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ON SOLUTIONS OF THE LAGERSTROM EQUATION

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Abstract. The paper deals with the behaviour of certain classes of Cauchy's solutions of the Lagerstrom equation. It considers the behaviour of solutions in the neighbourhood of an arbitrary or integral curve on a finite or infinite interval. It presents the respective sufficient conditions which grant certain behaviour of some classes of Cauchy's solutions. The results obtained refer also the approximation and asymptotic behaviour of solutions.

Key words. The Lagerstrom equation, behaviour of solutions, classes of Cauchy's solutions.

MS Classification. 34 C 05.

In this paper we shall consider the Lagerstrom equation of the form

$$y'' + \left(\frac{2}{t} + y\right)y' = 0 \qquad (' = d/\mathrm{d}t) \tag{1}$$

on interval $I = (a, b)$, where $-\infty \leqq a < b < 0$ or $0 < a < b \leqq +\infty$.

The Lagerstrom equation, in different forms, has been considered by many authors. The papers [1 – 4], as well as many others, deal with the behaviour and approximation of the solution which satisfies the boundary conditions $y(\varepsilon) = 0$, $y(\infty) = 1$ (or $y(\infty) = C \geqq 0$), where $\varepsilon > 0$ is small.

Let $\Gamma = \{(y, t)\colon y = \psi(t),\ t \in I\}$, $\psi \in C^2(I)$, be an arbitrary or integral curve of (1) and let

$$\sigma = \{(y, t)\colon |\, y - \psi(t)\,| \leqq \varrho(t), t \in I\}$$

be its neighbourhood. This paper deals with the behaviour of solutions of equation (1) with respect to set σ. There is a particularly interesting case when function ϱ is sufficiently small and tends to 0 as $t \to \infty$.

We are going to use functions $\alpha, \beta, \delta, \varphi, \varrho \in C^1(I)$, $\alpha \neq 0$, $\delta, \varrho > 0$ on I, and functions

$$f \equiv -\frac{\alpha'}{\alpha} - \frac{2}{t} - \beta - y, \qquad g \equiv -\frac{1}{\alpha}\left[\beta' + \beta^2 + \beta\left(\frac{2}{t} + y\right)\right]$$

and notations $y_0 = y(t_0)$, $y_0' = y'(t_0)$, $\alpha_0 = \alpha(t_0)$, ..., $t_0 \in I$.

Depending on the form of the Cauchy's conditions the study will be carried out in three parts.

1. Let us first consider the solutions of equation (1) which satisfy the initial conditions

$$(2) \qquad | y_0 - \psi_0 | \leqq \varrho_0, \qquad | y_0' - \beta_0 y_0 - \alpha_0 \varphi_0 | \leqq | \alpha_0 | \delta_0, \qquad t_0 \in I.$$

Theorem 1. *Let Γ be an arbitrary curve.*

(i) *If there exist functions α, β, δ, φ and ϱ such that*

$$(3) \qquad | f\varphi + g\psi - \varphi' | + | g | \varrho < \delta' - f\delta,$$

$$(4) \qquad | \alpha\varphi + \beta\psi - \psi' | + | \alpha | \delta < \varrho' - \beta\varrho$$

on σ, then all solutions $y(t)$ of problem (1)–(2) *satisfy condition*

$$(5) \qquad | y(t) - \psi(t) | < \varrho(t) \qquad \textit{for every } t \in (t_0, b).$$

(ii) *If*

$$(6) \qquad | f\varphi + g\psi - \varphi' | + | g | \varrho < f\delta - \delta',$$

$$(7) \qquad | \alpha\varphi + \beta\psi - \psi' | + | \alpha | \delta < \beta\varrho - \varrho'$$

on σ, then at least one solution of the problem (1)–(2) *satisfies condition* (5).

(iii) *In view of conditions* (3) *and* (7) *or* (4) *and* (6) *problem* (1)–(2) *has one-parameter class of solutions which satisfy condition* (5).

