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CONVERGENCE OF SEQUENCES OF INVERSE FUNCTIONS

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Abstract. The paper deals with a partial solution of the problem: given a convergence $f_n \to f_0$ of mappings, state conditions under which $f_n^{-1} \to f_0^{-1}$.

Notation. Let $\mathbf{N}$ be the set of natural numbers. Let $f : (M, \varrho) \to (N, \sigma)$, $f_n : (M, \varrho) \to (N, \sigma)$ denote mappings of a metric space (M, ϱ) into a metric space (N, σ) for $n \in \mathbf{N}$;$f_n \text{loc} \rightrightarrows f$ means that f_n converges locally uniformly to f on M as $n \to \infty$, i.e. for each x of M, there is a sphere $\Omega(x, r)$ with centre x and radius $r > 0$ such that f_n converges uniformly to f on $\Omega(x, r)$ i.e. $f_n \rightrightarrows f$ on $\Omega(x, r)$. By f^{-1} we denote the inverse mapping of f, provided f is an injection. Finally, if $M_1 \subseteq M$, int M_1 and $\overline{M}_1$ denote the interior of M_1 and the closure of M_1, respectively.

Main results

Theorem 1. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real injection functions defined on $\langle a, b\rangle \subseteq \cap_{n=1}^{\infty} Dom\ f_n$. If the sequence converges uniformly to a function f_0 on this interval, and if f_0 is a continuous injection on $\langle a, b\rangle$ and $\langle \alpha, \beta\rangle \subseteq \cap_{k=0}^{\infty} f_k\ (\langle a, b\rangle)$, then $f_n^{-1} \rightrightarrows f_0^{-1}$ on $\langle \alpha, \beta\rangle$.*

Proof. Suppose that f_0 is increasing on $\langle a, b\rangle$ and $\alpha = f_0(a), \beta < f_0(b)$. The proof in the cases when $\alpha > f_0(a)$, $\beta = f_0(b)$ or $\alpha > f_0(a)$, $\beta < f_0(b)$ or $\alpha = f_0(a)$, $\beta = f_0(b)$ is similar. Let $0 < \varepsilon < \varepsilon_0 = \min(f_0(b) - \beta,\ \beta - f_0(a))$,

$$g_\varepsilon(x) = \begin{cases} f_0^{-1}(f_0(x) - \varepsilon) & \text{for } x \in (f_0^{-1}(\alpha + \varepsilon), f_0^{-1}(\beta)\rangle, \\ a & \text{for } x \in \langle f_0^{-1}(\alpha), f_0^{-1}(\alpha + \varepsilon)\rangle. \end{cases}$$

It is easy to see that the function g_ε is continuous on $J = \langle f_0^{-1}(\alpha), f_0^{-1}(\beta)\rangle$. We denote $\max(f_0^{-1}(f_0(x) + \varepsilon) - x, x - g_\varepsilon(x))$ by $A(x, \varepsilon)$ for $x \in J, \varepsilon \in (0, \varepsilon_0)$ and define $A(\varepsilon)$ by $A(\varepsilon) = \max_{x \in J} A(x, \varepsilon)$. The function $A(\varepsilon)$ is non-negative and

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nondecreasing on $(0, \varepsilon_0)$. Thus $\lim_{\varepsilon \to 0_+} A(\varepsilon) = \inf \{A(\varepsilon) | \varepsilon \in (0, \varepsilon_0)\} \geqq 0$. As f_0^{-1} is uniformly continuous on $\langle f_0(a), f_0(b) \rangle$ and $x - g_\varepsilon(x) = x - a \leqq f_0^{-1}(\alpha + \varepsilon) - a = f_0^{-1}(\alpha + \varepsilon) - f_0^{-1}(\alpha)$ holds for $x \in \langle f_0^{-1}(\alpha), f_0^{-1}(\alpha + \varepsilon)\rangle$, we can easily show that $\inf \{A(\varepsilon) | \varepsilon \in (0, \varepsilon_0)\} = 0$.

Let ε^* be an arbitrary but fixed positive number. Then there exist $\varepsilon > 0$ such that $A(\varepsilon) < \varepsilon^*, \varepsilon < \varepsilon_0$ and a positive integer $n_0(\varepsilon)$ so that $f_0(x) - \varepsilon < f_n(x) < f_0(x) + \varepsilon$ whenever $n \geqq n_0$ for all x belonging to $\langle a, b \rangle$. Firstly we shall verify that $f_0^{-1}(y) - \varepsilon^* < f_n^{-1}(y)$ for every $n \geqq n_0$ and $y \in \langle \alpha, \beta \rangle$. Clearly, $f_0^{-1}(y) - \varepsilon^* < f_0^{-1}(y) - A(\bar{x}, \varepsilon)$ for $y \in \langle \alpha, \beta \rangle$, where $\bar{x}$ is a point satisfying $\bar{x} \in J, f_0(\bar{x}) = y$.

