Josef Diblík; Denis Khusainov

Asymptotic estimation of the convergence of solutions of the equation $\dot{x}(t) = b(t)x(t - \tau(t))$

Archivum Mathematicum, Vol. 37 (2001), No. 4, 279--287

Persistent URL: `http://dml.cz/dmlcz/107805`

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 37 (2001), 279 – 287

ASYMPTOTIC ESTIMATION OF THE CONVERGENCE OF SOLUTIONS OF THE EQUATION $\dot{x}(t) = b(t)x(t - \tau(t))$

JOSEF DIBLÍK AND DENYS KHUSAINOV

Abstract. The main result of the present paper is obtaining new inequalities for solutions of scalar equation $\dot{x}(t) = b(t)x(t-\tau(t))$. Except this the interval of transient process is computed, i.e. the time is estimated, during which the given solution $x(t)$ reaches an ε - neighbourhood of origin and remains in it.

1. Introduction

Problems concerning the asymptotic bounds, the stability or the asymptotic stability of solutions of differential equations with deviating argument were studied in many papers (e.g. [1] – [10], [14]). At present there are various approaches to the investigation of linear as well as nonlinear delayed systems. Basically, mostly constructive investigation methods were obtained for linear systems (e.g. [1, 5, 7, 8, 11, 12]). In many investigations the coefficient conditions of stability properties were given that depend on delays and that are uniform with respect to the deviation of the argument (e.g. [1, 5, 13]).

Note should be taken that in applied sciences and in engineering calculations not only a statement concerning the stability (asymptotic stability, unstability) is important but also the numerical characterization of transient processes and domains of asymptotic stability (in phase space and in parameter space), establishment of dependence of corresponding domains on parameters and delays, etc. play a major role too. One of the fundamental characterizations is the estimation of convergence solutions to equilibrium state and the computing of interval of the transient process (i.e. the time is computed, during which the solution $x(t)$ reaches an ε - neighbourhood of origin and remains in it provided the zero solutions is stable). In our opinion, not too much attention is paid to the investigation of these problems in available literature.

2000 *Mathematics Subject Classification*: 34K20, 34K25.
Key words and phrases: stability of trivial solution, estimation of convergence of nontrivial solutions.
Received February 14, 2000.

In this paper we consider linear scalar differential equations with delay of the form

$$\dot{x}(t) = b(t)x(t - \tau(t)), \quad t \geq t_0 \,, \tag{1}$$

where the delay $\tau : [t_0, \infty) \to \mathbb{R}^+$, $\mathbb{R}^+ = (0, \infty)$ and the coefficient $b : [t_{-1}, \infty) \to \mathbb{R}$ are continuous functions, $t_{-1} = \inf_{t \geq t_0}\{t - \tau(t)\}$ and $\lim_{t\to\infty}(t - \tau(t)) = \infty$. These assumptions are used throughout this paper. In the next we will investigate the solution $x = x(t)$ of (1) which is defined by means of the initial problem

$$x(t) \equiv \varphi(t) \not\equiv 0, \quad t \in [t_{-1}, t_0] \,. \tag{2}$$

Equation (1) was investigated e.g. in [4]-[8], [12, 14]. If $b(t) = -b < 0$ and $\tau(t) = \tau$ are constants, then for the asymptotic stability of the trivial solution the inequality

$$b\tau < \pi/2$$

is sufficient ([4, 6]). If $b(t)$ is continuous, nonpositive and bounded, and the delay is constant, i.e. $\tau(t) = \tau$ then (see [9, p. 145]) inequalities

$$\beta = \sup_{t \geq t_0} \left\{ -\int_t^{t+\tau} b(s)\, ds \right\} < 1 \quad \text{and} \quad b = \inf_{t \geq t_0} \{-b(t)\} > 0$$

are sufficient for the asymptotic stability. Conditions of asymptotic stability can be formulated in the form

$$\sup_{t \geq t_0}\{-b(t)\} \cdot \sup_{t \geq t_0}\{\tau(t)\} < 3/2$$

as well ([9, p. 97]). Some close results are contained e.g. in [6, 12].