Proof. Let us introduce the substitute

$$(8) \qquad y' = \alpha(t)\, x + \beta(t)\, y,$$

where $x = x(t)$ is a new unknown function. Equation (1) is transformed into a system of equations

$$(9) \qquad x' = f(y, t)\, x + g(y, t)\, y, \qquad y' = \alpha(t)\, x + \beta(t)\, y,$$

where functions f and g have already been defined.

Let $\Omega = R^2 \times I$ and

$$\omega_1 = \{(x, y, t) \in \Omega \colon | x - \varphi(t) | < \delta(t), | y - \psi(t) | < \varrho(t)\},$$
$$X_i = \{(x, y, t) \in Cl\omega_1 \colon x_i \equiv (-1)^i [x - \varphi(t)] - \delta(t) = 0\},$$
$$Y_i = \{(x, y, t) \in Cl\omega_1 \colon y_i \equiv (-1)^i [y - \psi(t)] - \varrho(t) = 0\}, \qquad i = 1, 2.$$

Let $\tau = (x'(t), y'(t), 1)$ be a tangential vector for the integral curve $(x(t), y(t), t)$ of system (9) on $\partial\omega_1$.

Let us consider the scalar products

$$\pi_{x_i} = (\text{grad}\, x_i, \tau), \qquad \pi_{y_i} = (\text{grad}\, y_i, \tau), \qquad i = 1, 2$$

on $\partial\omega_1$. We have

$$\pi_{x_i} = (-1)^i (fx + gy) - (-1)^i \varphi' - \delta' = (-1)^i (f\varphi + g\psi - \varphi') + \\ + (-1)^i g(y - \psi) + f\delta - \delta',$$

$$\pi_{y_i} = (-1)^i (\alpha x + \beta y) - (-1)^i \psi' - \varrho' = (-1)^i (\alpha\varphi + \beta\psi - \psi') + \\ + (-1)^i \alpha(x - \varphi) + \beta\varrho - \varrho'.$$

(i) In view of (3) and (4) we have

$$\pi_{x_i} \leqq |f\varphi + g\psi - \varphi'| + |g|\varrho + f\delta - \delta' < 0,$$

$$\pi_{y_i} \leqq |\alpha\varphi + \beta\psi - \psi'| + |\alpha|\delta + \beta\varrho - \varrho' < 0, \qquad i = 1, 2.$$

Accordingly, $\delta\omega_1$ is a set of points of strict entrance of integral curves of system (9) with respect to sets ω_1 and Ω. Hence, all solutions of system (9) which satisfy initial conditions

$$|x_0 - \varphi_0| \leqq \delta_0, \qquad |y_0 - \psi_0| \leqq \varrho_0$$

satisfy also conditions

$$|x(t) - \varphi(t)| < \delta(t), \qquad |y(t) - \psi(t)| < \varrho(t) \qquad \text{for every } t \in (t_0, b).$$

Since, in view of (8), $x_0 - \varphi_0 = \frac{1}{\alpha_0}(y_0' - \beta_0 y_0 - \alpha_0\varphi_0)$, then all solutions of problem (1)–(2) satisfy condition (5).

(ii) In view of (6) and (7) we have

$$\pi_{x_i} \geqq -|f\varphi + g\psi - \varphi'| - |g|\varrho + f\delta - \delta' > 0,$$

$$\pi_{y_i} \geqq -|\alpha\varphi + \beta\psi - \psi'| - |\alpha|\delta + \beta\varrho - \varrho' > 0, \qquad i = 1, 2.$$

Accordingly, $\delta\omega_1$ is a set of points of strict exit of integral curves of system (9) with respect to sets ω_1 and Ω. Hence, according to *T.* Waževski's retraction method ([5–7]), system (9) has at least one solution belonging to set ω_1 for every $t \in I$. Consequently, problem (1)–(2) has at least one solution which satisfies condition (5).

(iii) In this case $X = X_1 \cup X_2$ is a set of points of strict entrance, and $Y = Y_1 \cup Y_2$ is a set of points of strict exit (or reverse) of integral curves of system (9) with respect to sets ω_1 and Ω. Hence, according to the retraction method, system (9) has one-parameter class of solutions which belong to set ω_1 for every $t \in I$. Accordingly, problem (1)–(2) has one-parameter class of solutions which satisfy condition (5).

