assignment $\boldsymbol{\chi} h \to \chi^* h\omega \in T$ from

$$\begin{tikzcd}
I^* \arrow[r, "h"] & A \arrow[d, "\chi^*"'] \arrow[dr, "\chi"] & \\
& T \arrow[r, "\xi"'] & X
\end{tikzcd}$$

is injective, its inverse being $\chi^* h\omega \mapsto$ the unique admissible $\chi^* h$ (see 1.5) $\mapsto$ $\mapsto (\xi\chi^* h\omega, (k_i'\chi^* h)) = \boldsymbol{\chi} h$.

On $\boldsymbol{I^*}$ there exists another family of unary operations, $(r_i)_{i \in I}$, of adding its indices from the right: $r_i s = si$, which makes them fulfilling the admissibility condition of 1.4.

2.2. Construction. (Extending that of [9], pg. 521). *Let $\boldsymbol{K}$ be a quasicovariety of coalgebras and let $X \in Set$. Denote by H_X the collection of all compositions $\boldsymbol{\chi} h = (\chi h, (f_i'h))$ where $h : \boldsymbol{I^*} \to \boldsymbol{A}$ is any admissible map with $\boldsymbol{A} = (A, (f_i')) \in \boldsymbol{K}$ and $\boldsymbol{\chi} = (\chi, (f_i'))$ has $\chi : A \to X$ arbitrary.*

This collection becomes a (possibly large) coalgebra under the unary operations $k_i : H_X \to H_X$ of $\chi h \to \chi h r_i$ and (well defined) labellings $k_i' : H_X \to n_i$ of $\chi h \to f_i'h$; denote it by $\boldsymbol{H}_X$. Finally define $\xi : H_X \to X$ by $\chi h \to \chi h\omega$.

This construction can be proved to yield all the cofree coalgebras the quasicovariety $\boldsymbol{K}$ possesses:

2.3. Theorem. *Let $\boldsymbol{K}$ be a quasicovariety. Then for every $X \in Set$ it holds: In $\boldsymbol{K}$ there is a cofree coalgebra cogenerated by X if and only if H_X of 2.2 is a set, and then it is isomorphic to $\boldsymbol{H}_X$, with the cogenerating map ξ.*

Proof: It suffices to show that $\boldsymbol{H}_X$ is cofree in $\boldsymbol{K}$ since the smallness condition became proved during the proof of 2.1. Thus, let $\boldsymbol{A} = (A, (\bar{f}_i)) \in \boldsymbol{K}$ and let $\varphi : A \to$ $\to X$ be arbitrary. For every $a \in A$ define $h_a : \boldsymbol{I^*} \to \boldsymbol{A}$ as the unique admissible map with $h_a\omega = a$ (see 1.5), then put $\varphi = (\varphi, (f_i'))$ and define $\varphi^* : a \mapsto \varphi h_a \in H_X$. Clearly $h_{f_i a} = h_a r_i$ for all $i \in I$, whence $\varphi^+ : \boldsymbol{A} \to \boldsymbol{H}_X$ is a homomorphism. Also $\xi\varphi^* a = \varphi h_a\omega = \varphi a$, as desired. Our proof of $\boldsymbol{H}_X \in \boldsymbol{K}$ is constructive:

2.4. Construction (in fact that of SAFT). Let $\boldsymbol{K}$ be a quasicovariety of coalgebras, let X be a set. Assume, that cofree coalgebras in $\boldsymbol{K}$ actually exist, so that, by 3°

of 2.1, there is a representive set of pairs (A_ι, ξ_ι) with each $A_\iota \in K$ and ξ_ι mapping it into X. Let $\xi_* : \sum_\iota A_\iota \to X$ be composed of the maps ξ_ι and put $T_X = (T_X, (\bar{k}_i)) = \sum_\iota A_\iota/\underline{\ker \xi_*}$. Evidently ξ_* factors to well-defined $\xi : T_X \to X$.