The main result of this paper (Section 2, Theorem 1 below) presents inequalities for solutions and indicates stability conditions. In Section 3 this result is used for finding the transient process interval if the assumptions of Theorem 1 are satisfied.

2. Main Result

It is known that for the stability of the trivial solution of linear equation without delay

$$\dot{x}(t) = b(t)x(t) \,, \quad t \geq t_0 \tag{3}$$

the condition

$$\limsup_{t\to\infty} \int_{t_0}^t b(s)\, ds < \infty$$

is necessary and sufficient while for the asymptotic stability of it the condition

$$\limsup_{t\to\infty} \int_{t_0}^t b(s)\, ds = -\infty$$

is necessary and sufficient. The estimate of the convergence rate for solution of Eq. (3) with $x(t_0) = x_0$ is:

$$|x(t)| \leq |x(t_0)| \exp\left\{ \int_{t_0}^t b(s)\, ds \right\} , \quad t \geq t_0 \,.$$

We can expect that under appropriate assumptions on b and τ similar results will hold for Eq. (1) as well. We will indicate some sufficient conditions for it.
Let us define the number $\bar{\tau}$ as the maximal (positive) root of the equation

$$\tau^* = \tau(t_0 + \tau^*)\,.$$

The role of this number becomes clear in the formulation of the main result and in its proof since expressions which contain an argument of the form of a composite function of $t - \tau(t)$ are well defined for $t \geq t_0 + \bar{\tau}$.

Theorem 1 (Main result)**.** *Let a given function $g : [t_{-1}, \infty) \to \mathbb{R}^+$ be continuously differentiable and satisfy the inequality*

$$\frac{g'(t)}{g^2(t)} + |b(t)| \cdot \int_{t-\tau(t)}^{t} \frac{|b(s)|}{g(s-\tau(s))} \cdot \exp\left\{-\int_{s-\tau(s)}^{t} b(\xi)\,d\xi\right\} ds < 0 \tag{4}$$

if $t \geq t_0 + \bar{\tau}$. Then the solution $x = x(t)$ of the problem (1), (2) *satisfies the inequality*

(5)

$$|x(t)| \leq \begin{cases} \left(1 + \int_{t_0}^{t} |b(s)|\,ds\right) \cdot \|x(t_0)\|_\tau \quad \text{if } \ t_0 \leq t \leq t_0 + \bar{\tau}\,, \\ \left(1 + \int_{t_0}^{t_0+\bar{\tau}} |b(s)|\,ds\right) \cdot \|x(t_0)\|_\tau \cdot \exp\left\{\int_{t_0+\bar{\tau}}^{t} \left[b(s) - \frac{g'(s)}{g(s)}\right] ds\right\} \\ \text{if } t > t_0 + \bar{\tau} \end{cases}$$

with $\|x(t_0)\|_\tau = \max\limits_{[t_{-1},t_0]}\{|\varphi(t)|\} > 0$. If, moreover,

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[b(s) - \frac{g'(s)}{g(s)}\right] ds = -\infty, \tag{6}$$

then the solution $x(t) \equiv 0$ is asymptotically stable; in the case when

$$\int_{t_0}^{t} \left[b(s) - \frac{g'(s)}{g(s)}\right] ds \leq M \tag{7}$$

for a constant M and any $t \in [t_0, \infty)$, the solution $x(t) \equiv 0$ is Liapunov-stable and each solution $x(t)$ satisfies the inequality

$$|x(t)| < \varepsilon \quad \text{for} \quad t > t_0 \quad \text{if} \quad \|x(t_0)\|_\tau < \delta(\varepsilon, t_0) \tag{8}$$

with

$$\delta(\varepsilon, t_0) = \frac{\varepsilon}{1 + \int_{t_0}^{t_0+\bar{\tau}} |b(s)|\,ds} \cdot \frac{g(t_0)}{g(t_0+\bar{\tau})} \cdot \exp\left(-M + \int_{t_0}^{t_0+\bar{\tau}} b(s)\,ds\right).$$

