Felix M. Arscott

Multiplicative solution theory for three-path combinations

Archivum Mathematicum, Vol. 26 (1990), No. 2-3, 73--81

Persistent URL: http://dml.cz/dmlcz/107373

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 26, No. 2–3 (1990), 73–82

MULTIPLICATIVE SOLUTION THEORY FOR THREE-PATH COMBINATIONS

F. M. ARSCOTT

(Received June 7, 1989)

To Professor Otakar Borůvka, whose elegance and sublety in research on differential equations command my everlasting respect

Abstract. A multiplicative solution of an ordinary linear differential equation is one which, when continued analytically along a closed path in the complex plane, returns to its starting-point multiplied by a constant. This paper extends the work in [1] to give a number of results relating to combinations of three paths. This has particular relevance to doubly-periodic equations and to equations of Heun type. An application to Lamé's equation is discussed.

Keywords. Multiplicative solutions, Lamé's equation.

MS Classification. 34 A 20, 34 B 30.

1. INTRODUCTION

The basis of multiplicative solution theory is given in [1], [2], and the notation here used is given in [1]. To summarize: we have a linear, homogeneous second-order equation

$$L_t(y) = 0, \tag{1.1}$$

for which C_1 and C_2 are basic paths. Our concern is with the case when C_1, C_2 are both of type *I*, and we are in the "regular" case ([1], para. 7) so that the combination path C_{12} is also of type I and there is no solution multiplicative for both C_1 and C_2.

We have the two **standardized solution vectors** $\mathbf{y}_{1,2}$ and $\mathbf{y}_{2,1}$, each defined up to a multiplicative constant, such that $y_{i,j}$ is multiplicative for C_i and has specially simple behaviour on C_j, though not multiplicative there. Precisely

$$y_{i,j} \xrightarrow{C_i} S_i y_{i,j}, \qquad y_{i,j} \xrightarrow{C_j} L_{i,j} y_{i,j}, \tag{1.2}$$

where

(1.3) $$S_i = \begin{pmatrix} s_i^{(1)} & 0 \\ 0 & s_i^{(2)} \end{pmatrix}, \qquad L_{i,j} = p_j I + q_j M(\theta)$$

in which we use the notation

(1.4) $$2p_i = s_i^{(1)} + s_i^{(2)}, \quad 2q_i = s_i^{(1)} - s_i^{(2)}, \quad r_i = s_i^{(1)} s_i^{(2)}$$

and

(1.5) $$M(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

The quantity $\theta = \theta_{ij}$ is called the **link parameter** between C_i and C_j: it is known that $\theta_{ij} = \theta_{ji}$.

Between these two standardized solution vectors there holds the relation

(1.6) $$y_{2,1} = cM\left(\frac{1}{2}\theta\right) y_{1,2},$$

where c is a scalar constant, necessarily undetermined since $y_{1,2}$ and $y_{2,1}$ are each defined only up to a scalaı constant multiple.

2. RELATION BETWEEN STANDARDIZED AND MULTIPLICATIVE SOLUTIONS

No component of $y_{1,2}$ or of $y_{2,1}$ is multiplicative for the combined path C_{12}. Yet C_{12} is a basic path, so has always at least one multiplicative solution and, on the assumptions made heıe, a multiplicative solution vector which we denote by z_{12}. (This is, of course, determined only up to a constant diagonal matrix.) We now investigate the relation between this vector and the standardized vector $y_{1,2}$.

In principle, the connection is simple. A path matrix for C_{12} is the product $X_{1,2} := S_1 L_{1,2}$. Since, by hypothesis, C_{12} is of type I, the Jordan form of $X_{1,2}$ is diagonal, and we have matrices $W_{1,2}$, D_{12} such that

(2.1) $$X_{1,2} = S_1 L_{1,2} = W_{1,2} D_{12} W_{1,2}^{-1}.$$

Then a multiplicative solution vector $z_{1,2}$ for C_{12} is given by

(2.2) (2.3) $$z_{1,2} := W_{1,2}^{-1} y_{1,2} \Rightarrow z_{1,2} \xrightarrow{C_{12}} D_{12} z_{1,2}.$$

The matrix $X_{1,2}$ is found to be (writing $\theta_{12} = \theta$ for short)

(2.4) $$\begin{pmatrix} s_1^{(1)}(p_2 + q_2\cos\theta) & s_1^{(1)} q_2 \sin\theta \\ s_1^{(2)} q_2 \sin\theta & s_1^{(2)}(p_2 - q_2\cos\theta) \end{pmatrix}$$

with eigenvalues given by the roots of

$$s^2 - 2(p_1p_2 + q_1q_2 \cos\theta)\, s + r_1r_2 = 0. \tag{2.5}$$

Thus the path matrix D_{12} for C_{12} can be obtained.

