Archivum Mathematicum

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Archivum Mathematicum, Vol. 21 (1985), No. 4, 229--233

Persistent URL: http://dml.cz/dmlcz/107238

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 21, No. 4, (1985), 229–234

ON f-BEST APPROXIMATION IN TOPOLOGICAL SPACES

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(Received December 2, 1983)

Abstract. If K is a non-empty closed subset of a Hausdorff topological space X and f a continuous real-valued function on $X \times X$ then an element $k_0 \in K$ is said to be an f-best approximation to x in K if $f(x, k_0) = \inf \{f(x, k) \colon k \in K\}$. The set-valued map which takes each $x \in X$ to its set of its f-best approximants is called the f-best approximation map. In this paper we discuss the existence of f-best approximation, uniqueness of f-best approximation and the continuity of the f-best approximation map in Hausdorff topological spaces.

Key words. f-best approximation, f-projection, f-proximinal, f-Chebyshev, f-boundedly compact, γ-compact and f-convex set.

By using the existence of elements of f-best approximation in Hausdorff topological spaces, certain results on fixed points were proved by Pai and Veermani in [6]. Here we shall also discuss the existence of f-best approximation, uniqueness of f-best approximation and the continuity of the f-best approximation map in Hausdorff topological spaces. We start with a few definitions.

Let X be a Hausdorff topological space and f a continuous real-valued function on $X \times X$. Let K be a non-empty closed subset of X.

An element $k_0 \in K$ is said to be *f-nearest to x in K* or *f-best approximation* to x in K [6] if $f(x, k_0) = f(x, K) \equiv \inf \{f(x, k) : k \in K\}$.

The set-valued mapping $P_f \colon x \to P_f(x) \equiv \{k_0 \in K : f(x, k_0) = f(x, K)\}$ is called the *f-best approximation map* or *f-projection* [6] supported on K.

The set K is said to be *f-proximinal* (respectively *f-Chebyshev*) [6] if $P_f(x) \neq \emptyset$ (respectively $P_f(x)$ is a singleton set) for each x in X.

The set K is said to be *inf-compact at a point* $x \in X$ [6] if each minimizing net $\{k_\alpha\}$ in K (i.e. $f(x, k_\alpha) \to f(x, K)$) has a convergent subnet converging in K.

K is said to be *inf-compact* [6] if it is inf-compact at each point $x \in X$.

In case X is a metric space and $f = d$, the metric on X, the notion of *inf-compactness* of K coincides with the well known notion of approximative compactness (see [7]) of K. In this case, f-nearest elements to x in K are usually called elements of best approximation to x in K.

* The author is thankful to the U.G.C., India for financial support.

The mapping f is said to be *inf-compact at a point* $x \in X$ if the sub-level sets

$$S_r = \{y \in X : f(x, y) \leqq r\}$$

are compact for each $r \in \boldsymbol{R}$. f is said to be *inf-compact* if it is inf-compact for each $x \in X$.

The set K is said to be *f-boundedly compact* if for each $x \in X$ and $r \in \boldsymbol{R}$, $K \cap S_r$ is compact.

The set K is said to be *γ-compact* if for each $x \in X$, there exists $\gamma > f(x, K)$ such that $K \cap S_\gamma$ is compact.

Let X and Y be two topological spaces, then a mapping $g : X \to 2^Y$ (the collection of all subsets of Y) is called *upper-Kuratowski semi-continuous* if the relations

$$\lim_\alpha x_\alpha = x, \qquad y_\alpha \in g(x_\alpha), \qquad \lim_\alpha y_\alpha = y$$

imply $y \in g(x)$.

g is called *upper-semi-continuous* (*lower-semi-continuous*) if the set

$$g^{-1}(A) = \{x \in X : g(x) \cap A \neq \emptyset\}$$

is closed (open) for each closed (open) set A in Y.

Throughout the following, we assume that X is a Hausdorff topological space, f is a continuous real-valued function on $X \times X$ and K is a non-empty closed subset of X.

Proposition 1. *Consider the following statements:*

(i) *f is inf-compact,*

(ii) *K is f-boundedly compact,*

(iii) *K is γ-compact,*

(iv) *K is inf-compact,*

(v) *K is f-proximinal.*

We have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v).

Proof. (i) $\Rightarrow$ (ii). Since f is inf-compact, $S_r = \{y \in X : f(x, y) \leqq r\}$ is compact for each $x \in X$ and $r \in \boldsymbol{R}$. This implies that $K \cap S_r$ is compact for each $x \in X$ and $r \in \boldsymbol{R}$ as K is closed.

(ii) $\Rightarrow$ (iii). Let $x \in X$. Choose any $\gamma > f(x, K)$. Consider the set $K \cap S_\gamma$. This is compact, so K is γ-compact.

(iii) $\Rightarrow$ (iv). Let $x \in X$ and $\{k_\alpha\}$ be a minimizing net in K i.e. $f(x, k_\alpha) \to f(x, K)$. Since K is γ-compact, there exists $\gamma > f(x, K)$ such that $K \cap S_\gamma$ is compact. Since $\gamma > f(x, K) = \lim_\alpha f(x, k_\alpha)$, $\{k_\alpha\}$ is eventually in $K \cap S_\gamma$. Compactness of $K \cap S_\gamma$ implies that the new net, obtained by deleting those k_α's which do not lie in $K \cap S_\gamma$, will have a convergent subnet in K. Hence K is inf-compact.