Proof. The proof is based on the construction of an appropriate Liapunov function $v(t, x)$ and on using of Razumikhin type reasonings. Corresponding level curves $v(t, x) = \alpha$ with a suitable parameter $\alpha > 0$ define the boundary of a domain $\mathcal{D}$ in the (t, x) plane. It is proved that the graph of solution $x(t)$ of the problem (1), (2) lies in $\mathcal{D}$ on the interval $(t_0 + \bar{\tau}, \infty)$. This fact is verified by means of computing the full derivative of $v(t, x)$ along the trajectories of the equation (1). Its sign is estimated (in accordance with well-known Razumikhin approach (see e.g. [6]–[10], [14])) assuming that the graph of the solution $x(t)$ reaches the boundary $\partial\mathcal{D}$ of the domain $\mathcal{D}$ at the moment $t = t^* \geq t_0 + \bar{\tau}$ and lies in $\mathcal{D}$ for $t < t^*$. On the interval $[t_0, t_0 + \bar{\tau}]$ the solution $x(t)$ is estimated directly with the aid of (1). The negative sign of $\dot{v}(t,x)$ for $t = t^*$ leads to a contradiction. From the form of the domain $\mathcal{D}$ the desired inequalities are obtained and conclusions concerning the asymptotic stability or Liapunov stability are formulated.

a) Construction of Liapunov function. In the following we will use a quadratic function of the form

$$v(t,x) = g^2(t) \cdot \exp\left\{-2\int_{t_0}^{t} b(s)\,ds\right\} \cdot x^2 \tag{9}$$

with a weight function g. This function will be a Liapunov function if the condition for positive definiteness holds. This condition is expressed by the inequality

$$g(t) \cdot \exp\left\{-\int_{t_0}^{t} b(s)\,ds\right\} \geq N > 0, \ \ t \geq t_0\,,$$

where N is a constant, or by the inequality

$$\exp\left\{-\int_{t_0}^{t}\left[b(s) - \frac{g'(s)}{g(s)}\right]ds\right\} \geq \frac{N}{g(t_0)} > 0, \ \ t \geq t_0\,.$$

These inequalities hold due to (7) provided $N \leq g(t_0)\exp\{-M\}$.

b) Investigation of the sign of the derivative of the Liapunov function and of the asymptotic behaviour of the solution of the problem (1), (2). The full derivative of the Liapunov function v along solutions of Eq. (1) equals

$$\begin{aligned}\dot{v}(t,x) = {} & 2g(t) \cdot \exp\left\{-2\int_{t_0}^{t} b(s)\,ds\right\} \\ & \times \left\{g'(t)x^2(t) + g(t)b(t)x(t)[x(t-\tau(t)) - x(t)]\right\}\end{aligned} \tag{10}$$

for $t \geq t_0$. Let us estimate the sign of this derivative. For this we integrate Eq. (1) over the interval $[t - \tau(t), t]$ to obtain

$$x(t) = x(t-\tau(t)) + \int_{t-\tau(t)}^{t} b(\xi)x(\xi - \tau(\xi))\,d\xi, \ \ t \geq t_0 + \bar{\tau}\,.$$

Then

$$|x(t) - x(t-\tau(t))| \leq \int_{t-\tau(t)}^{t} |b(\xi)|\,|x(\xi-\tau(\xi))|\,d\xi\,, \ \ t \geq t_0 + \bar{\tau}. \tag{11}$$

Let us suppose that the constant α is taken so large that the graph of the solution of the problem (1), (2) remains on the interval (t_{-1}, t^*) with $t^* \geq t_0 + \bar{\tau}$ in the domain $v(t,x) < \alpha$, i.e.