If $\bar{x} \in (f_0^{-1}(\alpha + \varepsilon), f_0^{-1}(\beta)\rangle$, we have $f_0^{-1}(y) - A(\bar{x}, \varepsilon) \leqq f_0^{-1}(f_0(\bar{x}) - \varepsilon) = x_1$. Hence $y = f_0(x_1) + \varepsilon > f_0(x) + \varepsilon > f_n(x)$ for $x \in \langle a, x_1)$, which leads to the result $f_n^{-1}(y) \notin \langle a, x_1)$, i.e. $x_1 \leqq f_n^{-1}(y)$.

If $\bar{x} \in \langle f_0^{-1}(\alpha), f_0^{-1}(\alpha + \varepsilon)\rangle$ we get $f_0^{-1}(y) - A(\bar{x}, \varepsilon) \leqq \bar{x} - (\bar{x} - a) = a \leqq f_n^{-1}(y)$. Analogically we can deduce the inequalities $f_0^{-1}(y) + \varepsilon^* > f_0^{-1}(f_0(\bar{x}) + \varepsilon) = x_2 \geqq f_n^{-1}(y)$. This implies that $f_n^{-1} \rightrightarrows f_0^{-1}$ on $\langle \alpha, \beta \rangle$.

By the same way we should prove the assertion of the theorem for any decreasing function f_0 on $\langle a, b \rangle$. □

Example. Consider the nonincreasing sequence $\{n(\sqrt[n]{x} - 1)\}_{n=1}^\infty$ of the increasing continuous functions on $\langle 1, \infty)$. Evidently $\lim_{n \to \infty} n(\sqrt[n]{x} - 1) = \ln x$ on every closed interval $\langle 1, b \rangle$, where $b > 1$. Applying Theorem 1 to this sequence we obtain that $(1 + \frac{x}{n})^n \rightrightarrows e^x$ on $\langle 0, \ln b \rangle \cap \bigcap_{n=1}^\infty \langle 0, n(\sqrt[n]{b} - 1)\rangle = \langle 0, \ln b \rangle$.

Further we shall formulate the result of Theorem 1 in metric spaces.

Theorem 2. *If $\{f_n\}_{n=0}^\infty$ is a sequence of injection mappings on a metric space (M, ϱ) and taking values in a locally compact metric space (N, σ), $f_n \rightrightarrows f_0$ on M, and if f_0^{-1} is a continuous mapping on $N_1 \subseteq N$, then $f_n^{-1} \rightrightarrows f_0^{-1}$ on every compact set K_0 contained in $N_0 = \operatorname{int} N_1 \cap \bigcap_{n=1}^\infty \operatorname{Im} f_n$.*

Proof. Let K_0 be a compact subset of N_0 and K_1 a compact set such that $K_0 \subseteq \operatorname{int} K_1 \subseteq K_1 \subseteq \operatorname{int} N_1$, then

$$\sigma(K_0, \overline{N - K_1}) = \Delta > 0 \,. \tag{1}$$

(Since N is locally compact, $K_0 \subseteq \operatorname{int} N_1, \operatorname{int} N_1$ is open, the existence of K_1 is warranted.)

We shall show that for every positive value of ε, there exists a natural number n_0 such that $\varrho(f_n^{-1}(y), f_0^{-1}(y)) < \varepsilon$, whenever $n \geqq n_0$, for all points y of K_0.

The mapping f_0^{-1} is continuous on the compact set $K_1 \subseteq N_1$ and therefore it is uniformly continuous. Thus, for any fixed $\varepsilon > 0$ we can find a number $\delta > 0$ such that

$$\sigma(y_1, y_2)\delta \Longrightarrow \varrho(f_0^{-1}(y_1), f_0^{-1}(y_2)) < \varepsilon \tag{2}$$

whenever $y_1, y_2 \in K_1$. The sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f_0 on M, therefore, for $\delta_0 = \min(\delta, \Delta)$, there is a positive integer n_0 such that

$$(3) \qquad \sigma(f_n(x), f_0(x)) < \delta_0 \ \forall n \geqq n_0, \ \forall x \in M.$$

Let $y \in K_0$, $n \geqq n_0$ and $x_n = f_n^{-1}(y)$, $y_n = f_0(x_n)$. From (3), with $x = x_n$, we get $\sigma(f_n(x_n), f_0(x_n)) < \delta_0$. Since $f_n(x_n) = y$, $f_0(x_n) = y_n$ and at the same time from (1) it follows $y_n \in K_1$, the assumptions of (2) are satisfied. The inequality $\varrho(f_0^{-1}(y_n), f_0^{-1}(y)) < \varepsilon$ completes the proof because $f_0^{-1}(y_n) = x_n = f_n^{-1}(y)$ is true. □

Note. Theorem 1 seems to be a special case of Theorem 2. Nevertheless the authors decided to present this result because it was proved by using a different technique and besides it can give a "richer" domain of a convergence $(f_0(\langle a, b\rangle) \cap_{k=1} \infty f_k(\langle a, b\rangle))$ than the assertion of Theorem 2 (int $N_1 \cap \bigcap_{n=1}^{\infty} Im\, f_n$ and int $N_1 \subseteq \subseteq f_0(M) \subseteq N$).