(iv) $\Rightarrow$ (v). Let $x \in X$. By the definition of $f(x, K)$, we can extract a net $\{k_\alpha\}$ in K such that $\lim_\alpha f(x, k_\alpha) = f(x, K)$. Now K being inf-compact at x, $\{k_\alpha\}$ has a convergent subnet $\{k_\beta\}$ converging to $k_0 \in K$. Then

$$f(x, K) = \lim_\beta f(x, k_\beta)$$
$$= \underline{\lim} f(x, k_\beta)$$
$$\geqq f(x, k_0), \text{ as } f \text{ being continuous, is lower-semi-continuous}$$
$$\geqq f(x, K).$$

Hence $f(x, k_0) = f(x, K)$ and so $k_0 \in P_f(x)$.

It is well known (see e.g. [7]) that for a proximinal set in a metric space, the metric projection is upper-Kuratowski-semicontinuous and for approximatively compact sets it is upper-semicontinuous. For f-proximinal sets we have the following two propositions:

Proposition 2. *If a subset K of X is f-proximinal then P_f is upper-Kuratowski-semicontinuous.*

Proof. Let $\{x_\alpha\}$ be a net in X such that $x_\alpha \to x_0$, $y_\alpha \in P_f(x_\alpha)$, and $y_\alpha \to y_0$. Since K is closed, $y_0 \in K$. We claim that $y_0 \in P_f(x_0)$.

$y_\alpha \in P_f\ (x_\alpha) \Rightarrow f(x_\alpha, y_\alpha) = \inf_{z\in K} f(x_\alpha, z) \Rightarrow \lim_\alpha f(x_\alpha, y_\alpha) = \lim_\alpha \inf_{z\in K} f(x_\alpha, z) \Rightarrow$
$\Rightarrow f(x_0, y_0) \Rightarrow \inf_{z\in K} f(x_0, z)$, as f is continuous $\Rightarrow y_0 \in P_f(x_0)$.

Proposition 3. *If K is inf-compact then P_f is upper-semicontinuous.*

Proof. Let A be a closed subset of X. We want to show that the set $F = \{x \in X : P_f(x)\, A \cap \neq \emptyset\}$ is closed. Let $\{x_\alpha\}$ be a net in F such that $x_\alpha \to x_0$. Then $P_f(x_\alpha) \cap A \neq \emptyset$ for each α. Let $y_\alpha \in P_f(x_\alpha) \cap A$. Then we have $f(x_\alpha, y_\alpha) = f(x_\alpha, K)$. This implies that $\lim_\alpha f(x_\alpha, y_\alpha) = \lim_\alpha f(x_\alpha, K)$ i.e. $\lim_\alpha f(x_0, y_\alpha) = f(x_0, K)$ as f is continuous and $x \to f(x, K)$ is continuous i.e. $\{y_\alpha\}$ is a minimizing net for x_0 in K. Since K is inf-compact, $\{y_\alpha\}$ has a convergent subnet $\{y_\beta\}$ converging to $k_0 \in K$. Continuity of f gives $f(x_0, y_0) = f(x_0, K)$ i.e. $y_0 \in P_f(x_0) \cap A$ whence $x_0 \in F$ and F is closed.

Now we shall discuss conditions under which f-best approximation is unique.

A subset A of X is said to be *f-convex* if $x, y \in A$ imply $z \in A$ where $z \in X$ is such that $f(x, z) + f(z, y) = f(x, y)$ i.e.

$$[x, y] = \{z \in X : f(x, z) + f(z, y) = f(x, y)\}$$

is a subset of A for all $x, y \in A$.

f is said to be a *convex function* if

$$f(x_0, x) \leqq r, \quad f(x_0 y) \leqq r \quad \text{imply} \quad f(x_0, z) \leqq r$$

for all $z \in [x, y]$, where x_0 is arbitrary but fixed point of X.

f is said to be a *strictly convex function* if

$$f(x_0, x) = r = f(x_0, y), \quad x \neq y \quad \text{imply} \quad f(x_0, z) < r.$$

We have the following theorem on uniqueness of f-best approximation:

Theorem 1. *Let K be f-convex subset of X and f a strictly convex function on $X \times X$. Then $P_f(x)$ is atmost singleton for each $x \in X$.*

Proof. Let if possible, $k_1, k_2 \in P_f(x)$ i.e. $k_1, k_2 \in K$ and $f(x, k_1) = f(x, k_2) = f(x, K) \equiv r$. Since K is f-convex, $[k_1, k_2] \subset K$. Since f is strictly convex, $f(x, z) < r$ for all $z \in]k_1, k_2[$, a contradiction.

Remark. From Theorem 1, we get the following: If f is a strictly convex function on $X \times X$ and K an f-proximinal, f-convex subset of X then K is f-Chebyshev. This is similar to the result: A proximinal convex subset of a strictly convex metric space is Chebyshev [5].

The following theorem gives conditions under which the mapping P_f is continuous.

Theorem 2. *If K is inf-compact, f-Chebyshev set and f a continuous mapping of $X \times X \to R$ then P_f is continuous.*

Proof. The proof of this theorem follows from Proposition 3 using the facts that for f-Chebyshev sets, the mapping P_f is single-valued and for single-valued maps the two concepts of upper-semi-continuity and continuity coincide.

Theorem 2 is analogous to the following result:

If K is an approximatively compact, Chebyshev subset of a metric space then the metric projection is continuous [7].

Remark. The notion of ε-approximation (see [7]), best simultaneous approximation (see [1]), proximinal points for pair of sets (see [4]), best co-approximation (see [3]), strong approximation (see [2]) and strong co-approximation can be extended to Hausdorff topological spaces relative to the function f and can be further investigated.

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