$$(t, x(t)) \in \mathcal{D}\left\{(t,x) \in \mathbb{R} \times \mathbb{R}, t \geq t_{\,-1}, |x| < \frac{\sqrt{\alpha}}{g(t)}\exp\left\{\int_{t_0}^{t} b(s)\,ds\right\}\right\}$$

for $t \in [t_{-1}, t^*)$. This is always possible due to the linearity of the equation considered. Suppose, moreover, that at the moment $t = t^*$ the graph of $x(t)$ reaches the boundary $\partial\mathcal{D}$ of the domain $\mathcal{D}$, i.e. $(t^*, x(t^*)) \in \partial\mathcal{D}$. Let us show that this assumption leads to a contradiction. In this case $v(t^*, x(t^*)) = \alpha$ and

$$v(s, x(s)) < \alpha = v(t^*, x(t^*)), \quad t_{-1} \le s < t^*$$

or

$$g^2(s) \cdot \exp\left\{-2\int_{t_0}^{s} b(\xi)d\xi\right\} \cdot x^2(s) < \alpha = g^2(t^*) \cdot \exp\left\{-2\int_{t_0}^{t^*} b(\xi)d\xi\right\} \cdot x^2(t)$$

with $s \in [t_{-1}, t^*)$. Consequently

$$|x(s)| < \frac{g(t^*)}{g(s)} \cdot \exp\left\{-\int_{s}^{t^*} b(\xi)d\xi\right\} \cdot |x(t)|, \quad t_{-1} \le s < t^*\,.$$

Let us use this inequality for the integrand of the inequality (11) with $t = t^*$. Then

$$\begin{aligned}&|x(t^*) - x(t^* - \tau(t^*))|\\ &\qquad < \left[\int_{t^*-\tau(t^*)}^{t^*} \frac{|b(\xi)|}{g(\xi - \tau(\xi))} \cdot \exp\left\{-\int_{\xi-\tau(\xi)}^{t^*} b(s)ds\right\} d\xi\right] g(t^*)|x(t^*)|\,.\end{aligned}$$

Taking into account the last inequality, we can estimate the sign of the full derivative (10) for $t = t^*$. Then (with the aid of (4)) we get

$$\begin{aligned}&\dot v(t^*, x(t^*)) \le 2g(t^*) \cdot \exp\left\{-2\int_{t_0}^{t^*} b(s)\,ds\right\} \cdot \left\{g'(t^*) + g^2(t^*)|b(t^*)|\right.\\ &\quad\left.\times\left[\int_{t^*-\tau(t^*)}^{t^*} \frac{|b(\xi)|}{g(\xi - \tau(\xi))} \exp\left\{-\int_{\xi-\tau(\xi)}^{t^*} b(s)ds\right\} d\xi\right]\right\} \cdot x^2(t^*) < 0\,.\end{aligned} \tag{12}$$

This means that the velocity vector of the motion $x(t)$ is at the moment $t = t^*$ directed into the domain $\mathcal{D}$. Consequently, the graph of the solution $x(t)$ cannot leave this domain and it cannot reach its boundary at the moment $t = t^*$, i.e. $(t^*, x(t^*)) \in \mathcal{D}$ and

$$|x(t^*)| < \frac{\sqrt{\alpha}}{g(t^*)} \cdot \exp\left\{\int_{t_0}^{t^*} b(\xi)d\xi\right\}\,.$$

This contradicts the above assumption. Since the moment t^* can be taken arbitrarily in the interval $[t_0 + \bar\tau, \infty)$ we conclude that $(t, x(t)) \in \mathcal{D}$ for $t \in [t_0 + \bar\tau, \infty)$.

c) Concrete choice of the parameter α and validity of the inequalities (5). Let us determine the quantity α. First, we estimate the maximal deviation of solution $x(t)$ on interval $t_0 \le t \le t_0 + \bar\tau$. Integrating (1) with limits t_0 and t, we get

$$|x(t)| \le \|x(t_0)\|_\tau + \int_{t_0}^{t} |b(s)||x(s - \tau(s)|\,ds \le \left(1 + \int_{t_0}^{t} |b(s)|\,ds\right) \|x(t_0)\|_\tau\,.$$