Corollary 1. *If $f_n : M \to N$, $n = 1, 2, \dots$ are bijection mappings of a compact metric space (M, ϱ) onto a metric space (N, ϱ) and f_0 is a continuous mapping, $f_n \rightrightarrows f_0$ on M, then ${f_n^{-1}}_{n=1}^{\infty}$ converges uniformly to f_0^{-1} on N.*

Proof. The fact that the continuous bijection f_0 is defined on the compact metric space (M, ϱ) means that f_0 is a homeomorphism and (N, ϱ) is a compact. The assumption of Theorem 2 are fulfilled. Put $Imf_n = N_1 = K_0 = N_0 = N$. □

We can also obtain Corollary 1 from [2].

Theorem 3. *If $f_n : (M, \varrho) \to (N, \sigma)$, $n = 1, 2, \dots$ are injections, and if there exists a constant $\gamma > 0$ such that the condition $\gamma\varrho(x_1, x_2) \leqq \sigma(f_0(x_1), f_0(x_2))$ holds for every pair of points x_1 and x_2 of M, then from $f_n \rightrightarrows f_0$ it follows $f_n^{-1} \rightrightarrows f_n^{-1}$ on $Y = \bigcap_{n=0}^{\infty}$ Im f_n*

Proof. Let $y \in Y$, $f_0^{-1}(y) = x$, $f_n^{-1}(y) = x_n$. Then we have $\varrho(f_0^{-1}(y), f_n^{-1}(y)) = \varrho(x, x_n) \leqq \frac{1}{\gamma}\sigma(f_0(x), f_0(x_n)) = \frac{1}{\gamma}\sigma(f_n(x_n), f_0(x_n))$. Taking into account the assumption $f_n \rightrightarrows f_0$ as $n \to \infty$, we obtain $f_n^{-1} \rightrightarrows f_0^{-1}$ on Y. Indeed, for every $\varepsilon > 0$, we can find $n_0 \in \mathbf{N}$ such that $\sigma(f_n(x), f_0(x)) < \varepsilon\gamma$, where $n \geqq n_0$, $x \in M$. Hence $\varrho(f_n^{-1}(y), f_0^{-1}(y)) < \varepsilon$ holds for each point y of Y. □

Corollary 2. *Assume that $f_n : (M, \varrho) \to (M, \sigma)$, $n = 0, 1, 2, \dots$ are bijections and there exist positive constants $\gamma \leqq \Gamma$ satisfying the condition $\gamma\varrho(x_1, x_2) \leqq \leqq \sigma(f_0(x_1), f_0(x_2)) \leqq \Gamma\varrho(x_1, x_2)$ for $x_1 \in M$ and $x_2 \in M$. Then $f_n \rightrightarrows f_0$ on M iff $f_n^{-1} \rightrightarrows f_0^{-1}$ on N.*

In the following theorem let's look on our problem from another point of view.

Theorem 4. *If f_n, $n = 1, 2, \ldots$ are one-to-one mappings of (M, ϱ) onto (N, σ), $\{f_n\}_{n=1}^{\infty}$ and $\{f_n^{-1}\}_{n=1}^{\infty}$ converge locally uniformly to f and g, respectively, where $f : M \to N, g : N \to M$ are continuous, then the mappings f, g are both bijections and $f = g^{-1}$.*

Proof. Let $\varepsilon > 0$ be an arbitrary but fixed number and x a point of M. From $f_n^{-1} \overset{\text{loc}}{\rightrightarrows} g$ it follows that for $f(x) \in M$ and $\varepsilon/2$, there are $r > 0$ and a positive integer n_1 so that

$$\sigma(y, f(x)) < r \Rightarrow \varrho(g(y) f_n^{-1}(y)) < \varepsilon/2 \tag{4}$$

whenever $n \geqq n_1$, $y \in N$.

The mapping g is continuous at the point $f(x)$ of N. Therefore for $\varepsilon/2$, there exists $0 < \delta \leqq r$ such that

$$\sigma(y, f(x)) < \delta \Rightarrow \varrho(g(y), g(f(x))) < \varepsilon/2\ , \tag{5}$$

where $y \in N$.

Further, because $f_n(x) \to f(x)$, for $\delta > 0$ we can find a natural number $n_2(\delta)$ such that $\sigma(f_n(x), f(x)) < \delta$ whenever $n \geqq n_2$. Thus, we have $\sigma(f_n(x), f(x)) < < \delta$ for every $n \geqq \max(n_1, n_2)$ and by using (4), (5) we obtain $\varrho(g(f(x)), x) \leqq \leqq \varrho(g(f(x)), g(f_n(x))) + \varrho(g(f_n(x)), f_n^{-1}(f_n(x))) < \varepsilon$. Make $\varepsilon \to 0_+$, then $\varrho(g(f(x)), x) = 0$, i.e. $g(f(x)) = x$ for all x belonging to M.

In view of the symmetry properties of the assumptions, we can also prove $f(g(y)) = y$ for every $y \in N$. Evidently, f and g are both bijections and $f = g^{-1}$. The proof of the theorem is finished. □

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