For especially selected functions α, β, δ and φ we can obtain very interesting results.

Corollary 1 $\left(\beta \equiv \dfrac{\psi'}{\psi}, \psi \neq 0, \delta \equiv 1, \varphi \equiv 0, \alpha > 0\right)$. (i) *If*

$$(10) \quad \frac{1}{\alpha}\left|\psi'' + \left(\frac{2}{t} + y\right)\psi'\right|\left(1 + \frac{\varrho}{|\psi|}\right) < \frac{\psi'}{\psi} + \psi - \varrho + \frac{\alpha'}{\alpha} + \frac{2}{t} \quad \text{on } \sigma,$$

$$(11) \quad \frac{\varrho'}{\varrho} > \frac{\psi'}{\psi} + \frac{\alpha}{\varrho} \quad \text{on } I,$$

then all solutions $y(t)$ of (1) *which satisfy the initial conditions*

$$(12) \quad |y_0 - \psi_0| \leqq \varrho_0, \quad \left|y_0' - \frac{\psi_0'}{\psi_0} y_0\right| \leqq \alpha_0, \quad t_0 \in I$$

also satisfy condition (5).

(ii) *If*

$$(13) \quad \frac{1}{\alpha}\left|\psi'' + \left(\frac{2}{t} + y\right)\psi'\right|\left(1 + \frac{\varrho}{|\psi|}\right) < -\frac{\psi'}{\psi} - \psi - \varrho - \frac{\alpha'}{\alpha} - \frac{2}{t} \quad \text{on } \sigma,$$

$$(14) \quad \frac{\varrho'}{\varrho} < \frac{\psi'}{\psi} - \frac{\alpha}{\varrho} \quad \text{on } I,$$

then at least one solution of problem (1)–(12) *satisfies condition* (5).

(iii) *In view of conditions* (10) *and* (14) *or* (11) *and* (13) *problem* (1)–(12) *has one-parameter class of solutions which satisfy condition* (5).

Corollary 2 ($\beta \equiv 0$, $\delta \equiv 1$, $\varphi \equiv 0$, $\alpha > 0$). (i) *If*

$$\psi > \varrho - \frac{\alpha'}{\alpha} - \frac{2}{t}, \quad |\psi'| < \varrho' - \alpha \quad \text{on } I,$$

then all solutions $y(t)$ of (1) *which satisfy the initial conditions*

$$(15) \quad |y_0 - \psi_0| \leqq \varrho_0, \quad |y_0'| \leqq \alpha_0, \quad t_0 \in I$$

satisfy also condition (5).

(ii) *If*

$$\psi < -\varrho - \frac{\alpha'}{\alpha} - \frac{2}{t}, \quad |\psi'| < -\varrho' - \alpha \quad \text{on } I,$$

then at least one solution of problem (1)–(15) *satisfies condition* (5).

(iii) *If*

$$\psi > \varrho - \frac{\alpha'}{\alpha} - \frac{2}{t}, \quad |\psi'| < -\varrho' - \alpha$$

or

$$\psi < -\varrho - \frac{\alpha'}{\alpha} - \frac{2}{t}, \quad |\psi'| < \varrho' - \alpha$$

on I, then problem (1)–(15) *has one-parameter class of solutions which satisfy condition* (5).

Example 1. (i) All solutions $y(t)$ of (1) which satisfy initial conditions

$$|y(\varepsilon) - (1-\varepsilon^2)| \leqq \varepsilon, \qquad |y'(\varepsilon)| \leqq \varepsilon^3 e^{-\frac{\varepsilon}{2}}, \qquad 0 < \varepsilon \leqq \frac{1}{4}$$

satisfy also condition

$$\left| y(t) - \left(1 - \frac{\varepsilon^3}{t}\right)\right| < \varepsilon\left(2 - \frac{\varepsilon}{t}\right) \qquad \text{for every } t \in (\varepsilon, \infty).$$