So the first part of (5) holds. As was noted, the graph of the solution $x(t)$ of the problem does not leave the domain $\mathcal{D}$ if $t \in [t_0+\bar{\tau},\infty)$. Since the derivative $\dot{v}(t,x)$ is negative here, the maximum of $v(t,x)$ is reached at the left-hand point of the interval considered, i.e. at the point $t=t_0$. Then

$$\max_{[t_0+\bar{\tau},\infty)} v(t,x(t)) = v(t_0+\bar{\tau},x(t_0+\bar{\tau})) \le$$

$$g^2(t_0+\bar{\tau})\cdot\exp\left\{-2\int_{t_0}^{t_0+\bar{\tau}} b(s)ds\right\}\cdot\left(1+\int_{t_0}^{t_0+\bar{\tau}}|b(s)|\,ds\right)^2\cdot\|x(t_0)\|_\tau^2 .$$

Now it is possible to put

$$\alpha=\alpha^*=g^2(t_0+\bar{\tau})\cdot\exp\left\{-2\int_{t_0}^{t_0+\bar{\tau}} b(s)ds\right\}\cdot\left(1+\int_{t_0}^{t_0+\bar{\tau}}|b(s)|\,ds\right)^2\cdot\|x(t_0)\|_\tau^2 .$$

As was proved above, $(t,x(t))\in\mathcal{D}$ if $t\in[t_0+\bar{\tau},\infty)$. This gives (for $\alpha=\alpha^*$ and $t\ge t_0+\bar{\tau}$) the estimates

$$\begin{aligned}|x(t)| &< \frac{\sqrt{\alpha^*}}{g(t)}\cdot\exp\left\{\int_{t_0}^t b(s)\,ds\right\}=\frac{1}{g(t)}\cdot\exp\left\{\int_{t_0}^t b(s)\,ds\right\}\\ &\quad\times g(t_0+\bar{\tau})\cdot\exp\left\{-\int_{t_0}^{t_0+\bar{\tau}} b(s)ds\right\}\cdot\left(1+\int_{t_0}^{t_0+\bar{\tau}}|b(s)|\,ds\right)\cdot\|x(t_0)\|_\tau\end{aligned}$$

or

(13)
$$|x(t)|<\frac{g(t_0+\bar{\tau})}{g(t)}\exp\left\{\int_{t_0+\bar{\tau}}^t b(s)ds\right\}\left(1+\int_{t_0}^{t_0+\bar{\tau}}|b(s)|\,ds\right)\|x(t_0)\|_\tau .$$

Consequently, the second part of (5) holds.

d) Asymptotic stability and Liapunov stability of the trivial solution of Eq. (1). From (13) and (6) it follows immediately that $\lim_{t\to\infty}x(t)=0$. This means that the condition (6) ensures the asymptotic stability of the trivial solution. Condition (7) expresses the property of Liapunov stability since the relation (8) immediately follows from (13) and (7).

The proof of the theorem is complete. □

Remark 1. Assume that $g(t)=\mathrm{e}^{-\gamma t}$ with a constant $\gamma>0$. If, moreover, the inequality (4) holds, i.e. if

$$|b(t)|\int_{t-\tau(t)}^t |b(s)|\cdot\exp\left\{-\int_{s-\tau(s)}^t[b(\xi)+\gamma]\,d\xi\right\}ds<\gamma, \tag{14}$$

for $t\ge t_0+\bar{\tau}$, then the inequality (5) takes the form

$$|x(t)|\le\begin{cases}\left(1+\int_{t_0}^t|b(s)|\,ds\right)\cdot\|x(t_0)\|_\tau \text{ if } t_0\le t\le t_0+\bar{\tau},\\[2ex] \left(1+\int_{t_0}^{t_0+\bar{\tau}}|b(s)|\,ds\right)\cdot\|x(t_0)\|_\tau\cdot\exp\left\{\int_{t_0+\bar{\tau}}^t[b(s)+\gamma]\,ds\right\}\\ \text{if } t>t_0+\bar{\tau}\,.\end{cases}$$

