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Marián J. Fabián

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THEORY OF FRÉCHET CONES

MARIÁN FABIAN, Praha

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INTRODUCTION

If nonlinear analysis, when working with mappings, we often use their Fréchet differentials. This requires Fréchet differentiability of mappings and so the set of the mappings coming into question is restricted considerably.

In this paper, we introduce a more general concept – the Fréchet cone of a mapping. It is shown here that this concept is almost as useful as the Fréchet differentials. In this way, we have a possibility of generalizing many theorems in which the Fréchet differentiability of mappings is required.

Let us give a brief outline. Let X, Y be normed linear spaces and $F : X \to 2^Y$ a mapping, singlevalued and upper semicontinuous at some $x_0 \in \operatorname{int} D(F)$. Following Durdil [5], we construct a cone $C_0(F, x_0)$ (if it exists) in the space $X \times Y$ and call it the Fréchet cone of F at x_0. Its construction is based on the concept of the so called conic limit introduced in [5]. The relation to the Fréchet differentiability is expressed in the following

Theorem 4.1 (*Durdil* [5]). *F is Fréchet differentiable at x_0 if and only if the Fréchet cone $C_0(F, x_0)$ exists and is (the graph of) a linear continuous mapping from X into Y. In this case,*

$$(\textit{the graph of})\ \mathrm{d}F(x_0) = C_0(F, x_0).$$

This shows that the Fréchet cone is a natural generalization of the Fréchet differential. Moreover, it suggests that a calculus with the Fréchet cones, similar to that with the Fréchet differentials, can be developed. Indeed, the proofs of many theorems remain correct after replacing the Fréchet differentials by the Fréchet cones. Thus we can easily generalize a lot of results from nonlinear analysis in normed linear spaces.

Let us go through the paper briefly. The basic object we use is a cone in a normed linear space Z. In Section 1, we define a conic neighbourhood of a cone and derive some statements about it. By using this concept, we define in Section 2 the conic limit. In Section 3, on the basis of the conic limit, we introduce the Fréchet cone of a set $M \subset Z$ at a point $z \in M'$. The main result here is Theorem 3.1, which tells us what is the form of the Fréchet cone of $B(M)$ at Bz_0, where $B : Z \to 2^W$ is a mapping having fairly general properties. Bearing in mind that a mapping from X into Y is nothing else than a subset of $X \times Y$, we define, in Section 4, the Fréchet cone of a mapping at a point. Theorem 4.1 shows that the Fréchet cone is a generalization of the Fréchet differential. Section 5 contains some computing rules about the Fréchet cones. Namely, applying Theorem 3.1 we derive statements concerning the Fréchet cones of the inverses, linear combinations and compositions of mappings. It is also remarked that they at once yield the corresponding well known theorems from differential calculus in normed linear spaces. The last section is devoted to the formulation of a mean value theorem.

This paper is a shortened and rewritten form of the preprint [7].

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0. PRELIMINARIES

Throughout the paper $(X, \|.\|)$, $(Y, \|.\|)$, $(Z, \|.\|)$ $(U, \|.\|)$, $(W, \|.\|)$ mean (unless otherwise stated) nontrivial real normed linear spaces (abbreviated to n.l.s.). If we put $Z = X \times Y$, $W = X \times U$, etc., we always take the maximum norm, that is, for instance,

$$\|(x, y)\| = \max(\|x\|, \|y\|), \quad (x, y) \in X \times Y = Z .$$

Let M be a nonempty subset of, say, Z. The symbols int M, cl M, sp M and M' denote the interior, the closure, the linear span and the derived set of M, respectively.

When writing $F : X \to 2^Y$, we mean that F is a (multivalued) mapping from X into Y, i.e., $0 \neq F \subset X \times Y$, while $F : X \to Y$ means that F is a singlevalued mapping from X into Y, i.e., $0 \neq F \subset X \times Y$ and

$$((x, y_1) \in F, \quad (x, y_2) \in F) \Rightarrow y_1 = y_2 .$$

For $F : X \to 2^Y$, we set

$$D(F) = \{x \in X | \; \exists y \in Y \; (x, y) \in F\}, \quad R(F) = \{y \in Y | \; \exists x \in X \; (x, y) \in F\},$$

$$F^{-1} = \{(y, x) \in Y \times X | \; (x, y) \in F\}, \quad Fx = \{y \in Y | \; (x, y) \in F\}, \quad x \in X,$$

$$F(M) = \bigcup \{Fx | \; x \in M\}, \quad M \subset X .$$

If Fx consists of one point only, we denote this point also by Fx. Further, we see that F can also be defined by fixing Fx for each $x \in X$. We say that F is singlevalued at $x_0 \in X$ if Fx_0 is a singleton, and upper semicontinuous (abbreviated to u.s.c.) at x_0 if, to each $\varepsilon > 0$, there is $\delta > 0$ so that the set Fx lies in the ε-neighbourhood of the set Fx_0 whenever $\|x_0 - x\| < \delta$. Of course, if F is singlevalued, then the upper semicontinuity coincides with the continuity.

