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DOMATIC NUMBERS OF UNIFORM HYPERGRAPHS

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Abstract. E. J. Cockayne and S. T. Hedetniemi have defined the domatic number of a graph G s the maximum number of classes of a partition of the vertex set of G, all of whose classes are dominating sets in G. In this paper this concept is generalized for hypergraphs and its properties are studied.

Key words. Dominating set, partition, domatic number, uniform hypergraph, hyperforest.

In [1] E. J. Cockayne and S. T. Hedetniemi have introduced the domatic number of an undirected graph. Here we shall introduce an analogy of this numerical invariant for hypergraphs.

Let H be a hypergraph with the vertex set $V(H)$ and with the edge set $E(H)$. A dominating set in H is a subset D of $V(H)$ with the property that to each vertex $x \in V(H) - D$ there exists a vertex $y \in D$ such that there exists an edge of H containing both x and y. A partition of $V(H)$, all of whose classes are dominating sets in H, is called a domatic partition of H The maximum number of classes of a domatic partition of H is called the domatic number of H and denoted by $d(H)$.

This concept can be related to the domatic number of a graph in a simple way. For a hypergraph H we can introduce the graph $G(H)$. The vertex set of $G(H)$ is $V(H)$ and two vertices are adjacent in $G(H)$ if and only if they are distinct and there exists an edge of H containing both of them. The following assertion is easy to prove.

Proposition 1. *Let H be a hypergraph, let $G(H)$ be the above defined graph. Then* $d(H) = d(G(H))$.

Thus the investigation of domatic numbers of hypergraphs can be easily transferred to the investigation of domatic numbers of graphs. Nevertheless, we shall prove some results concerning uniform hypergraphs. A hypergraph H is called r-uniform for a positive integer r, if the cardinality of any edge of H is r. We shall take always $r \geqq 2$.

Proposition 2. *For an r-uniform hypergraph H, where $r \geqq 2$, we have $d(H) = 1$ if and only if H has an isolated vertex: otherwise $d(H) \geqq 2$.*

Proof. The hypergraph H has an isolated vertex if and only if so has $G(H)$. For graphs (i.e. for $r = 2$) the assertion holds [1] and thus according to Proposition 1 it holds for any r.

It may seem that for $r > 2$ the minimum value of the domatic number of an r-uniform hypergraph might be greater than 2. But this conjecture is disproved by the following theorem.

Theorem 1. *For every integer $r \geqq 2$ there exists an r-uniform hypergraph H without isolated vertices such that $d(H) = 2$.*

Proof. Let V_1 be a set of a cardinality $n \geqq 3r$, let V_2 be a set of the cardinality $\binom{n}{r-1}$ and $V_1 \cap V_2 = \emptyset$. Put $V = V_1 \cup V_2$. Let φ be a one-to-one mapping of V_2 onto the set of all subsets of V_1 having the cardinality $r - 1$. Let H be the hypergraph with the vertex set V and with the edge set consisting of all sets of the form $\{v\} \cup \varphi(v)$ for $v \in V_2$. The hypergraph H is evidently r-uniform without isolated vertices, thus $d(H) \geqq 2$. Suppose that $d(H) \geqq 3$. Evidently there exist domatic partitions of H of all cardinalities between 1 and $d(H)$; thus there exists a domatic partition $\mathscr{D}$ with three classes. As $|V_1| \geqq 3r$, there exists a class $D_1 \in \mathscr{D}$ such that $|D_1 \cap V_1| \geqq r$. Let D_0 be a subset of $D_1 \cap V_1$ of the cardinality $r - 1$, let $v = \varphi^{-1}(D_0)$. Then $E = D_0 \cup \{v\}$ is an edge of H. Let D_2 be the class of D containing the vertex v (it may be $D_2 = D_1$). As $\mathscr{D}$ has three classes, there exists $D_3 \in \mathscr{D} - \{D_1, D_2\}$. The vertex v is contained in only one edge, namely E. No vertex of E belongs to D_3, therefore D_3 is not a dominating set in H and $\mathscr{D}$ is not a domatic partition, which is a contradiction. Hence $d(H) = 2$.

A hypergraph H will be called a simple hyperforest, if it contains no isolated vertices and for any edges $E_1, \ldots, E_k$, where $k \geqq 3$ and $E_i \cap E_{i+1} \neq \emptyset$ for $i = 1, \ldots, k - 1$ we have $E_1 \cap E_k = \emptyset$.

Note that any vertex of a simple hyperforest has the degree 1 or 2. If a vertex had the degree 0, it would be isolated. If it had the degree greater than 2, then the edges incident with it would not satisfy the condition from the definition.