The coalgebra T_X jsut constructed clearly belongs to K, we show now how $T_X = H_X$. Define a map $\tau : T_X \to H_X$ by $\tau[t] : s \mapsto \xi_* k_s t$ where $\xi_* = (\xi_*, (k_i'))$ and $[t]$ denotes the $\underline{\ker \xi_*}$-block t lies in. By 1.6 $\tau[t_1] = \tau[t_2]$ if and only if $(t_1, t_2) \in \underline{\ker \xi_*}$ so that τ is injective. Its image is, by the universal property of I^*, $\mathrm{Im}\, \tau = \{\xi_* h;\, h : I^* \to \sum_\iota A_\iota \text{ admissible}\}$ which is clearly the whole H_X. This τ preserves operations: $k_i \tau[t] = k_i \xi_* h_t = \xi_* h_t r_i = \xi_* h_{f_i t} = \tau[f_i t] = \tau f_i[t]$ for every $t \in T$ and $i \in I$, and similarly for the labelling operations, so that τ is the desired isomorphism.

What remained is the uniqueness of φ^*. Then, suppose there were two coalgebra homomorphisms $p, q : A \to T_X$ satisfying $\xi p = \xi q$, let c be a coequalizer of p and q, constructed, as any colimit in quasicovarieties, at the level of *Set*. Then ξ must have a factorization through c, giving $\underline{\ker c} \subseteq \underline{\ker \xi}$, whence $\underline{\ker c} \subseteq \underline{\ker \xi} = \mathrm{id}_T$, as follows immediately from 2.4. Consequently, $p = q$.

2.5. Definition. Let $A = (A, (\bar{f}_i))$ be a coalgebra. Call A to be X-cogenerated via a cogenerating map $\xi : A \to X$ whenever $\underline{\ker \xi} = \mathrm{id}_A$ holds.

2.6. Assertion. *A coalgebra A is X-cogenerated if and only if it is embeddable into the cofree coalgebra T_X (cofree at least with respect to A).*

Proof: Clearly, every subcoalgebra of T_X is X-cogenerated via the restriction of the cogenerating map $\xi : T_X \to X$. On the other hand, an X-cogeneration $\alpha : A \to X$ of a coalgebra A admits a factorization $\alpha = \xi\alpha^*$ with α^* homomorphic, whence $\ker \alpha^* \subseteq \underline{\ker \xi} = \mathrm{id}_A$, so that α^* is injective.

In [6], 2-cogenerated unary algebras are of special interest, (cf. 1.7). Our 2.2 and 2.4 show, how to construct the 2-cogenerated cofree unary algebra T_2 of a given type (put $n_i = 1$). As a consequence of 2.6 we get:

2.7. Corollary. *Unary algebras possessing a distinguishing subset are just those, which are embeddable into the 2-cogenerated cofree unary algebra T_2.*

III. COVARIETIES AS BIRKHOFF CLASSES

In this section we proceed in analogy with the classical universal algebra. Standard proofs are omitted.

3.1. Let K be a class of coalgebras of a given type. Always $\Sigma \mathbf{S} K \subseteq \mathbf{S} \Sigma K$, $\Sigma \mathbf{H} K \subseteq \mathbf{H} \Sigma K$ and $\mathbf{SH} K \subseteq \mathbf{HS} K$ and that is why the quasicovariety closure of K is just $\mathbf{H}\Sigma K$ and why the covariety closure of K is $\mathbf{HS}\Sigma K$ provided K has all cofree coalgebras (which are then easily seen to be cofree in the whole class $\mathbf{HS}\Sigma K$).

3.2. Every covariety K admits a coreflection of any coalgebra of the same type. The coreflection $A \xleftarrow{\gamma A} A_K$ is obtained when embedding into A its largest subcoalgebra belonging to K.

3.3. *Every covariety is cotripleable over Set.*

Proof: Simply by Beck theorem, also by direct computation of the cotriple.