A similar analysis can be carried out, starting with the standardized vector $y_{2,1}$ which on C_{12} is multiplied by $L_{2,1}S_2$. In this way we get a matrix $W_{2,1}$ such that

$$L_{2,1}S_2 = W_{2,1}D_{12}W_{2,1}^{-1} \tag{2.6}$$

and, taking

$$z_{2,1} := W_{2,1}^{-1}y_{2,1}, \tag{2.7}$$

$z_{2,1}$ is another solution vector for C_{12}: in general, it is not the same as $z_{1,2}$ but a multiple of it by a diagonal matrix.

There is a special case, however, in which one can easily carry the analysis further, namely when C_1, C_2 are elementary paths, and this case is of sufficiently frequent occurence to be worth setting out as a theorem.

Theorem 1. *Let C_1, C_2 both be elementary paths (i.e. with $S = \operatorname{diag}(1, -1)$) and in the regular case with C_{12} also of type I. Let the link parameter between C_1, C_2 be $\theta_{12} = \theta$. Let $y_{1,2}, y_{2,1}$ be the standardized solution vectors. Then:*

(i) *the path factors for C_{12} are* $\exp(\pm i\theta)$,

(ii) *a multiplicative solution vector for C_{12} is*

$$z_{1,2} := Gy_{1,2}, \qquad \text{where} \qquad G = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \tag{2.8}$$

(iii) *another multiplicative solution for C_{12} is*

$$z_{2,1} := \bar{G}y_{2,1}, \tag{2.9}$$

where $\bar{G}$ denotes the complex conjugate of G.

The proof involves only straightforward linear algebra and is omitted.

Corollary. *Since $z_{1,2}$ and $z_{2,1}$ are both multiplicative solution vectors for the same path, we expect them to be related by a diagonal matrix. Use of* (2.8), (2.9) *and* (1.6) *confirms this, giving (with a little working)*

$$z_{2,1} = c\begin{pmatrix} \exp\left(\frac{1}{2}i\theta\right) & 0 \\ 0 & \exp\left(-\frac{1}{2}i\theta\right) \end{pmatrix} z_{1,2}, \tag{2.10}$$

where c is the constant occuring in (1.6).

3. THREE PATHS: THE DILATION PARAMETER AND THE SPHERICAL TRIANGLE THEOREM

For the equation (1.1) $L_t(y) = 0$, satisfying the conditions assumed in [1], para. 11, let C_i, $i = 1, 2, 3$, be basic paths, such that any number of these can be described successively, in any order. As usual, we denote the combination of paths C_i, C_j, described in that order, by C_{ij}, with the obvious extensions of this notation.

We now make the following assumptions:

(A) each path C_i is of Type I, with path factors $s_i^{(r)}$, $r = 1, 2$, $s_i^{(1)} \neq s_i^{(2)}$. We use again the p, q, r notation as in (1.4).

(B) each path combination C_{ij} $(i, j = 1, 2, 3, i \neq j)$ is also of type I, hence there is no solution multiplicative for both C_i and C_j; in the terminology of [1], para. 7, we are in the "regular" case. The „link parameter" between C_i and C_j is denoted by θ_{ij} (recall that $\theta_{ij} = \theta_{ji}$, [1], theorem 3).

We thus have the standardised solution vectors $y_{i,j}$, determined up to a scalar multiple, such that

$$\text{(3.1a, b)} \qquad y_{i,j} \xrightarrow{C_i} S_i y_{i,j}, \qquad y_{i,j} \xrightarrow{C_j} L_{i,j} y_{i,j},$$

where S_i, $L_{i,j}$ and $M(\theta)$ are as in (1.3), (1.5).

It should be noted that we make no assumption regarding the nature of the three-path combination C_{ijk}.

