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Boris Lavrič

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CONTINUITY OF MONOTONE FUNCTIONS

Boris Lavrič

Abstract. It is shown that a monotone function acting between euclidean spaces $\mathbf{R}^n$ and $\mathbf{R}^m$ is continuous almost everywhere with respect to the Lebesgue measure on $\mathbf{R}^n$.

As well known the set of all points of discontinuity of a real monotone function is at most countable. The paper deals with the set of all discontinuity points of a monotone function acting between euclidean spaces. We shall be concerned in order theoretic monotonicity, so let us agree that $\leq$ denotes the componentwise ordering of $\mathbf{R}^k$ ($x \leq y$ means $x_i \leq y_i$ for $i = 1, 2, \cdots, k$). If $a, b \in \mathbf{R}^k$, $a \leq b$, the set $[a, b] = \{x \in \mathbf{R}^k : a \leq x \leq b\}$ will be called a *k-cell.*

Let A be a nonempty subset of $\mathbf{R}^n$. Then A is said to be *solid*, if $a, b \in A$ and $a \leq x \leq b$ implies $x \in A$. The smallest solid set containing A is called the *solid cover of A* and equals

$$S(A) = \{x \in \mathbf{R}^n : a \leq x \leq b \quad \text{for some} \quad a, b \in A\}.$$

A function $f : A \longrightarrow \mathbf{R}$ is said to be nondecreasing (respectively nonincreasing) if

$$x, y \in A, \; x \leq y \implies f(x) \leq f(y) \quad (\text{ respectively } \; f(y) \leq f(x)).$$

A function $g = (g_1, \cdots, g_m) : A \longrightarrow \mathbf{R}^m$ is called *monotone* if each of its components $g_i : A \longrightarrow \mathbf{R}$ is either nondecreasing or nonincreasing.

The set D of all points of discontinuity of a monotone function $f : A \longrightarrow \mathbf{R}^m$ is not necessarily countable if $n > 1$. By way of example take the characteristic function $h_C : \mathbf{R}^n \longrightarrow \mathbf{R}$ of the cone $C = \{x \in \mathbf{R}^n : x \geq 0\}$. However, D remains small also for $n > 1$.

We need first a property of solid subsets of $\mathbf{R}^n$.

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Proposition. *The boundary of a solid subset of* $\mathbf{R}^n$ *is of Lebesgue measure zero.*

Proof. Let A be a solid subset of $\mathbf{R}^n$. Denote by K the interior of the cone $C = \{x \in \mathbf{R}^n : x \geq 0\}$, set

$$A^- = \{x \in \text{ cl}(A) : A \cap (x - K) = \emptyset\},$$
$$A^+ = \{x \in \text{ cl}(A) : A \cap (x + K) = \emptyset\},$$

and note that A^-, A^+ are closed subsets of the boundary $\text{bd}(A)$ of A. We claim that $\text{bd}(A) = A^- \cup A^+$. By way of contradiction suppose $x \in \text{ bd}(A) \setminus (A^- \cup A^+)$. Then there exists elements $y, z \in A$ such that $y \in x - K$, $z \in x + K$, hence x is an interior point of the n-cell $[y, z]$ which is contained in A. This contradicts $x \in \text{bd}(A)$, therefore $\text{bd}(A) \subset A^- \cup A^+$ and the claim follows.

Thus, we have to prove that A^- and A^+ are of Lebesgue measure zero. To this end suppose that A^- is nonempty, note that

$$y \notin (x - K) \cup (x + K) \quad \text{for all} \quad x, y \in A^-, \tag{1}$$

and denote by P the orthogonal projection of $\mathbf{R}^n$ onto the subspace $E = \{x \in \mathbf{R}^n : x_1 + x_2 + \cdots + x_n = 0\}$. Since by (1) P is injective on A^-, there exists a function $h : P(A^-) \longrightarrow \mathbf{R}$ such that

$$A^- = \{u + h(u)e : \quad u \in P(A^-),\ e = (1, 1, \cdots, 1)\}.$$

An easy computation shows that (1) implies

$$|h(u) - h(v)| \leq \|u - v\|_\infty, \quad u, v \in P(A^-),$$

hence h is continuous. It follows that A^- and similarly A^+ is of Lebesgue measure zero, as desired. □

Corollary. *Every bounded solid subset of* $\mathbf{R}^n$ *is Jordan measurable.*

Proof. It is well known (see for example [1]) that a subset A of $\mathbf{R}^n$ is Jordan measurable if and only if A is bounded and $\text{bd}(A)$ is of Lebesgue measure zero.□

We are now in a position to prove our main result.