3.4. *Every covariety K is complete and cocomplete category.*

Proof: Cotripleable categories over a complete category are known to be complete provided they have equalizers, which covarieties actually have (also constructed at the level of *Set*). A direct construction of the product $(P, (\bar{p}_i)) = \Pi_K(A_k, (\bar{f}_i)_k)$ is the following: Forget for a moment the unary operations of the coalgebras and form the product $(B, (b_i'))$ of the family $((A_k, (f_i')_k))$ simply as the common pullback of all the arrows $f_{ik}' : A_k \to n_i$. This B is a certain subset in the product ΠA_k and, similarly as in 3.2., contains the largest subalgebra $(P, (\bar{p}_i))$ of $\Pi(A_k, (f_i)_k)$ belonging, with the restricted labellings $p_i' = b_i' |_P$ to the given covariety. The projections are then obvious.

Thus, the products are not the cartesian ones when nontrivial labellings occur. Since right adjoints preserve limits it easily follows that free coalgebras are unavailable unless for unary algebras.

When dualizing the notion of subdirect irreducibility one sees that it reduces to the property of single-generation:

3.5. *Every* HS Σ*-closed class of coalgebras is fully determined by its single-generated members via $A \in K$ if and only if* $S_1 A \subseteq S_1 K$ (S_1 *for the closure under all single generated subcoalgebras).*

Indeed, $S_1 A \subseteq S_1 K$ implies $A \in H\Sigma S_1 A \subseteq H\Sigma S_1 K \subseteq K$.

As for the possible Birkhoff covariety theorems, our covarieties are Birkhoff subcategories in the sense of [8] and therefore the Birkhoff variety theorem 3.3.4 is valid for them. Also coidentities can be involved, to describe the nonsurjectivity of the occurring triple maps, the result being:

3.6. *Let K be a covariety. For $X \in Set$ define an X-ary coidentity as any of the elements of an X-cogenerated cofree coalgebra T_X. We say that a coalgebra A satisfies the X-ary coidentity $t \in T_X$ if $t \notin \operatorname{Im} h$ for any homomorphism $h : A \to T_X$. We call a subclass $L \subseteq K$ describeable by coidentities if there exists a class Z of coidentities such that for a coalgebra $A \in K$ it is $A \in L$ if and only if A satisfies all the coidentities of Z. Then it holds: Necessary and sufficient condition for a subclass $L \subseteq K$ to be describeable by coidentities is that L be* HS Σ*-closed.*

But, as was pointed out by J. Rosický, 3.5 makes the case of covarieties so simple that in fact neither our 3.6 nor the main theorem of [3] says anything more than 3.5 itself.

3.6′. *Let (I, n) be a type with constant arity map $n : i \to N \in$* Card. *Then* **3.6** *remains valid when defining*

a) an X-ary coidentity as any pair (τ, σ) of maps $\tau : I^* \to X$ and $\sigma : I^+ \to N$,

b) the validity of the coidentity (τ, σ) in a coalgebra $(A, (f_i))$ as the validity of a formula

$$\forall_{a \in A} [(\forall_{s \in I^+} f'_s a = \sigma s) \Rightarrow \exists_{u, v \in I} \tau u \neq \tau v \text{ and } f_u a = f_v a]$$

in $(A, (f_i), (f'_i))$.

Proof: Relate to an X-ary coidentity (τ, σ) of T_X the map $(\tau, (\sigma_i)) : I^* \to X \times N^I$, $s \mapsto (\tau s, (\sigma(is)))$ and consult the proof of 2.3 to see that the above formula means: There is no $a \in A$ and no map $h : A \to X$ such that $h^+ a = (\tau, (\sigma_i))$. Because of the coadjointness this is the same condition as that of 3.6.