It is convenient to introduce notation for a new matrix. We write

$$\text{(3.2)} \qquad N(\alpha) := \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}.$$

observing that

$$\text{(3.3)} \qquad N(\alpha)^{-1} = N(-\alpha).$$

Now consider two standardised solution vectors $y_{i,j}$ and $y_{i,k}$ $(i, j, k$ all different$)$. Since these vectors are each multiplicative for the path C_i, and this is of Type I, they must be related by a diagonal matrix, which we denote by $D_{i,jk}$, i.e. so that

$$\text{(3.4)} \qquad y_{i,j} = D_{i,jk} y_{i,k}.$$

It can be shown that $D_{i,jk}$ may be expressed in the form

$$\text{(3.5)} \qquad D_{i,j} = \lambda N\left(\frac{1}{2}\psi_{i,jk}\right),$$

where $\lambda \in \mathbf{C}$ and the parameter $\psi_{i,jk}$ is determined uniquely in the region

$$\text{(3.6)} \qquad \operatorname{Re} \psi_{i,jk} \in (-\pi, \pi].$$

Thus the parameter $\psi_{i,jk}$ expresses the connection between the standardized solution vectors $y_{i,j}$ and $y_{i,k}$ which have C_i as a common multiplicative path. We call $D_{i,jk}$ the **dilation matrix** and $\psi_{i,jk}$ the **dilation parameter** between C_{ij} and C_{ik}. Between the three link parameters θ_{ij}, θ_{jk}, θ_{ki} and the three dilation parameters $\psi_{i,jk}$, $\psi_{j,ki}$, $\psi_{k,ij}$ subsist relationships which form the basis of three-path theory.

We note, in passing, that from (3.4)

$$(D_{i,jk})^{-1} = D_{i,kj}.$$

Since $N(\psi_{i,jk})$ and $N(\psi_{i,kj})$ are appropriate multiples of $D_{i,jk}$ and $D_{i,kj}$, on taking account of (3.3)

$$\psi_{i,jk} = -\psi_{i,kj}. \tag{3.7}$$

The two following theorems are fundamental to the 3-path theory. They were given in [2], but it has been found that they hold only under stricter conditions than there stated. In [1] the theorems are correctly stated but without proof.

Theorem 2. (The spherical triangle theorem)
On the assumptions (A), (B) *above*

$$\cos\theta_{ij} = \cos\theta_{ik}\cos\theta_{jk} + \sin\theta_{ik}\sin\theta_{jk}\cos\psi_{i,jk}. \tag{3.8}$$

Proof. We consider the result of continuing the solution vector $y_{i,j}$ around the path C_{jk}. We have

$$y_{i,k} \xrightarrow{C_k} L_{i,k}y_{i,k}. \tag{3.9}$$

and from this and (3.4)

$$y_{i,j} \xrightarrow{C_k} D_{i,jk}L_{i,k}(D_{i,jk})^{-1}\, y_{i,j}.$$

From this and (3.1b)

$$y_{i,j} \xrightarrow{C_{jk}} L_{i,j}D_{i,jk}L_{i,k}(D_{i,jk})^{-1}\, y_{i,j}.$$

Recalling (3.5) and wiiting for short

$$N(\psi_{i,jk}) = N_{i,jk}, \tag{3.10}$$

the (non-zero) constant λ in (3.5) now falls out and we have

$$y_{i,j} \xrightarrow{C_{jk}} L_{i,j}N_{i,jk}L_{i,k}(N_{i,jk})^{-1}\, y_{i,j}. \tag{3.11}$$

Hence $L_{i,j}N_{i,jk}L_{i,k}(N_{i,jk})^{-1}$ is a path matrix for C_{jk}. But so also is $S_jL_{j,k}$, and these two matrices must be similar, hence

$$\operatorname{tr}\,[L_{i,j}N_{i,jk}L_{i,k}(N_{i,jk}^{-1})] = \operatorname{tr}\,[S_jL_{j,k}]. \tag{3.12}$$

Some tedious but quite straightforward working now gives the required result.

This result, in a different notation, was given in [2], as Theorem 6. It is fundamental to further work on three-path combinations; because of the similarity of (3.8) to the standard analogue of the "cosine formula" of spherical trigonometry, it is convenient to refer to this result as the „spherical triangle theorem".

We can now show how the path factors may be determined for the three-path combination C_{ijk}.