Theorem. *Let* A *be a nonempty subset of* $\mathbf{R}^n$ *and let* $f : A \longrightarrow \mathbf{R}^m$ *be a monotone function. Then the set of all points of discontinuity of* f *is of Lebesgue measure zero.*

Proof. The components f_i of $f = (f_1, f_2, \cdots, f_m)$ are real-valued monotone functions, f is continuous at $x \in A$ if and only if all f_i are continuous at x, hence it suffices to prove the theorem for $m = 1$. Furthermore, f can be extended on the solid cover $S(A)$ of A by

$$\widehat{f}(z) = \sup\{f(x) : \ x \in A, : x \leq z\}, \quad z \in S(A),$$

so we may suppose that A is solid. Finally, since $\mathrm{int}(A)$ is a countable union of n-cells, we may assume by Proposition and by a homothetic argument that $A=[0,e]$, $e=(1,\cdots,1)$, and that $f:[0,e]\longrightarrow \mathbf{R}$ is nondecreasing.

For every $x\in U=\mathrm{int}([0,e])$ set

$$g(x)=\inf\{f(x+te)-f(x-te),\quad 0<t\in\mathbf{R}\}.$$

Observe that for sufficiently small $s>0$ f maps the neighborhood $[x-se,x+se]\subset[0,e]$ of x into the real interval $[f(x-se),f(x+se)]$ containing $f(x)$. Therefore f is continuous at x if and only if $g(x)=0$. Put

$$D_k=\{x\in U:\quad g(x)\geq\frac{1}{k}\},\ \ k=1,2,\cdots,$$

and note that the set D of all discontinuity points of f satisfies $D\cap U=\bigcup_{k\in\mathbf{N}}D_k$. Thus, we have to prove that each D_k is of Lebesgue measure zero.

We claim that $D_k=\mathrm{cl}(D_k)\cap U$. Take any $x\in U\setminus D_k$, and pick $s>0$ such that

$$[x-se,x+se]\subset[0,e],\quad f(x+se)-f(x-se)<\frac{1}{k}.$$

Note that every $y\in[x-(s/2)e,x+(s/2)e]$ satisfies

$$[y-\frac{s}{2}e,y+\frac{s}{2}e]\subset[x-se,x+se],$$

hence $g(y)\leq f(y+(s/2)e)-f(y-(s/2)e)<1/k$, and so $y\notin D_k$. Therefore,

$$[x-\frac{s}{2}e,x+\frac{s}{2}e]\cap D_k=\emptyset,$$

and the claim follows.

Assume now that D_k is nonempty and let $\epsilon>0$. For each fixed $x\in[0,e]$ consider the real function $h:t\longmapsto f(x+te)$. Since h is nondecreasing and jumps for at least $1/k$ at every t satisfying $x+te\in D_k$, the set $D_k\cap(x+\mathbf{R}e)$ contains finitely many elements or it is empty.

Remove from the line $x+\mathbf{R}e$ finitely many disjoint relatively open intervals of common length less than ϵ and containing $\mathrm{cl}(D_k)\cap(x+\mathbf{R}e)$. Denote by $R(x)$ the remaining set, observe that $d=\mathrm{dist}(R(x),D_k)>0$ and put

$$T(x)=\{y\in\mathbf{R}^n:\ \mathrm{dist}(y,x+\mathbf{R}e)<d\}.$$

From the open covering $\{T(x):\ x\in[0,e]\}$ of $[0,e]$ extract a finite subcovering $\{T_i=T(x_i):\ i=1,\cdots,p\}$. Accept $T_0=\emptyset$ and set

$$U_i=T_i\setminus\bigcup_{j<i}T_j,\quad E=\{z\in\mathbf{R}^n:z_1+\cdots+z_n=0\}.$$

By construction

$$\mu_n(U_i \cap D_k) \le \epsilon \mu_{n-1}(U_i \cap E)$$

holds for all i (μ_m denotes the Lebesgue measure in $\mathbf{R}^m$). It follows from

$$\mu_n(D_k) = \sum_{i=1}^{p} \mu_n(U_i \cap D_k) \le \le \epsilon \sum_{i=1}^{p} \mu_{n-1}(U_i \cap E) = \mu_{n-1}\left(\bigcup_{i=1}^{p} U_i \cap E\right)$$

that $\mu_n(D_k) = 0$, so the proof is complete. □

Applying the Lebesgue's characterization of Riemann integrable functions (see [1] or [2]) we get the folowing result.

Corollary. *Let A be a nonempty Jordan measurable subset of $\mathbf{R}^n$ and let $f : A \longrightarrow \mathbf{R}^m$ be a bounded monotone function. Then f is Riemann integrable.*

References

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[2] Spivak, M., *Calculus on Manifolds*, Benjamin, New York, 1965.

Boris Lavrič
Department of Mathematics
University of Ljubljana
Jadranska 19
61 000 Ljubljana, SLOVENIA