Remark 2. Assume that $b(t) \equiv -b < 0$, $g(t) = \mathrm{e}^{-\gamma t}$ with a constant $\gamma > 0$, $b > \gamma$ and assume that the delay $\tau(t)$ is constant, i.e. $\tau(t) = \tau$. Then $\bar{\tau} = \tau$ and the inequality (4) turns into inequality

$$b^2 \mathrm{e}^{-(\gamma-b)\tau} \cdot \frac{1}{\gamma - b}\left[1 - \mathrm{e}^{-(\gamma-b)\tau}\right] < \gamma\,. \tag{15}$$

Then for $\tau < \tau_0$ with

$$\tau_0 = \frac{1}{b-\gamma}\ln\left[\frac{1}{2}\left(1 + \sqrt{1 + 4\gamma(b-\gamma)/b^2}\right)\right]$$

(τ_0 is the supremum of values τ, which satisfy the inequality (15)) the inequality (5) has the form

$$|x(t)| \leq \begin{cases} (1 + b(t - t_0)) \cdot \|x(t_0)\|_\tau & \text{if } \ t_0 \leq t \leq t_0 + \tau\,, \\ (1 + b\tau) \cdot \|x(t_0)\|_\tau \cdot \exp\{-(b-\gamma)(t - t_0 - \tau)\} & \text{if } \ t > t_0 + \tau\,. \end{cases}$$

Let us note that

$$\sup_{\gamma \in (0,b)} \tau_0 = \lim_{\gamma \to b-0} \frac{1}{b-\gamma}\ln\left[\frac{1}{2}\left(1 + \sqrt{1 + 4\gamma(b-\gamma)/b^2}\right)\right] = \frac{1}{b}\,.$$

Then the trivial solution $x(t) \equiv 0$ of Eq. (1) is asymptotically stable if $\tau < 1/b$ since for every b an appropriate value of γ can be taken.

Remark 3. Conditions of stability (asymptotic stability) and estimation of convergence depend, in general, upon the initial moment t_0. This means they are not uniform with respect to t_0.

3. Interval of Transient Process

Let us denote

$$\Phi(t) = \int_{t_0+\bar{\tau}}^{t}\left[b(s) - \frac{g'(s)}{g(s)}\right] ds\,, \quad A(t_0) = \left[1 + \int_{t_0}^{t_0+\bar{\tau}} |b(s)|\, ds\right] \|x(t_0)\|_\tau\,.$$

Suppose that all conditions of Theorem 1 hold and that the function Φ is monotone decreasing (not necessarily with the infinite limit). Let ε be a given positive number satisfying the inequality $\varepsilon < A(t_0)$ which expresses the desired accuracy. Consider the equation

$$A(t_0)\exp\{\Phi(t)\} = \varepsilon\,. \tag{16}$$

If

$$\inf_{t \geq t_0+\bar{\tau}} \Phi(t) < \ln\left(\frac{\varepsilon}{A(t_0)}\right)$$

then there exists a unique solution $t = T$ of equation (16). The quantity

$$T = \Phi^{-1}\left[\ln\left(\frac{\varepsilon}{A(t_0)}\right)\right], \tag{17}$$

where Φ^{-1} is the inverse function to Φ, is called *the overshoot quantity*. So interval (T, ∞) was defined such that

$$|x(t)| < \varepsilon \quad \text{if} \quad t > T\,.$$

In this way the following theorem was proved.

Theorem 2 (Overshoot quantity). *Let the assumptions of Theorem* 1 *hold. Let the function* Φ *be strictly decreasing within interval* $[t_0 + \bar{\tau}, \infty)$ *and let* $\varepsilon < A(t_0)$ *be a given positive number. Then the solution* $x(t)$ *of the problem* (1), (2) *satisfies the inequality* $|x(t)| < \varepsilon$ *on the interval* (T, ∞) *with* T *given by* (17).