Theorem 3. *The path factors for the path C_{ijk} are the roots of the equation*

$$s^2 - 2As - r_i r_j r_k = 0, \tag{3.13}$$

where

$$A = p_i p_j p_k - i q_i q_j q_k N + \sum_{i,j,k} (p_i q_j q_k \cos \theta_{jk}), \tag{3.14}$$

the symbol $\sum_{i,j,k}$ denoting the sum of the three terms with i, j, k cyclically permuted, and

$$N = \sin \theta_{ij} \sin \theta_{jk} \sin \psi_{j,ki} = \sin \theta_{jk} \sin \theta_{ki} \sin \psi_{k,ij} = \sin \theta_{ki} \sin \theta_{ij} \sin \psi_{i,jk}. \tag{3.15}$$

Proof. Consider the solution vector $y_{1,2}$ taken first around C_i, then around C_{jk}. Using (3.1), we see that a path matrix for C_{ijk} is

$$S_i L_{i,j} N_{i,jk} L_{i,k} (N_{i,jk})^{-1}. \tag{3.16}$$

The determinant of this matrix is easily found to be $r_i r_j r_k$; to evaluate its trace is longer but not difficult and readily yields the value 2A, where A is given in (3.14): in the evaluation, (3.8) must be used. Hence (3.13) is obtained with N given by the third expression in (3.15). The other two expressions follows from identical reasoning with i, j, k cyclically permuted.

4. APPLICATION TO THE LAMÉ EQUATION

Consider the Lamé equation in its Jacobian form ([3], Chap. IX, [4], Chap. XV)

$$w''(z) + (h - \nu(\nu + 1) k^2 sn^2 z) w(z) = 0, \tag{4.1}$$

where $k \in (0,1)$ is the modulus of the Jacobian elliptic function $snz = sn(z, k)$, ν is the order of the equation and h is a parameter.

Since the coefficients of (4.1) are doubly-periodic with real period $2K$, imaginary period $2iK'$, in the usual notation, we are naturally interested in the properties of solutions of (4.1) with respect to these periods.

By means of the transformations

$$t = sn^2 z, \qquad w(z) = y(t), \qquad a = k^{-2} (\in (1, \infty)) \tag{4.2}$$

the equation (4.1) takes the form

$$t(t-1)(t-a)\,y''(t) + \frac{1}{2}(3t^2 - 2(1+a)\,t + a)\,y'(t) +$$

$$+ \frac{1}{4}(ah - \nu(\nu+1)\,t)\,y(t) = 0. \tag{4.3}$$

This equation has three elementary finite singularities, namely at $t = 0$, 1 and a, (the exponents at each being 0 and $\frac{1}{2}$) and a regular singularity at ∞ with exponents $-\frac{1}{2}\nu$, $\frac{1}{2}(\nu+1)$. It is found that properties of parity about the points $z = 0, K, K + iK'$, i.e. the substitutions

$$z \to -z, \qquad K + z \to K - z, \qquad K + iK' + z \to K + iK' - z \tag{4.4}$$

correspond to properties on the elementary circuits C_0, C_1, C_a about $t = 0, 1$, a respectively.

Properties of periodicity with respect to $2K$, $2iK'$, i.e. the substitutions

$$z \to z + 2K, \qquad z \to z + 2iK' \tag{4.5a, b}$$

correspond respectively to the combination paths C_{01}, C_{1a}.

We now assume that C_{01}, C_{1a}, C_{0a} are all of type I. It follows that, on C_{01}, the path factors are complex conjugates $\exp(\pm i\omega)$, and on C_{1a} they are also complex conjugates $\exp(\pm i\omega')$, where

$$\omega := \theta_{01}, \qquad \omega' := \theta_{1a}. \tag{4.6}$$

These two parameters are fundamental to the development of a global theory of Lamé's equation.

Let us pause to consider the relation of these assumptions to the equation in the original z-form (4.1). The existence of solutions $z_{1,0}^{(j)}(t)$ of (4.3) which are multiplicative for C_{10} with path factors $\exp(\pm i\omega)$ implies the existence of solutions $w^{(j)}(z)$ of (4.1) such that

$$w^{(1),(2)}(z + 2K) = \exp(\pm i\omega)\, w^{(1),(2)}(z), \tag{4.7}$$

which we call **$2K$-multiplicative.** The assumption that C_{01} is of type I, with its consequence that $\omega \not\equiv 0 \bmod. \pi$, means that there is no solution of (4.1) which has $2K$ as period or antiperiod: instead, there is the pair of $2K$-multiplicative solutions given in (4.7).