Let λ be a real number and $G : X \to 2^Y$, $H : Y \to 2^U$ two mappings. Then we define

$$\lambda F = \{(x, \lambda y) | \ (x, y) \in F\}, \tag{0.1}$$

$$F + G = \{(x, u + v) | \ (x, u) \in F, \ (x, v) \in G\}, \tag{0.2}$$

$$H \circ F = \{(x, u) \in X \times U | \ \exists y \in Y \ (x, y) \in F, \ (y, u) \in H\}. \tag{0.3}$$

Hence, $\lambda F : X \to 2^Y$, $F + G : X \to 2^Y$, $H \circ F : X \to 2^U$.

The set of all linear continuous mappings $L: X \to Y$ with $D(L) = X$ is denoted by $\mathscr{L}(X, Y)$. Also, we set

$$\operatorname{Isom}(X, Y) = \{L \in \mathscr{L}(X, Y) | \ L^{-1} \in \mathscr{L}(Y, X)\}.$$

The real line with the usual linear structure and topology is denoted by $\mathbb{R}$ and we put $\omega = \{1, 2, \ldots\}$.

Finally, recall

Definition. Let $F : X \to 2^Y$ be a mapping with $\operatorname{int} D(F) \neq \emptyset$ and take some $x_0 \in \operatorname{int} D(F)$. We say that *$F$ is Fréchet differentiable at* x_0 if it is singlevalued at x_0 and there exists $L \in \mathscr{L}(X, Y)$ such that

$$\frac{1}{\|h\|} \sup \{\|y - Fx_0 - Lh\| \ | \ y \in F(x_0 + h)\} \to 0 \quad \text{as} \quad 0 \neq h \to 0 .$$

In this case we write $\mathrm{d}F(x_0) = L$ and call it the *Fréchet differential of F at* x_0.

Of course, F has at most one Fréchet differential at x_0 and the Fréchet differentiability of F at x_0 implies that F is u.s.c. at x_0.

1. CONIC NEIGHBOURHOODS

Let $(Z, \|\cdot\|)$ be a n.l.s. By a cone (in Z) we understand each nonempty subset C of Z such that $C \neq \{0\}$ and $\lambda z \in C$ whenever $\lambda \geqq 0$ and $z \in C$.

A conic neighbourhood of a cone can be defined in various, topologically equivalent ways, see [3], [5]. We prefer the following

Definition. (Daneš, Durdil [3]). For a cone $C \subset Z$ and $\varepsilon > 0$, we set

$$V_\varepsilon(C) = \{z \in Z | \ \exists c \in C \ \|z - c\| < \varepsilon \|z\|\} \cup \{0\}$$

and call it the *conic ε-neighbourhood of C* (with respect to the norm $\|\cdot\|$ in Z).

Of course, $V_\varepsilon(C)$ is a cone and $V_\varepsilon(C) = Z$ whenever $\varepsilon > 1$.
Some elementary properties of the conic neighbourhoods are expressed in

Proposition 1.1. *If C, D are cones in Z and $\varepsilon, \delta > 0$, then*

$$C \subset V_\delta(C) \subset V_\varepsilon(C) \quad \text{whenever} \quad \delta < \varepsilon\,,$$
$$V_\varepsilon(C) \subset V_\varepsilon(D) \quad \text{whenever} \quad C \subset D\,,$$
$$V_\varepsilon(V_\delta(C)) \subset V_{\varepsilon+\delta+\varepsilon\delta}(C)\,,$$
$$V_\varepsilon(\operatorname{cl} C) = V_\varepsilon(C)\,,$$
$$\forall \Delta > 0 \ \operatorname{cl}(V_\varepsilon(C)) \subset V_{\Delta+\varepsilon}(C)\,,$$
$$\operatorname{cl} C = \bigcap_{\Delta>0} V_\Delta(C)\,.$$

