## Archivum Mathematicum

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Archivum Mathematicum, Vol. 24 (1988), No. 2, 83--85

Persistent URL: http: //dml.cz/dmlcz/107313

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# THE SUFFICIENT CONDITION OF THE ASYMPTOTIC STABILITY OF TWO-DIMENSIONAL LINEAR SYSTEMS 

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(Received June 23, 1986)


#### Abstract

The differential system of second-order with variable coefficients is studied, and a sufficient condition of the asymptotic stability for solutions is given.


Key words. Asymptotic stability.
MS Classification. 34 DO 5

## 1. INTRODUCTION

In the present paper we consider a system of differential equations

$$
\begin{align*}
& \frac{\mathrm{d} x_{1}}{\mathrm{~d} t}=a_{11}(t) x_{1}+a_{12}(t) x_{2}  \tag{1.1}\\
& \frac{\mathrm{~d} x_{2}}{\mathrm{~d} t}=a_{21}(t) x_{1}+a_{22}(t) x_{2}
\end{align*}
$$

where $a_{i k}: R^{+} \rightarrow R(i, k=1,2)$ are functions summable on every finite segment.
It will be assumed throughout that

$$
\begin{equation*}
\sigma a_{12}(t)>0, \quad \sigma a_{21}(t)<0 \tag{1.2}
\end{equation*}
$$

if $t \in R^{+}$, where $\sigma \in\{-1,1\}$ and the function $\frac{a_{21}}{a_{12}}$ is summable on every finite
segment.
Let

$$
\begin{equation*}
c(t)=\sigma\left(\left|a_{12}(t) a_{21}(t)\right|\right)^{-1 / 2}\left[\frac{1}{2}\left(a_{11}(t)-a_{22}(t)\right)+\frac{1}{4}\left(\ln \left|\frac{a_{21}(t)}{a_{12}(t)}\right|\right)^{\prime}\right] \tag{1.3}
\end{equation*}
$$

$$
\begin{equation*}
l_{i}(t)=\left(\left|\frac{a_{i j}(t)}{a_{j i}(t)}\right|\right)^{1 / 4} \exp \left[\frac{1}{2} \int_{0}^{t}\left(a_{11}(\tau)+a_{22}(\tau)\right) \mathrm{d} \tau\right] \quad(i \neq j ; i, j=1,2) \tag{1.4}
\end{equation*}
$$

$$
\begin{equation*}
\psi(t)=\int_{0}^{t} \sqrt{\left|a_{12}(\tau) a_{21}(\tau)\right|} \mathrm{d} \tau \tag{1.5}
\end{equation*}
$$

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Lemma 1. By means of the transformations

$$
\begin{equation*}
x_{i}(t)=l_{i}(t) y_{i}(s) \quad(i=1,2), s=\psi(t) \tag{1.6}
\end{equation*}
$$

the system (1.1) will take the form

$$
\begin{aligned}
& \frac{\mathrm{d} y_{1}}{\mathrm{~d} s}=\sigma\left[\alpha(s) y_{1}+y_{2}\right] \\
& \frac{\mathrm{d} y_{2}}{\mathrm{~d} s}=-\sigma\left[y_{1}+\alpha(s) y_{2}\right]
\end{aligned}
$$

where

$$
\begin{equation*}
\alpha(s)=c\left(\psi^{-1}(s)\right) \quad \text { if } 0 \leqq s<s_{0}, s_{0}=\lim _{t \rightarrow \infty} \psi(t) \tag{1.8}
\end{equation*}
$$

and $\psi^{-1}$ is the inverse to $\psi$.
Proof. Let ( $x_{1}, x_{2}$ ) be an arbitrary solution of the system (1.1). In view of (1.4), (1.5) and (1.6)

$$
\begin{equation*}
l_{i}^{\prime}(t)=\left[\frac{1}{2}\left(a_{11}(t)+a_{22}(t)\right)+\frac{1}{4}\left(\ln \left|\frac{a_{i j}(t)}{a_{j i}(t)}\right|\right)^{\prime}\right] l_{i}(t), \tag{1.9}
\end{equation*}
$$

where $i \neq j, i, j=1,2$ and

$$
\begin{equation*}
x_{i}^{\prime}(t)=l_{i}^{\prime}(t) y_{i}(s)+l_{i}(t)\left(\left|a_{12}(t) a_{21}(t)\right|\right)^{1 / 2} y_{i}^{\prime}(s) \tag{1.10}
\end{equation*}
$$

( $i=1,2$ ). By substitung (1.6), (1.9) and (1.10) into (1.1) we obtain (1.7). The lemma is proved.