Theorem 2. *Let H be a simple hyperforest and let H be r-uniform for $r \geqq 2$. Then there exists a domatic partition $\{D_1, \ldots, D_r\}$ of H such that $|D_i \cap E| = 1$ for any $i = 1, \ldots, r$ and any edge E of H.*

Proof. We use the induction according to the number m of edges of H. If $m = 1$, then H consists of r vertices and one edge and $\{D_1, \ldots, D_r\}$ is the partition of the vertex set of H into one-element subsets. Let $m_0 \geqq 2$ and suppose that the assertion is true for any r-uniform simple hyperforest with $m_0 - 1$ edges. Let H be an r-uniform simple hyperforest with m_0 edges. Let E be an edge of H and let H' be the hypergraph obtained from H by deleting E. Let F_0 be the set of all vertices of H which are contained in E and in no other edge of H. Then F_0 is the set of all isolated vertices of H'. Let H'' be the hypergraph obtained from H' by deleting all vertices of F_0. Evidently H'' is an r-uniform simple hyperforest with $m_0 - 1$ edges. Thus there exists the required partition $\{D''_1, \ldots, D''_r\}$ of H''. We may consider it as a colouring of vertices of H'' by the colours $1, \ldots, r$ such that a vertex is coloured by the colour i if and only if it belongs to D''_i; this colouring will be denoted by $\mathscr{C}''$.

Let C be the connected component of H which contains the edge E. Let $E_1, \ldots, E_k$ be all edges of H having non-empty intersections with E, let $F_i = E_i \cap E$ for $i = 1, \ldots, k$. As H is a simple hyperforest, we have $F_i \cap F_j = \emptyset$ for $i \neq j$, $\{i, j\} \subseteq \{0, 1, \ldots, k\}$ and $E = \bigcup_{i=0}^{k} F_i$. Moreover, after deleting E from C a hypergraph consisting of k connected components $C_1, \ldots, C_k$ is obtained such that E_i is in C_i for $i = 1, \ldots, k$.

Let $f_i = | F_i |$ for $i = 0, 1, \ldots, k$, let $g_i = \sum_{j=1}^{i} f_j$ for $i = 1, \ldots, k$ and $g_0 = 0$. For $i = 1, \ldots, k$ let G_i be the set of all integers x such that $g_{i-1} + 1 \leqq x \leqq g_i$ and let K_i be the set of all colours which occur in F_i. Evidently $| G_i | = | K_i | = f_i$ for $i = 1, \ldots, k$. Let φ_i be a one-to-one mapping of K_i onto G_i. As both G_i and K_i are subsets of the set $\{1, \ldots, r\}$, there exists a permutation ψ_i of $\{1, \ldots, r\}$ whose restriction onto K_i is φ_i.

Now we shall colour the vertices of H. Any vertex not belonging to C will be coloured by the same colour as in $\mathscr{C}''$. Any vertex of C_i will be coloured by the colour $\psi_i(c)$, where c is its colour in $\mathscr{C}''$. The vertices of F_0 will be coloured by pairwise distinct colours from the set of all integers x such that $g_k + 1 \leqq x \leqq r$. This colouring will be denoted by $\mathscr{C}$. The colouring $\mathscr{C}$ has the property that the vertices of any edge of H are coloured by pairwise distinct colours. Thus there exists a partition $\{D_1, \ldots, D_r\}$ of the vertex set of H such that each of its classes has exactly one vertex in common with each edge of H. Evidently it is a domatic colouring of H.

Corollary 1. *Let H be an r-uniform hypergraph for $r \geqq 2$ which has a simple hyperforest as a spanning subhypergraph. Then $d(H) \geqq r$.*

Note that the condition from the definition of a simple hyperforest cannot be weakened so that if $E_1, \ldots, E_k$ are edges of H such that $E_i \cap E_{i+1} \neq \emptyset$ for $i = 1, \ldots, k - 1$, then either $E_1 \cap E_k = \emptyset$, or $\bigcap_{i=1}^{k} E_i \neq \emptyset$. Let r be even and let $s = \frac{1}{2} r - 1$. Let H be a hypergraph with the vertex set $\{v, w_1, w_2, w_3, x_1, \ldots, x_s, y_1, \ldots, y_s, z_1, \ldots, z_s\}$ and with the edge set $\{E_1, E_2, E_3\}$, where $E_1 = \{v, w_1, x_1, \ldots, x_s, y_1, \ldots, y_s\}$, $E_2 = v, w_2, y_1, \ldots, y_s, z_1, \ldots, z_s\}$, $E_3 = \{v, w_3, x_1, \ldots, x_s, z_1, \ldots, z_s\}$. The reader may verify himself that this hypergraph satisfies the weakened condition and the assertion of Theorem 2 does not hold for it.

The following proposition is the generalisation of an analogous assertion for graphs from [1].

Proposition 3. *Let H be an r-uniform hypergraph for $r \geqq 2$, let $\delta(H)$ be the minimum degree of a vertex of H, let $d(H)$ be the domatic number of H. Then $d(H) \leqq \leqq (r - 1)\,\delta(H) + 1$.*

Proof. For any vertex v of H let $N(v)$ be the set of all vertices x of the property that there exists an edge of H containing both v and x. Evidently the cardinality of $N(v)$ is at most $(r - 1)\,\delta(v) + 1$, where $\delta(v)$ is the degree of v. The set $N(v)$ for any v must have non-empty intersections with all classes of any domatic partition of H, therefore $d(H)$ cannot be greater than $(r - 1)\,\delta(H) + 1$.