Proposition 1.2. *Let C be a cone in Z, let W be another n.l.s. and assume that there exist $L \in \mathscr{L}(Z, W)$ and $\alpha > 0$ such that*

$$\forall z \in C \quad \alpha\|z\| \leqq \|Lz\|\,. \tag{1.1}$$

Furthermore, let $\varepsilon \in (0, \alpha/(\alpha + \|L\|))$ and put

$$\Delta = \frac{\|L\|\,\varepsilon}{\alpha - \alpha\varepsilon - \|L\|\,\varepsilon}\,.$$

Then $L(C)$ is a cone in W and

$$L(V_\varepsilon(C)) \subset V_\Delta(L(C))\,.$$

Proof. (1.1) implies $L(C) \neq \{0\}$ and the linearity of L then gives that $L(C)$ is a cone in W. Now take $w \neq 0$ in $L(V_\varepsilon(C))$. Then $w = Lz$ with $z \in V_\varepsilon(C)$ and we can find $c \in C$ so that $\|z - c\| < \varepsilon\|z\|$. Hence

$$\|c\| \geqq \|z\| - \|z - c\| > (1 - \varepsilon)\,\|z\|\,,$$

$$\|L\|\,\|c\| > (1 - \varepsilon)\,\|L\|\,\|z\| > \frac{1 - \varepsilon}{\varepsilon}\,\|L\|\,\|z - c\| \geqq \frac{1 - \varepsilon}{\varepsilon}\,\|Lz - Lc\|\,,$$

$$\|Lz\| \geqq \|Lc\| - \|Lz - Lc\| \geqq \alpha\|c\| - \|Lz - Lc\| > \left(\frac{\alpha}{\|L\|}\,\frac{1 - \varepsilon}{\varepsilon} - 1\right)\|Lz - Lc\|\,,$$

$$\|Lz - Lc\| < \Delta\|Lz\|\,,$$

i.e., $w = Lz \in V_\Delta(L(C))$.

If $W = Z$ with an equivalent norm and L is the identity mapping, then we have a stronger result.

Proposition 1.3. *Let C be a cone in Z, $\varepsilon > 0$ and let $|||\cdot|||$ be an equivalent norm on Z, i.e., there are $\alpha, \beta > 0$ such that*

$$\forall z \in Z \quad \alpha\|z\| \leqq |||z||| \leqq \beta\|z\| . \tag{1.2}$$

Put $\Delta = (\beta/\alpha)\,\varepsilon$, $\delta = (\beta/\alpha)\,\Delta$ and let $V'_\Delta(C)$ denote the conic Δ-neighbourhood of C with respeτt to the norm $|||\cdot|||$.

Then

$$V_\varepsilon(C) \subset V'_\Delta(C) \subset V_\delta(C) .$$

Let X, Y be n.l.s.. For a cone $C \subset X \times Y$, we define (taking $1/0 = +\infty$, if necessary)

$$\|C\|_X = \sup\left\{\frac{\|y\|}{\|x\|} \,\middle|\, (0,0) \neq (x, y) \in C\right\} . \tag{1.3}$$

It should be noted that $L \in \mathscr{L}(X, Y)$ is a cone in $X \times Y$ and $\|L\|_Y$ is equal to the usual norm $\|L\|$ of the mapping L.

Proposition 1.4. *Let C be a cone in $X \times Y$ such that $\gamma = \|C\|_X < +\infty$.*

Then, for $\varepsilon \in (0, 1/(1 + \gamma))$,

$$\|V_\varepsilon(C)\|_X < \max\left(\gamma + \gamma\varepsilon + \varepsilon,\ \frac{\gamma}{1 - \varepsilon - \gamma\varepsilon}\right). \tag{1.4}$$

Proof. Let $(0, 0) \neq (x, y) = z \in V_\varepsilon(C)$. As we deal with cones, we may assume that $\|z\| = \max(\|x\|, \|y\|) = 1$. Choose $c = (a, b) \in C$ so that

$$\|z - c\| = \max(\|x - a\|, \|y - b\|) < \varepsilon . \tag{1.5}$$

Hence $c \neq (0, 0)$. We shall distinguish two cases:

1. $\|x\| = 1$. Then we have from (1.5)

$$\frac{\|y\|}{\|x\|} = \|y\| \leqq \|b\| + \|y - b\| < \gamma\|a\| + \varepsilon \leqq$$
$$\leqq \gamma(\|x\| + \|a - x\|) + \varepsilon \leqq \gamma + \gamma\varepsilon + \varepsilon .$$