(ii) Equation (1) has one-parameter class of solutions which satisfy the initial conditions

$$|y(\varepsilon)| \leqq 1 + \varepsilon, \qquad |y'(\varepsilon)| \leqq \varepsilon^2 e^{-\varepsilon}, \qquad 0 < \varepsilon \leqq \frac{2}{3}$$

and condition

$$\left| y(t) - \left(1 - \frac{\varepsilon}{t}\right)\right| < \frac{\varepsilon(1+\varepsilon)}{t} \qquad \text{for every } t \in (\varepsilon, \infty).$$

Corollary 3 $\left(\beta \equiv 0,\ \delta \equiv \varrho,\ \varphi \equiv \frac{\psi'}{\alpha}\right)$. *Let $\psi(t)$ be an arbitrary solution of equation* (1). (*i*) *If*

$$\psi > \frac{|\psi'|}{\alpha} + \varrho - \frac{\varrho'}{\varrho} - \frac{\alpha'}{\alpha} - \frac{2}{t}, \qquad \frac{\varrho'}{\varrho} > \alpha > 0 \qquad \text{on } I,$$

then all solutions $y(t)$ of equation (1) *with initial conditions*

(16) $$|y_0 - \psi_0| \leqq \varrho_0, \qquad |y_0' - \psi_0'| \leqq \alpha_0 \varrho_0, \quad t_0 \in I$$

satisfy also condition (5).

(ii) *If*

$$\psi > \frac{|\psi'|}{\alpha} + \varrho - \frac{\varrho'}{\varrho} - \frac{\alpha'}{\alpha} - \frac{2}{t}, \qquad \frac{\varrho'}{\varrho} < -\alpha < 0$$

or

$$\psi < -\frac{|\psi'|}{\alpha} - \varrho - \frac{\varrho'}{\varrho} - \frac{\alpha'}{\alpha} - \frac{2}{t}, \qquad \frac{\varrho'}{\varrho} > \alpha > 0$$

on I, then problem (1)–(16) *has one-parameter class of solutions satisfying condition* (5).

Example 2. Let $A, T, r, \vartheta \in R$, $r > 0$, $A, T > \vartheta > 0$ and $\alpha \equiv \frac{\vartheta^3}{t^2}$. (i) Let

$$\varrho \equiv r e^{A(t-T)}, \qquad I_1 = [-\infty, -\vartheta], \qquad I_2 = [\vartheta, T].$$

The statement (i) of Corollary 3 is valid for every solution

$$\psi(t) > -A + r + \frac{t^2}{\vartheta^3}|\psi'| \qquad \text{on } I_1 \text{ and } I_2.$$

The statement (ii) of Corollary 3 is valid for every solution

$$\psi(t) < -A - r - \frac{t^2}{\vartheta^3}|\psi'| \qquad \text{on } I_1 \text{ and } I_2.$$

(ii) Let

$$\varrho \equiv re^{-A(t+T)}, \qquad I_1 = [-T, -\vartheta], \qquad I_2 = [\vartheta, \infty).$$

The statement (ii) of Corollary 3 is valid for every solution

$$\psi(t) > A + r + \frac{t^2}{\vartheta^3}|\psi'| \qquad \text{on } I_1 \text{ and } I_2.$$

Corollary 4 $\left(\alpha \equiv \frac{1}{|t|}, \beta \equiv -\frac{1}{t}, \delta \equiv \varrho, \varphi \equiv |t|\psi' + \frac{|t|}{t}\psi\right)$. *Let* $\psi(t)$ *be an arbitrary solution of* (1) *and let* $I \subseteq (0, \infty)$. *If*

$$\psi \geqq \varrho, \qquad \varrho' > |t\psi'|\varrho \qquad or \qquad \psi \geqq 0,$$

$$\varrho' > 2\varrho^2 + |t\psi'|\varrho \qquad or \qquad \frac{1}{2}|t\psi'| + \varrho - \frac{\varrho'}{2\varrho} < \psi \leqq 0$$

on I, *then all solutions* $y(t)$ *of* (1) *with initial conditions*

$$|y_0 - \psi_0| \leqq \varrho_0, \qquad |t_0(y_0' - \psi_0') + y_0 - \psi_0| \leqq \varrho_0, \qquad t_0 \in I$$

satisfy also condition (5). *This statement is valid also when* $I \subseteq (-\infty, 0)$ *if function* ϱ *also satisfies condition* $\frac{\varrho'}{\varrho} > -\frac{2}{t}$ *on* I.