A hypergraph H with the property that $d(H) = (r - 1)\,\delta(H) + 1$ will be called domatically full. (This is an analogy of a domatically full graph introduced by the quoted authors.)

Theorem 3. *Let H be a domatically full r-uniform hypergraph which is regular of degree δ. Then any two edges of H have at most one common vertex.*

Proof. Suppose that two edges E_1, E_2 of H have two distinct common vertices u, v. Let $N(v)$ denote the same as in the proof of Proposition 3. All edges containing u have the common vertex u and moreover two of them have a vertex v in common, thus $|N(v)| \leqq (r - 1)\,\delta < (r - 1)\,\delta + 1 = d(H)$ and there exists a class of a domatic partition of H which has the empty intersection with $N(v)$, which is a contradiction.

Theorem 4. *If a domatically full r-uniform hypergraph H with n vertices and regular of degree δ exists, then there exists a bipartite graph B on the vertex sets U, V such that $|U| = n$, all vertices of U have the degree δ, all vertices of V have the degree r and any two vertices of B have at most one common neighbour.*

Proof. Let H be the described hypergraph. We may denote its vertex set by U and its edge set by V. We construct a bipartite graph B on the sets U, V such that a vertex of U is adjacent to a vertex of V if and only if it is contained in the edge of H corresponding to this vertex. Evidently each vertex of U has the degree δ and each vertex of V has the degree r. If two vertices of V had at least two common neighbours, this would mean that two edges of H would have at least two common vertices, which is a contradiction with Theorem 3. If two vertices of U had at least two common neighbours, this would mean that two vertices of H would be contained both in two distinct edges, i.e. again that there would exist two edges having two vertices in common; this is again a contradiction. A vertex of U with a vertex of V cannot have a common neighbour, because B is bipartite.

Theorem 5. *Let H be a domatically full r-uniform hypergraph which is regular of degree δ, let n be its number of vertices. Then n is divisible by $(r - 1)\,\delta + 1$ and by $r/(r, \delta)$, where (r, δ) is the greatest common divisor of r and δ.*

Proof. Let $N(v)$ denote the same as in the proof of Proposition 3. Let $\mathcal{D}$ be a domatic partition of H with $d(H) = (r - 1)\,\delta + 1$ classes. Suppose that there are vertices v_1, v_2 such that $v_1 \neq v_1$, $N(v_1) \cap N(v_2) \neq \emptyset$ and v_1, v_2 belong to the same class of $\mathscr{D}$. Let $w \in N(v_1) \cap N(v_2)$. Then $v_1 \in N(w)$, $v_2 \in N(w)$. The set $N(w)$ contains $d(H)$ vertices. As two of them belong to the same class of $\mathscr{D}$, there exists a class of $\mathscr{D}$ disjoint with $N(w)$, which is a contradiction. Therefore if $D \in \mathscr{D}$, then the sets $N(v)$ for $v \in D$ are pairwise disjoint. As D is a dominating set in H, the union of all these sets is the vertex set of H. Therefore we have a partition of the vertex set of H, all of whose classes have the cardinality $(r - 1)\,\delta + 1$ and n must be divisible by this number. Now consider the graph B from Theorem 4. If m is the number of edges of H, then $|U| = n$, $|V| = m$ and the number of edges of B is $n\delta = mr$, which gives $n = mr/\delta$. Let a be the greatest common divisor of r and δ. Then $r = ar_0$, $\delta = a\delta_0$, where r_0 and δ_0 are relatively prime, and $n = mr_0/\delta_0$. Then m/δ_0 is an integer and n is the product of integers m/δ_0 and r_0, hence it is divisible by $r_0 = r/a = r/(r, \delta)$.

At the end we show an example of a domatically full 3-uniform hypergraph which is regular of degree 2 and which has 15 vertices. (This is the least possible number of vertices of such a hypergraph.)

Let the vertex set of H be $\{a, a', a'', b, b', b'', c, c', c'', d, d', d'', e, e', e''\}$ and let the edges of H be $\{a, b, e'\}$, $\{b, d, c'\}$, $\{d, e, a'\}$, $\{c, e, b'\}$, $\{a, c, d'\}$, $\{a', b'', c''\}$, $\{b', a'', d''\}$, $\{c', a'', e''\}$, $\{d', b'', e''\}$, $\{e', c'', d''\}$. Then H is a 3-uniform hypergraph and is regular of degree 2. The sets $\{a, a', a''\}$, $\{b, b', b''\}$, $\{c, c', c''\}$, $\{d, d', d''\}$, $\{e, e', e''\}$ form a domatic partition of H of the cardinality 5.

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