2. $\|y\| = 1$. If $b = 0$, then (1.5) would imply $1 = \|y\| < \varepsilon$, which is impossible. Hence, $b \neq 0$ and so $\gamma\|a\| \geqq \|b\| > 0$, $\gamma > 0$. Thus

$$\|x\| \geqq \|a\| - \|x - a\| > \gamma^{-1}\|b\| - \varepsilon \geqq$$
$$\geqq \gamma^{-1}(\|y\| - \|b - y\|) - \varepsilon > \gamma^{-1}(1 - \varepsilon) - \varepsilon ,$$
$$\frac{1}{\|x\|} = \frac{\|y\|}{\|x\|} < \frac{1}{\gamma^{-1}(1 - \varepsilon) - \varepsilon} = \frac{\gamma}{1 - \varepsilon - \gamma\varepsilon} ,$$

and (1.4) is proved.

Let Z be a n.l.s. The set of all cones of Z is denoted by $\mathscr{C}(Z)$.

Definition (see Durdil [5]). Let $\{C_r\}_{r>0}$ be a net of cones in Z and let there exist a closed cone $C_0 \subset Z$ such that

$$\forall \varepsilon > 0 \ \exists r > 0 \ \forall s \in (0, r] \ C_0 \subset V_\varepsilon(C_s) \ \& \ C_s \subset V_\varepsilon(C_0).$$

Then C_0 is called the *conic limit of the net* $\{C_r\}_{r>0}$ and we write

$$C_0 = \operatorname*{c-lim}_{r \downarrow 0} C_r .$$

Of course, it follows from Propositions 1.1 and 1.3 that the conic limit is at most one and is independent of which equivalent norm is taken on Z.

The conic limit can also be introduced with help of the Hausdorff distance on the unit sphere S of Z: For $A, B \subset S$ put

$$d(A, B) = \inf\{\varepsilon > 0 \mid A \subset H_\varepsilon(B) \ \& \ B \subset H_\varepsilon(A)\} ,$$

where

$$H_\varepsilon(A) = \{z \in S \mid \exists a \in A \ \|z - a\| < \varepsilon\} .$$

Proposition 2.1. *A closed cone $C_0 \subset Z$ is a conic limit of $\{C_r\}_{r>0} \subset \mathscr{C}(Z)$ if and only if*

$$\lim_{r \downarrow 0} d(C_r \cap S, \ C_0 \cap S) = 0 .$$

Proof. For $C, D \in \mathscr{C}(Z)$ put

$$\varrho(C, D) = \inf\{\varepsilon > 0 \mid C \subset V_\varepsilon(D) \ \& \ D \subset V_\varepsilon(C)\} . \tag{2.1}$$

Then $C_0 = \operatorname*{c-lim}_{r \downarrow 0} C_r$ if and only if $\lim_{r \downarrow 0} \varrho(C_r, C_0) = 0$. Moreover, we have

$$\varrho(C, D) \leqq d(C \cap S, \ D \cap S) \leqq 2\,\varrho(C, D) , \quad C, D \in \mathscr{C}(Z) . \tag{2.2}$$

So in order to complete the proof it remains to show (2.2).

Let $d(C \cap S, \ D \cap S) < \varepsilon$ and $0 \neq c \in C$. As $C \cap S \subset H_\varepsilon(D \cap S)$, there is $d \in D \cap S$ such that $\|c/\|c\| - d\| < \varepsilon$ and so $\|c - \|c\| \, d\| < \varepsilon\|c\|$, i.e., $c \in V_\varepsilon(D)$. Hence $C \subset V_\varepsilon(D)$. In the same way we get that $D \subset V_\varepsilon(C)$ and therefore $\varrho(C, D) < \varepsilon$. We have proved the implication

$$d(C \cap S, \ D \cap S) < \varepsilon \Rightarrow \varrho(C, D) < \varepsilon ,$$

from which the left hand inequality in (2.2) follows.

Let us prove the right hand inequality. If $\varrho(C, D) = 1$, it holds because we always have $d(C \cap S, D \cap S) \leqq 2$. Further, let $\varrho(C, D) < 1$ and choose an arbitrary ε such that $\varrho(C, D) < \varepsilon < 1$. Then $C \subset V_\varepsilon(D)$. Take $c \in C \cap S$. Then there is $d \in D$

such that $\|c - d\| < \varepsilon\|c\| = \varepsilon$. Consequently, $|1 - \|d\|| < \varepsilon$ and $d \neq 0$. Let us estimate

$$\left\|c - \frac{d}{\|d\|}\right\| \leqq \|c - d\| + \left\|d - \frac{d}{\|d\|}\right\| < \varepsilon + \|d\| \frac{|\|d\| - 1|}{\|d\|} < 2\varepsilon .$$

Hence $c \in H_{2\varepsilon}(D \cap S)$, $C \cap S \subset H_{2\varepsilon}(D \cap S)$. Symmetrically, we get $D \cap S \subset$ $\subset H_{2\varepsilon}(C \cap S)$. So we have shown that $d(C \cap S,\ D \cap S) < 2\varepsilon$ provided that $\varrho(C, D) < \varepsilon < 1$, from which the right hand inequality in (2.2) follows.