Example 3. Let $\psi(t) \equiv C \geqq 0$, $C \in R$. The statement of Corollary 4 is valid on the interval $I = (0, T]$ ($T > 0$ is an arbitrarily large number) when

$$\varrho \equiv \frac{r}{T + 1 - t}, \qquad 0 < r < \frac{1}{2}.$$

2. Let us consider now the solutions of equation (1) satisfying the initial condition

$$(17) \qquad |y_0 - \psi_0| + \frac{1}{|\alpha_0|}|y_0' - \beta_0 y_0 - \alpha_0\varphi_0| \leqq \varrho_0, \qquad t_0 \in I.$$

Theorem 2. *Let* Γ *be an arbitrary curve.*

(i) *Let there exist functions* α, β, φ *and* ϱ *such that*

$$U \equiv (|f-\beta|+|g-\alpha|)\varrho+|f\varphi+g\psi-\varphi'|+|\alpha\varphi+\beta\psi-\psi'| < \varrho'-(|\alpha|+f)\varrho$$

or

$$U < \varrho' - (\beta + |g|)\varrho$$

on σ, *then all solutions of problem* (1)–(17) *satisfy condition* (5).

(ii) *If*

$$U < -\varrho' + (f - |\alpha|)\varrho \quad \text{or} \quad U < -\varrho' + (\beta - |g|)\varrho$$

on σ, *then problem* (1)–(17) *has at least one solution satisfying condition* (5).

Proof. In this case equation (1) is also transformed into system (9) by a substitute (8). Let $\Omega = R^2 \times I$,

$$\omega_2 = \{(x, y, t) \in \Omega: |x - \varphi(t)| + |y - \psi(t)| < \varrho(t)\},$$
$$P_i = \{(x, y, t) \in Cl\omega_2: p_i \equiv (-1)^i[x + y - \varphi(t) - \psi(t)] - \varrho(t) = 0\},$$
$$Q_i = \{(x, y, t) \in Cl\omega_2: q_i \equiv (-1)^i[x - y - \varphi(t) + \psi(t)] - \varrho(t) = 0\}, \quad i = 1, 2.$$

Let τ be a tangential vector on the integral curves of system (9) on $\partial\omega_2$. Let us consider the scalar products

$$\pi_{p_i} = (\operatorname{grad} p_i, \tau), \qquad \pi_{q_i} = (\operatorname{grad} q_i, \tau), \qquad i = 1, 2.$$

We have

$$\pi_{p_i} = (-1)^i(fx + gy + \alpha x + \beta y - \varphi' - \psi') - \varrho',$$
$$\pi_{q_i} = (-1)^i(fx + gy - \alpha x - \beta y - \varphi' + \psi') - \varrho'.$$

This can be written also as

$$\pi_{p_i} = (-1)^i[(-f + \beta + g - \alpha)(y - \psi) + (f\varphi + g\psi - \varphi') + (\alpha\varphi + \beta\psi - \psi')] + \\ + (f + \alpha)\varrho - \varrho'$$

or

$$\pi_{p_i} = (-1)^i[(f - \beta - g + \alpha)(x - \varphi) + (f\varphi + g\psi - \varphi') + (\alpha\varphi + \beta\psi - \psi')] + \\ + (g + \beta)\varrho - \varrho'$$

and

$$\pi_{q_i} = (-1)^i[(f - \beta + g - \alpha)(y - \psi) + (f\varphi + g\psi - \varphi') - (\alpha\varphi + \beta\psi - \psi')] + \\ + (f - \alpha)\varrho - \varrho'$$

or

$$\pi_{q_i} = (-1)^i[(f - \beta + g - \alpha)(x - \varphi) + (f\varphi + g\psi - \varphi') - (\alpha\varphi + \beta\psi - \psi')] - \\ - (g - \beta)\varrho - \varrho'.$$

Now it is sufficient to note that we have in case (i) $\pi_{p_i} < 0$, $\pi_{q_i} < 0$ on $\partial\omega_2$, and in case (ii) $\pi_{p_i} > 0$, $\pi_{q_i} > 0$ on $\partial\omega_2$ $(i = 1, 2)$.