Remark 4. Let the assumptions of Theorem 2 hold. Then, as follows from (17), the overshoot quantity is defined as the solution of the equation

$$\int_{t_0+\bar{\tau}}^{T} b(s)\,ds = \ln\left[\frac{\varepsilon g(T)}{g(t_0+\bar{\tau})A(t_0)}\right]. \tag{18}$$

Remark 5. Let the assumptions of Theorem 2 hold. If, moreover, the function $g(t)$ is given in the form $g(t) = \mathrm{e}^{-\gamma \mathrm{t}}$ with $\gamma > 0$ then the overshoot quantity T (see (18)) is defined as the solution of the equation

$$\int_{t_0+\bar{\tau}}^{T} b(s)\,ds + \gamma(T - t_0 - \bar{\tau}) = \ln\left(\frac{\varepsilon}{A(t_0)}\right).$$

In the partial case when the function $b(t)$ is bounded by a negative constant, i.e. if $b(t) \le -b < 0$ and $b \ge \gamma$, we get

$$T \le t_0 + \bar{\tau} + \frac{1}{b-\gamma}\ln\left(\frac{A(t_0)}{\varepsilon}\right).$$

Acknowledgement. The authors are very grateful to the referee for his/her remarks and comments which led to a valuable improvement of the presentation of the paper.
The first author was supported by the Grant 201/99/0295 of Czech Grant Agency (Prague) and by the plan of investigations MSM 262200013 of the Czech Republic. The second author was supported by the grant of Ministry of Education of the Czech Republic No. 1058/1999 FRVŠ (Fund of Development of Czech Universities).

References

[1] Bellman, R., Cooke, K. L., *Differential-Difference Equations,* Acad. Press, New-York-London, 1963.
[2] Čermák, J., The asymptotic bounds of solutions of linear delay systems, *J. Math. Anal. Appl.* **225** (1998), 373–338.
[3] Diblík, J., Stability of the trivial solution of retarded functional equations, *Differ. Uravn.* **26** (1990), 215–223. (In Russian).
[4] Elsgolc, L. E., Norkin, S. B., *Introduction to the Theory of Differential Equations with Deviating Argument,* Nauka, Moscow, 1971. (In Russian).
[5] Györi, I., Pituk, M., Stability criteria for linear delay differential equations, *Differential Integral Equations* **10** (1997), 841–852.
[6] Hale, J., Lunel, S. M. V., *Introduction to Functional Differential Equations,* Springer-Verlag, 1993.

[7] Kolmanovskij, V., Myshkis, A., *Applied Theory of Functional Differential Equations,* Kluwer Acad. Publ., 1992.

[8] Kolmanovskij, V., Myshkis, A., *Introduction to the Theory and Applications of Functional Differential Equations,* Kluwer Acad. Publ., 1999.

[9] Kolmanovskij, V. B., Nosov, V. R., *Stability and Periodic Modes of Regulated Systems with Delay,* Nauka, Moscow, 1981. (In Russian).

[10] Krasovskii, N. N., *Stability of Motion,* Stanford Univ. Press, 1963.

[11] Krisztin, T., Asymptotic estimation for functional differential equations via Lyapunov functions, *Colloq. Math. Soc. János Bolyai, Qualitative Theory of Differential Equations, Szeged, Hungary,* 1988, 365–376.

[12] Myshkis, A. D., *Linear Differential Equations with Delayed Argument,* Nauka, Moscow, 1972. (In Russian).

[13] Pituk, M., Asymptotic behavior of solutions of differential equation with asymptotically constant delay, *Nonlinear Anal.* **30** (1997), 1111–1118.

[14] Razumikhin, B. S., *Stability of Hereditary Systems,* Nauka, Moscow, 1988. (In Russian).

J. Diblík, Department of Mathematics
Faculty of Electrical Engineering and Computer Science
Brno University of Technology, Technická 8
616 00 Brno, CZECH REPUBLIC
E-mail: diblik@dmat.fee.vutbr.cz

D. Khusainov, Department of Complex Systems Modelling
Faculty of Cybernetics, Kiev University
Vladimirskaja 64, Kiev 252033, UKRAINE
E-mail: denis@dh.cyb.univ.kiev.ua