The corresponding assumption that C_{1a} is of type I, with the consequence that $\omega' \not\equiv 0 \bmod \pi$, leads to similar statements with respect to the $2iK'$ – periodicity of (4.1); there exist solutions $\hat{w}^{(j)}(z)$ such that

$$\hat{w}^{(1),(2)}(z + 2iK') = \exp(\pm i\omega')\, \hat{w}^{(1),(2)}(z). \tag{4.8}$$

Considerable interest attaches to the possibility of coincidence of one (perhaps two) of $w^{(j)}(z)$ with one (possibly two) of $\hat{w}^{(j)}(z)$, giving a doubly-multiplicative solution of (4.1), i.e. multiplicative for both $2K$ and $2iK'$. A rather intricate analysis ([5] para. 15.6) by complex-variable methods shows that such coincidence can happen only for integral values of ν, but this fact emerges simply from the discussion which follows.

We return to consideration of (4.3), and make use of theorem 3, identifying C_i, C_j, C_k with C_0, C_1, C_a respectively. It is clear that the combined path C_{01a} is equivalent to a negative circuit about ∞. Now, provided only that ν is not half an odd integer, $\bar{C}_\infty$ is a type I path, with path factors $\exp(\pi\nu)$, $-\exp(-\pi\nu)$, and, since the components of C_{01a} are all elementary paths for which $p = 0$, $q = 1$, $r = -1$, it follows from theorem 3 that

$$N = \sin \pi\nu. \tag{4.9}$$

The common element to the paths C_{10}, C_{1a} is, of course, C_1, so we naturally consider the standardized solutions $y_{1,0}$ and $y_{1,a}$. The connection between these standardized solutions depends on the dilation parameter $\psi_{1,0a}$, which we write as ψ for short. Then (3.15) gives

$$\sin \omega \sin \omega' \sin \psi = \sin \pi\nu. \tag{4.10}$$

More precisely, the connection is (by (3.4))

$$y_{1,0} = N\left(\frac{1}{2}\psi\right)y_{1,a},$$

the arbitrary constant λ being irrelevant, and since C_0, C_1, C_a are all elementary, by Theorem 1 the multiplicative solutions for C_{10}, C_{1a} are linked by

$$z_{1,0} = GN\left(\frac{1}{2}\psi\right)G^{-1}z_{1,a},$$

which reduces to

$$z_{1,0} = \begin{pmatrix} \cos\left(\frac{1}{2}\psi\right) & i\sin\left(\frac{1}{2}\psi\right) \\ i\sin\left(\frac{1}{2}\psi\right) & \cos\left(\frac{1}{2}\psi\right) \end{pmatrix} z_{1,a}. \tag{4.11}$$

The formula (4.10) is a key result. Of the four parameters involved, ν is the order of the equation, ω and ω' are fundamental to the equation since (by (4.7), (4.8)) they govern the behaviour of solutions of the z-equation with respect to the periods $2K$, $2iK'$. The remaining parameter ψ expresses, in essence, the connection between the two vectors of solutions with this special behaviour.

Development of this analysis must, unfortunately be deferred to a subsequent paper, but one result can be noted. Let ν be nonintegral; then (4.10) shows that

$\sin\psi \neq 0$, and (from (4.11)) it is impossible for $y_{1,0}$ and $y_{1,a}$ to coincide. Hence a doubly - multiplicative solution cannot exist if ν is non-integral.

It should be stressed that this section has been concerned only with the most regular case of Lamé's equation, when there are no solutions even singly-periodic. But there is reason to expect that suitable modifications to the analysis—allowing C_{01}, C_{1a} or C_∞ to be type II paths, for instance—will enable the more intricate special cases to be handled on similaı lines.

REFERENCES

[1] F. M. Arscott, *Studies in multiplicative solutions of linear differential equations,* Proc. Roy. Soc., Edinburgh, 106A, (1987), 277–305.

[2] F. M. Arscott and B. D. Sleeman, *Multiplicative solutions of linear differential equations,* J. Lond. Math. Soc. (2), 43 (1968), 263–270.

[3] F. M. Arscott, *Periodic Differential Equations,* Pergamon Press, 1964.

[4] Erdélyi et al. *Higher Transcendental Functions,* (Bateman Manuscript Project) Vol. III, McGraw-Hill, 1955.

[5] E. L. Ince, *Ordinary Differential Equations,* Longmans–Dover, 1927.

Felix M. Arscott
Department of Applied Mathematics
University of Manitoba
Winnipeg, Manitoba, Canada R3T 2N2