Further possible modification of 3.6 involves notions coming from elementary automata theory:

3.6″. *For the type as in* **3.6′**, **3.6** *remains valid when defining:*

a) a coidentity as a couple $(\sigma, \boldsymbol{H})$ with $\sigma : I^+ \to N$ (an N-sorted language), $\boldsymbol{H}$ being a covariety of unary algebras (with I-indexed operations),

b) a coalgebra $(A, (f_i))$ to satisfy such a coidentity if its underlying unary algebra belongs to $\boldsymbol{H}$ whenever there is $a \in A$ such that $f'_s a = \sigma s$ for every $s \in I^+$ (an automaton $(A, (f_i), (f'_i))$ accepts the language σ).

Proof: For covarieties of unary algebras the coidentities are simply maps τ: $I \to X$ and for them the formula of 3.6′ looses its antecedent part.

The approach of [3], in turn, leads to

3.7. Theorem. *Let $\boldsymbol{K}$ be a covariety, let r be a cardinal number such that every single-generated coalgebra of $\boldsymbol{K}$ is r-cogenerated. Then any subcovariety $\boldsymbol{L}$ of $\boldsymbol{K}$ admits a description by coidentities at most r-ary.*

Proof: When using cofree coalgebras instead of general injective objects for the Drbohlav's [3] description of a subcovariety $\boldsymbol{L}$ of $\boldsymbol{K}$, one needs only to embed all the coalgebras of $\mathbf{S}_1\boldsymbol{K} - \mathbf{S}_1\boldsymbol{L}$ into some cofree coalgebras of $\boldsymbol{K}$. The result follows from 2.6 now.

Example: The rank of any covariety of monounary algebras is thus established to be at most two.

Also the dual part of the Birkhoff variety theorem has its covariety analogy:

3.8. *Given a covariety $\boldsymbol{K}$ of the rank r (the least number r of 3.7.), the coidentities non valid in some given subcovariety $\boldsymbol{L} \subseteq \boldsymbol{K}$ form a fully (i.e. under endomorphisms) invariant subcoalgebra of the cofree coalgebra $\boldsymbol{T}_r$ of $\boldsymbol{K}$.*

3.9. *All the subcovarieties of $\boldsymbol{K}$ form a complete lattice under the ordering by inclusion, which is* i s o m o r p h i c *to that of all fully invariant subcoalgebras of $\boldsymbol{T}_r$.*

3.10. *Corollary: The lattice of all subcovarieties of a given covariety is complete and completely distributive.*

IV. COVARIETIES AS EQUATIONAL CLASSES

Let $\mathfrak{D}$ denote the equational theory of complete atomic Boolean algebras, with operation symbols $\bigcup$, $\bigcap$ (infinite), $'$, $\emptyset$, 1 and with equations those of complete Boolean algebras plus complete distributivity (see Tarski [12].)

4.1. The equational class $\mathfrak{D}$-*alg* is well known to be equivalent to the category Set^{op}.

Linton extended this algebraicity of Set^{op} to arbitrary tripleable category over Set^{op} (so that also to covarieties): we know from [7] that for every cotriple $\mathbf{G}$ in Set the composition of the tripleable functors

$$(Set^{\mathbf{G}})^{op} \xrightarrow{(U^{\mathbf{G}})^{op}} Set^{op} \xrightarrow{2^{(-)}} Set$$

is tripleable again, i.e. equivalent to the forgetful functor from some equational class to Set. We are going to make this equivalence explicit, in order to present the covariety theory as a part of (infinitary) universal algebra.

4.2. Definition. Le (I, n) be a (coalgebra) type. Enrich the theory $\mathfrak{D}$ by

a) unary operations F_i, one for each $i \in I$, satisfying the equations of being endomorphisms of complete Boolean algebras,

b) families $(Q^i_k)_{k\in n_i}$ of constants, one for each $i \in I$, satisfying the equations $\bigcup_{k\in n_i} Q^i_k = 1$ and $Q^i_k \cap Q^i_j = \emptyset$ whenever $k \neq j \in n_i$.