If we take that for functions α and β we have $\alpha \equiv \dfrac{\vartheta}{t^2}$, $\vartheta \in R$, $0 < \vartheta < 1$, $\beta \equiv 0$ it can be shown that the following corollaries are valid.

Corollary 5 (for $\varphi \equiv 0$). *Let Γ be an arbitrary curve and let*

$$(18) \qquad |y_0 - \psi_0| + \frac{t_0^2}{\vartheta}|y_0'| \leqq \varrho_0, \qquad t_0 \in I.$$

(i) *If*

$$\psi \geqq \varrho, \qquad |\psi'| < \varrho' - \frac{2\vartheta}{t^2}\varrho$$

or

$$|\psi'| + \left(|\psi| + \varrho + \frac{\vartheta}{t^2}\right)\varrho < \varrho'$$

on I, then all solutions of problem (1)–(18) *satisfy condition* (5).

(ii) *If*

$$\psi \leqq -\varrho, \qquad |\psi'| < -\varrho' - \frac{2\vartheta}{t^2}\varrho$$

or

$$|\psi'| + \left(|\psi| + \varrho + \frac{\vartheta}{t^2}\right)\varrho < -\varrho'$$

on I, then at least one solution of problem (1)–(18) *satisfies condition* (5).

Corollary 6 $\left(\text{for } \varphi \equiv \frac{t^2}{\vartheta}\psi'\right)$. *Let $\psi(t)$ be an arbitrary solution of* (1) *and let $\psi(t) \geqq \varrho(t)$. For $\vartheta > 0$ $(\vartheta \in R)$ and function ϱ such that*

$$\frac{\varrho'}{\varrho} > \frac{2\vartheta}{t^2} + \frac{t^2}{\vartheta}|\psi'| \qquad \text{on } I$$

all solutions $y(t)$ of (1) *which satisfy the initial condition*

$$|y_0 - \psi_0| + \frac{t_0^2}{\vartheta}|y_0' - \psi_0'| \leqq \varrho_0, \qquad t_0 \in I$$

satisfy also condition (5).

Example 4. Let $\psi(t) \equiv C \geqq r > 0$, $C, r \in R$. Corollary 6 is valid for

$$\varrho \equiv r e^{-\frac{3\vartheta}{t}} \qquad \text{on } I = (0, \infty).$$

3. Let us consider also solutions of equation (1) with the initial condition

$$(19) \qquad (y_0 - \psi_0)^2 + \frac{1}{\alpha_0^2}(y_0' - \beta_0 y_0 - \alpha_0\varphi_0)^2 \leqq \varrho_0^2, \qquad t_0 \in I.$$

Theorem 3. *Let Γ be an arbitrary curve.*

(i) *If there exist functions α, β, φ and ϱ such that*

$$H \equiv \frac{1}{2}|g+\alpha|\varrho + |f\varphi + g\psi - \varphi'| + |\alpha\varphi + \beta\psi - \psi'| < \varrho' - \beta\varrho, \tag{20}$$
$$f - \beta \leqq 0$$

or

$$H < \varrho' - f\varrho, \qquad \beta - f \leqq 0 \tag{21}$$

on σ, then all solutions of problem (1)–(19) *satisfy condition* (5).

(ii) *If*

$$H < -\varrho' + \beta\varrho, \qquad f - \beta \geqq 0 \tag{22}$$

or

$$H < -\varrho' + f\varrho, \qquad \beta - f \geqq 0 \tag{23}$$

on σ, then problem (1)–(19) *has at least one solution satisfying condition* (5).