Lemma 2. Let the function $\alpha$ in (1.8) be absolutely continous on every finite segment and let there exist $s_{1} \in\left(0, s_{0}\right)$ and $\delta \in(0,1)$ such that $|\alpha(s)|<\delta$ for $s_{1} \leqq s<s_{0}$. Then every solution $\left(y_{1}, y_{2}\right)$ of (1.7) satisfies the estimate

$$
\begin{equation*}
y_{1}^{2}(s)+y_{2}^{2}(s) \leqq \frac{1+\delta}{1-\delta}\left[y_{1}^{2}\left(s_{1}\right)+y_{2}^{2}\left(s_{1}\right)\right] \exp \left[\frac{1}{1-\delta} \int_{s_{1}}^{s}\left|\alpha^{\prime}(\xi)\right| \mathrm{d} \xi\right] \tag{1.11}
\end{equation*}
$$

for $s_{1} \leqq s<s_{0}$.
Proof. Let ( $y_{1}, y_{2}$ ) be an arbitrary solution of the system (1.7). From (1.7) we have

$$
-y_{1}(s) y_{1}^{\prime}(s)-\alpha(s) y_{2}(s) y_{1}^{\prime}(s)=\alpha(s) y_{1}(s) y_{2}^{\prime}(s)+y_{2}(s) y_{2}^{\prime}(s)
$$

Therefore

$$
\left(y_{1}^{2}(s)+y_{2}^{2}(s)\right)^{\prime}=-2 \alpha(s)\left(y_{1}(s) y_{2}(s)\right)^{\prime}
$$

Integrating of this equality from $s_{1}$ to $s$ yields

$$
\begin{gather*}
y_{1}^{2}(s)+y_{2}^{2}(s)=y_{1}^{2}\left(s_{1}\right)+y_{2}^{2}\left(s_{1}\right)+2 \alpha\left(s_{1}\right) y_{1}\left(s_{1}\right) y_{2}\left(s_{1}\right)-  \tag{1.12}\\
-2 \alpha(s) y_{1}(s) y_{2}(s)+2 \int_{s_{1}}^{s} \alpha^{\prime}(\xi) y_{1}(\xi) y_{2}(\xi) \mathrm{d} \xi .
\end{gather*}
$$

Let $u(s)=y_{1}^{2}(s)+y_{2}^{2}(s)$. Then $2\left|y_{1}(s) y_{2}(s)\right| \leqq u(s)$ and from (1.12) we get

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$$
\begin{aligned}
u(s) \leqq(1+\delta) u\left(s_{1}\right)+\delta u(s)+\int_{s_{1}}^{s}\left|\alpha^{\prime}(\xi)\right| u(\xi) \mathrm{d} \xi \\
u(s) \leqq \frac{1+\delta}{1-\delta} u\left(s_{1}\right)+\frac{1}{1-\delta} \int_{1_{1}}^{s}\left|\alpha^{\prime}(\xi)\right| u(\xi) \mathrm{d} \xi
\end{aligned}
$$

for $s_{1} \leqq s<s_{0}$. Hence according to the Gronwall-Bellman lemma

$$
u(s) \leqq \frac{1+\delta}{1-\delta} u\left(s_{1}\right) \exp \left[\frac{1}{1-\delta} \int_{s_{1}}^{s}\left|\alpha^{\prime}(\xi)\right| \mathrm{d} \xi\right]
$$

for $s_{1} \leqq s<s_{0}$. The lemma is proved.

## 2. THE ASYMPTOTIC STABILITY OF THE SYSTEM (1.1)

Theorem 1. Let for large the inequality

$$
\begin{equation*}
\delta_{1}<\left|\frac{a_{21}(t)}{a_{12}(t)}\right|<\delta_{2} \tag{2.1}
\end{equation*}
$$

where $\delta_{1}$ and $\delta_{2}$ are positive constans, hold. Moreover, let the function $c$ be absolutely continuous on every finite segment,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \sup |c(t)|<1, \quad \int_{0}^{\infty}\left|c^{\prime}(t)\right| d t<\infty \tag{2.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \int_{0}^{t}\left(a_{11}(\tau)+a_{22}(\tau)\right) \mathrm{d} \tau=-\infty \tag{2.3}
\end{equation*}
$$

Then the system (1.1) is asymptotically stable.
Proof. According to (2.2) the conditions of Lemma 2 hold. Therefore, from (1.0) by means of (2.1) and (2.3) we conclude that (1.1) is asymptotically stable. This completes the proof.

Corollary 1. Let $a_{11}(t)=0, a_{12}(t)=-a_{21}(t)>0$ for $t \in R^{+}$,

$$
\lim _{t \rightarrow \infty} \sup \left(-\frac{a_{22}(t)}{2 a_{12}(t)}\right)<1, \quad \int_{0}^{\infty}\left|d\left(-\frac{a_{22}(t)}{2 a_{12}(t)}\right)\right|<\infty
$$

and

$$
\lim _{t \rightarrow \infty} \int_{0}^{t} a_{22}(\tau) \mathrm{d} \tau=-\infty
$$

Then the system (1.1) is asymptotically stable.

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