Denote the arising theory by $\mathfrak{E}_{I,n}$. Further, define $2^{(-)}$ as an assignment to a coalgebra $A = (A, (f_i))$ the $\mathfrak{E}_{I,n}$-algebra $2^A = (2^A, \bigcap, \bigcup, ', \emptyset, A, (f_i^{-1}), (f_i'^{-1}\{k\}))$ (cf. 4.1.) and to a homomorphism h: $A \to B$ the (obviously homomorphic) map 2^h: $2^B \to 2^A$.

4.3. Observation. Let K be a category of coalgebras of a type (I, n), and all homomorphisms between them. Then $2^{(-)}$ of 4.2 is a full embedding of the dual category K^{op} into the category $\mathfrak{E}_{I,n}-alg$.

The kind reader will forgive us to refer to the collection like $\mathfrak{E}_{I,n}-alg$ as to category.

Proof: A sort of such an embedding, having an advantage of being immediate, can be obtained in the following way: The faithfullness of $2^{(-)}$ ($2^f = 2^g$ iff $f = g$) and 4.1 prove a $1 - 1$ correspondence between all

1° coalgebra homomorphisms

$$(A, f\colon A \to nA) \xrightarrow{h} (B, g\colon B \to nB),$$

2° all commutative squares

$$\begin{tikzcd} 2^{nA} \arrow[d, "2^{\bar f}"'] & & 2^{nB} \arrow[ll, "2^{nh}"'] \arrow[d, "2^{\bar g}"] \\ 2^{A} & & 2^{B} \arrow[ll, "2^{h}"] \end{tikzcd}$$

3° all commutative squares

$$\begin{tikzcd} (2^{A})^{n} \arrow[d, "\bar F"'] & & (2^{B})^{n} \arrow[ll, "H^{n}"'] \arrow[d, "\bar G"] \\ 2^{A} & & 2^{B} \arrow[ll, "H"] \end{tikzcd}$$

in the category $\mathfrak{D}$-*alg*, and, up to isomorphism.

4° all homomorphisms of the enriched complete atomic Boolean algebras

$$(\bar A, \cap, \cup, ', \emptyset, A, F: \bar A^n \to A) \leftarrow (\bar B, \cap, \cup, ', \emptyset, B, \bar G: \bar B^n \to B),$$

where F, $\bar G$ are required to be homomorphisms of the complete Boolean algebras.

Clearly, in the above structures I-tuples of operations may occur. Then these correspondences composed give the desired embedding.

To obtain arities at most one perform this procedure only for unary operations, for labellings consider $1-1$ correspondences between all

1′ homomorphisms $(A, A \xrightarrow{f'} n) \xrightarrow{h} (B, B \xrightarrow{g'} n)$,

2′ commutative triangles

$$\begin{tikzcd} & 2^{n} \arrow[dl, "2^{f'}"'] \arrow[dr, "2^{g'}"] & \\ 2^{A} & & 2^{B} \arrow[ll, "2^{h}"] \end{tikzcd}$$

3′ homomorphisms of enriched Boolean algebras

$$(2^A, \cap, \cup, ', \emptyset, A, F': 2^n \to 2^A) \xleftarrow{H} (2^B, \cap, \cup, ', \emptyset, B, G': 2^n \to 2^B)$$

satisfying $F' = HG'$, F', G', being homomorphisms of complete Boolean algebras.

But such homomorphisms F': $2^n \to 2^A$ are just complement preserving homomorphisms of the underlying complete semilattices $(2^n, \cup)$, $(2^A, \cup)$, the first being known to be the free one. Thus the maps like F' are just free extensions of maps Q: $n \to 2^A$ satisfying $Q_i' = \bigcup_{j \neq i} Q_j$ i.e. satisfying $\bigcup_{i \in n} Q_i = A$ and $Q_i \cap Q_j = \emptyset$ for all $i \neq j$. The condition $F' = HG'$ in subset notation looks like $Q_j^A = HQ_j^B$ for all j, so that we can go on with

4′ homomorphisms of $\mathfrak{C}_{I,n}$-algebras

$$(2^A, \cap, \cup, ', \emptyset, A, (Q_j^A)) \xleftarrow{H} (2^B, \cap, \cup, ', \emptyset, B, (Q_j^B)),$$

where

$$Q_j^A = (f')^{-1}\{j\}, \qquad Q_j^B = (g')^{-1}\{j\},$$

which finishes the proof.