Proof. In this case equation (1) is also transformed into system (9). Let $\Omega = R^2 \times I$ and

$$\omega_3 = \left\{(x, y, t) \in \Omega \colon k \equiv \frac{(x-\varphi(t))^2}{\varrho^2(t)} + \frac{(y-\psi(t))^2}{\varrho^2(t)} < 1\right\}.$$

Let τ be a tangential vector for integral curves of system (9) on $\partial\omega_3$. Let us consider the scalar product $\pi = \left(\frac{1}{2}\operatorname{grad} k, \tau\right)$. We have

$$\pi = (fx + gy)\frac{x-\varphi}{\varrho^2} + (\alpha x + \beta y)\frac{y-\psi}{\varrho^2} -$$
$$-\frac{1}{\varrho^3}[(x-\varphi)\varrho\varphi' + (x-\varphi)^2\varrho' + (y-\psi)\varrho\psi' + (y-\psi)^2\varrho'] =$$
$$= \left(f - \frac{\varrho'}{\varrho}\right)X^2 + (g+\alpha)XY + \left(\beta - \frac{\varrho'}{\varrho}\right)Y^2 +$$
$$+ \frac{1}{\varrho}[(f\varphi + g\psi - \varphi')X + (\alpha\varphi + \beta\psi - \psi')Y],$$

where

$$X = \frac{x-\varphi}{\varrho}, \qquad Y = \frac{y-\psi}{\varrho}.$$

(i) We have

$$\pi \leqq \left(f - \frac{\varrho'}{\varrho}\right)X^2 + \frac{1}{2}|g+\alpha|(X^2+Y^2) + \left(\beta - \frac{\varrho'}{\varrho}\right)Y^2 +$$
$$+ \frac{1}{\varrho}[|f\varphi + g\psi - \varphi'| + |\alpha\varphi + \beta\psi - \psi'|] \equiv \bar{\pi}.$$

In view of (20) it is valid

$$\pi = \left(\beta - \frac{\varrho'}{\varrho} + \frac{1}{2}|g + \alpha|\right)(X^2 + Y^2) + (f - \beta) X^2 +$$
$$+ \frac{1}{\varrho}[|f\varphi + g\psi - \varphi'| + |\alpha\varphi + \beta\psi - \psi'|] =$$
$$= \beta - \frac{\varrho'}{\varrho} + \frac{1}{2}|g + \alpha| + (f - \beta) X^2 +$$
$$+ \frac{1}{\varrho}[|f\varphi + g\psi - \varphi'| + |\alpha\varphi + \beta\psi - \psi'|] < 0,$$

and in view of (21) we have

$$\pi = \left(f - \frac{\varrho'}{\varrho} + \frac{1}{2}|g + \alpha|\right)(X^2 + Y^2) + (\beta - f) Y^2 +$$
$$+ \frac{1}{\varrho}[|f\varphi + g\psi - \varphi'| + |\alpha\varphi + \beta\psi - \psi'|] =$$
$$= f - \frac{\varrho'}{\varrho} + \frac{1}{2}|g + \alpha| + (\beta - f) Y^2 +$$
$$+ \frac{1}{\varrho}[|f\varphi + g\psi - \varphi'| + |\alpha\varphi + \beta\psi - \psi'|] < 0.$$

(ii) We have

$$\pi \geqq \left(f - \frac{\varrho'}{\varrho}\right) X^2 - \frac{1}{2}|g + \alpha|(X^2 + Y^2) + \left(\beta - \frac{\varrho'}{\varrho}\right) Y^2 -$$
$$- \frac{1}{\varrho}[|f\varphi + g\psi - \varphi'| + |\alpha\varphi + \beta\psi - \psi'|] = \underline{\pi}.$$

In view of (22) it is valid

$$\underline{\pi} = \beta - \frac{\varrho'}{\varrho} - \frac{1}{2}|g + \alpha| + (f - \beta) X^2 -$$
$$- \frac{1}{\varrho}[|f\varphi + g\psi - \varphi'| + |\alpha\varphi + \beta\psi - \psi'|] > 0,$$

and in view of (23) we have

$$\underline{\pi} = f - \frac{\varrho'}{\varrho} - \frac{1}{2}|g + \alpha| + (\beta - f) y^2 -$$
$$- \frac{1}{\varrho}[|f\varphi + g\psi - \varphi'| + |\alpha\varphi + \beta\psi - \psi'|] > 0.$$

Accordingly, in case (i) $\partial\omega_3$ is a set of points of strict entrance, and in case (ii) $\partial\omega_3$ is a set of points of strict exit of integral curves of system (9) with respect to sets ω_3 and Ω. This conclusion grants the validity of Theorem 3.