In fact, what this observation tells for small types ($I \in Set$) is that the category of all (I, n)-coalgebras is dual to that of all $\mathfrak{C}_{I,n}$-algebras. The case of general covarieties is a matter of classical Birkhoff variety theorem. First an auxiliary theorem:

4.4. Theorem. *Let* $\mathbf{K}$ *be a class of coalgebras, let* $\mathbf{E}$ *be the class of the corresponding* $\mathfrak{C}_{I,n}$*-algebras. Then* $\mathbf{K}$ *possesses all cofree coalgebras if and only if* $\mathbf{E}$ *possesses all free algebras. More precisely: For* $X \in Set$ *a coalgebra* T *is cofree in* $\mathbf{K}$*, cogenerated into* 2^X *if and only if* 2^T *is free in* $\mathbf{E}$ *generated by* X.

Proof: All is clear from the following diagram of functors and their adjoints

$$\begin{tikzcd}
 & K^{op} \equiv E \arrow[dl, shift left, "U^{op}"] \arrow[d, shift left, "U^*"] \\
Set^{op} \arrow[ur, shift left, "F^{op}"] \arrow[r, shift left, "2^{(-)}"] & Set \arrow[u, shift left, "F^*"] \arrow[l, shift left, "2^{(-)}"]
\end{tikzcd}$$

where the left-hand arrows are left adjoints to the right-hand ones. All assumption of the "Sandwich theorem" 3.1.29 of [8] are satisfied, therefore F^{op} exists if and only if F^* does. Other cofree coalgebras exist e.g. by 2.2 or can be computed as certain quotients of the larger ones in the expected way.

4.5. Theorem. *Let* L *be a covariety of coalgebras of a type* (I, n)*. Then sufficient and necessary condition for its subclass* $\mathbf{K}$ *to be also covariety is that* $\mathbf{K}^{op}$ *as the concrete category* $\mathbf{K}^{op} \xrightarrow{U^{op}} Set^{op} \xrightarrow{2^{(-)}} Set$ *be equivalent to some* $\mathfrak{C}$*-alg* $\to Set$ *with* $\mathfrak{C}$ *obtainable by adding new equations to those of* $\mathfrak{C}_{I,n}$.

Proof: First observe, that there is a $1-1$ correspondence between all **HS** Σ-closed subclasses in $\boldsymbol{L}$ and all **HSP**-closed subclasses of the class $\boldsymbol{E}$ of all $\mathfrak{C}_{I,n}$-algebras 2^A corresponding to the coalgebras A of $\boldsymbol{L}$. This follows at once from the functor properties of $2^{(-)}$ ($\mathbf{H} \leftrightarrow \mathbf{S}$ because of U-split monos and U-split epis). In order to be able to apply the Birkhoff variety theorem on $\boldsymbol{E}$ we need to have enough free algebras here, but we have all by 4.4.

4.6. Corollary. *Let (I, n) be a small type. By a set equation of the type (I, n) let us mean any pair of set polynomials, which are defined to be polynomials (in subsets of coalgebras as variables) whose operation symbols are those of infinitary Boolean operations, f_i^{-1} (for the inverse images under the unary operations) and $f_i'^{-1}\{j\}$, $j \in n_i$ (as constants). Call a coalgebra satisfying such a set equation if the obvious realizations of the two set polynomials coincide. Call a class $\boldsymbol{K}$ of (I, n)-coalgebras to be defined by set equations if there exists a class Z of set equations such that $\boldsymbol{K}$ contains those and only those coalgebras of the type (I, n), which satisfy all the set equations of Z. Then it holds: Covarieties of coalgebras of a small type are just classes of coalgebras of the same type which are defined by set equations.*

4.7. Remark on regular covarieties. Regular covarieties, i.e. equational classes over Set^{op} are easily recognizable among others when deriving 4.5 for the theory obtained from only $1°-4°$ of 4.3 instead of $\mathfrak{C}_{I,n}$ as just those, whose defining equations do not link the Boolean operations and the added ones more than the requirement of 4° does. Thus we see that regular covarieties are indeed rare.

4.8. Remark on distinguishing subsets: To decide whether $D \subseteq A$ is a distinguishing subset in a coalgebra A is the same thing as to decide whether D as an element of 2^A generates it. This is an obvious consequence of 4.4 and 2.7.

It follows now, that if D is a distinguishing subset of a coalgebra A, then a necessary and sufficient condition on another subset X of A to be also distinguishing is that D be a value in X of some set polynomial on A.

V. COALGEBRAS AS LINTON ALGEBRAS

Following Davis [1], different algebras we can convert coalgebras into are Linton algebras over Set^{op}, which have an advantage of keeping the carriers. The only thing we want to show in this section is that this conversion can be easily derived in a way analogous to that of 4.3. The reader is assumed to be familiar with the fundamentals of Linton theories, for instance from [10].

Define the Linton theory $\mathfrak{L}$ in the following way: For every $u, v \in \mathrm{Card}$ let $\mathfrak{L}$ possess all the possible $(2^u, 2^v)$-ary operation symbols (maps) p: $2^u \to 2^v$ and no equations but those of equal composition. Thus, the $\mathfrak{L}$-algebras are just sets equipped with all the possible Boolean operations, write them $A^* = (A, (2^u \xrightarrow{p} 2^v))$, and the $\mathfrak{L}$-homomorphisms are just all the possible maps between them, which makes $\mathfrak{L}$-*alg* isomorphic to *Set*.

5.1. Definition. Let (I, n) be a coalgebraic type, define a Linton theory $\mathfrak{L}_{I,n}$ as $\mathfrak{L}$ enriched by a) new $(2^{n_i}, 2)$-ary operations F_i, one for each $i \in I$. b) new equations of F_i commuting with all the possible maps p of $\mathfrak{L}$.

5.2. Observation. Similarly to 4.3 there are $1 - 1$ correspondences between
1″ all coalgebra homomorphisms

$$(A, \bar{f}; A \to nA) \xrightarrow{h} (B, \bar{g}; B \to nB),$$

2″ All commutative squares

$$\begin{tikzcd} 2^{nA} \arrow[d, "2^{\bar f}"'] & & 2^{nB} \arrow[ll, "2^{nh}"'] \arrow[d, "2^{\bar g}"] \\ 2^A & & 2^B \arrow[ll, "2^h"] \end{tikzcd}$$

3″ all commutative squares

$$\begin{tikzcd} (nA)^* \arrow[rr, "nh"] \arrow[d, "\bar f"'] & & (nB)^* \arrow[d, "\bar g"] \\ A^* \arrow[rr, "h"'] & & B^* \end{tikzcd}$$

in $\mathfrak{L}$-*alg*.
4″ all homomorphisms of $\mathfrak{L}_{I,n}$-algebras

$$(A, (2^{nA} \xrightarrow{2^{\bar f}} 2^A), (2^{unA} \xrightarrow{p^{nA}} 2^{vnA})) \xrightarrow{h} (B, (2^{nB} \xrightarrow{2^{\bar g}} 2^B), (2^{unB} \xrightarrow{p^{nB}} 2^{vnB})),$$

which proves that for any class of coalgebras of a type (I, n) $(-)^*$ defined by $1'' \to 4''$ embeds it into $\mathfrak{L}_{I,n}$-*alg*.

5.3. Remark. Similar simplification for labelling operations as that of 4.3. is also possible in this case; the corresponding Linton operations turn out to be F': $1^A \to n^A$, with f' as the image of the unique element of 1^A; this can be easily verified directly.

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