For $\beta \equiv 0$ the following corollaries can be proved.

Corollary 7 (for $\varphi \equiv 0$). *Let Γ be an arbitrary curve, $\alpha(t) > 0$ on I and let*

$$(y_0 - \psi_0)^2 + \frac{1}{\alpha_0^2}(y_0')^2 \leqq \varrho_0^2, \qquad t_0 \in I. \tag{24}$$

(i) *If*

$$\psi \geqq \varrho - \frac{\alpha'}{\alpha} - \frac{2}{t}, \qquad |\psi'| < \varrho' - \frac{1}{2}\alpha\varrho$$

or

$$\psi \leqq -\varrho - \frac{\alpha'}{\alpha} - \frac{2}{t}, \quad |\psi'| < \varrho' + \left(\psi - \varrho + \frac{\alpha'}{\alpha} + \frac{2}{t} - \frac{\alpha}{2}\right)\varrho$$

on I, then all solutions of problem (1)–(24) *satisfy condition* (5).

(ii) *If*

$$\psi \leqq -\varrho - \frac{\alpha'}{\alpha} - \frac{2}{t}, \qquad |\psi'| < -\varrho' - \frac{1}{2}\alpha\varrho$$

or

$$\psi \geqq \varrho - \frac{\alpha'}{\alpha} - \frac{2}{t}, \qquad |\psi'| < -\varrho' - \left(\psi + \varrho + \frac{\alpha'}{\alpha} + \frac{2}{t} + \frac{\alpha}{2}\right)\varrho$$

on I, then problem (1)–(24) *has at least one solution satisfying condition* (5).

Example 5. Let

$$\psi \equiv \eta - r\vartheta e^{-t}, \qquad \eta \geqq r(1 + \vartheta), \qquad r, \vartheta > 0, \qquad \eta, r, \vartheta \in R.$$

The statement (i) of Corollary 7 is valid on $I = (\vartheta, \infty)$ for

$$\varrho \equiv r\left(1 - \frac{\vartheta}{t}\right), \qquad \alpha \equiv \frac{\vartheta}{4t^2}.$$

Corollary 8 $\left(\text{for } \varphi \equiv \dfrac{\psi'}{\alpha}\right)$. *Let $\psi(t)$ be an arbitrary solution of* (1) *and $\alpha(t) > 0$ on I. If*

$$\psi \geqq \varrho - \frac{\alpha'}{\alpha} - \frac{2}{t}, \qquad \frac{\varrho'}{\varrho} > \frac{\alpha}{2} + \frac{|\psi'|}{\alpha}$$

or

$$\frac{|\psi'|}{\alpha} + \varrho - \frac{\varrho'}{\varrho} - \frac{\alpha'}{\alpha} - \frac{2}{t} + \frac{\alpha}{2} < \psi \leqq -\varrho - \frac{\alpha'}{\alpha} - \frac{2}{t}$$

on I, then all solutions $y(t)$ of (1) *with the initial condition*

$$(y_0 - \psi_0)^2 + \frac{1}{\alpha_0^2}(y_0' - \psi_0')^2 \leqq \varrho_0^2, \qquad t_0 \in I$$

satisfy also condition (5).

Example 6. The statement of Corollary 8 is valid for every solution

$$\psi(t) \geqq r - \frac{2}{t},$$

$|\psi'(t)| \leqq A$ on $I_1 = (-\infty, 0)$ and $I_2 = (0, T)$, $r, A, T \in R, r, A, T > 0$, with functions

$$\varrho \equiv re^{(A+1)(t-T)}, \qquad \alpha \equiv 1.$$

Remark. The obtained results directly refer to the approximation and asymptotic behaviour of solutions when function ϱ is sufficiently small and tends to 0 as $t \to \infty$. The obtained results can also be used for the analysis of solutions of equation (1) with respect to its stability.

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