In the next sections we will often work with nested nets, that is, with the nets $\{C_r\}_{r>0}$ which satisfy the following condition

(2.3) $$0 < s < r \Rightarrow C_s \subset C_r .$$

Proposition 2.2. *Let $\{C_r\}_{r>0} \subset \mathscr{C}(Z)$ be a nested net and denote*

(2.4) $$C_0 = \bigcap_{r>0} \operatorname{cl} C_r .$$

Then the net has a conic limit if and only if $C_0 \neq \{0\}$ and

(2.5) $$\forall \varepsilon > 0 \quad \exists r > 0 \quad C_r \subset V_\varepsilon(C_0) .$$

In this case $C_0 = \operatorname*{c-lim}_{r \downarrow 0} C_r$.

Proof. Let $C_0 \neq \{0\}$ and let (2.5) be satisfied. From (2.4) it follows by Proposition 1.1 that

$$\forall \varepsilon > 0 \quad \forall r > 0 \quad C_0 \subset \operatorname{cl} C_r \subset V_\varepsilon(C_r) .$$

Now (2.5), (2.3) together with the definition of the conic limit yield that $C_0 =$ $= \operatorname*{c-lim}_{r \downarrow 0} C_r$.

Conversely, let D_0 be the conic limit of $\{C_r\}_{r>0}$. It means by the definition that $D_0 \neq \{0\}$ is closed and that

$$\forall \varepsilon > 0\ \exists r > 0\ \forall s \in (0, r]\ D_0 \subset V_\varepsilon(C_s)\ \&\ C_s \subset V_\varepsilon(D_0).$$

Now using (2.3), (2.4) and Proposition 1.1 we get

$$D_0 \subset \bigcap_{s>0} \bigcap_{\varepsilon>0} V_\varepsilon(C_s) = \bigcap_{s>0} \operatorname{cl} C_s = C_0 \subset \bigcap_{\varepsilon>0} \operatorname{cl} V_\varepsilon(D_0) = D_0.$$

Hence $C_0 \neq \{0\}$ and (2.5) holds.

For a nested net $\{C_r\}_{r>0}$ let C_0 be defined by (2.4). It may happen that $C_0 = \{0\}$ and so, by Proposition 2.2, the corresponding net has no conic limit. Such a net is constructed in Example 2.1. Even if $C_0 \neq \{0\}$, $\{C_r\}_{r>0}$ need not have a conic limit. This is shown in Example 2.2. Hence, the condition (2.5) is substantial.

Example 2.1. Let Z be a real separable Hilbert space with a total orthonormal system $\{e_n\}_1^\infty$ and set

$$C_r = \bigcup_{n \geqq 1/r} \mathbb{R}^+ e_n\,, \quad r > 0\,.$$

If $z \in \bigcap_{r>0} \operatorname{cl} C_r$, then $(z, e_n) = 0$ for all $n \in \omega$ and so $z = 0$. Thus according to Proposition 2.2, $\{C_r\}_{r>0}$ has no conic limit.

Example 2.2. Z being as in Example 2.1, set

$$C_r = \mathbb{R}^+ e_1 \cup \bigcup_{n \geqq 1/r} \mathbb{R}^+ e_n\,, \quad r > 0\,.$$

Then $\bigcap_{r>0} \operatorname{cl} C_r = \mathbb{R}^+ e_1$ but C_r is included in $V_1(\mathbb{R}^+ e_1)$ for no $r > 0$. Hence, by Proposition 2.2, $\{C_r\}_{r>0}$ has no conic limit.

Further examples will be given in Section 4. However, if Z is of finite dimension, a compactness argument yields the following

Proposition 2.3. *Every nested net of cones in a finite dimensional n.l.s. has a conic limit.*

Proposition 2.4 (*Cauchy condition*). *Let Z be a complete n.l.s.. Then a nested net $\{C_r\}_{r>0} \subset \mathscr{C}(Z)$ has a conic limit (equal to $\bigcap_{r>0} \operatorname{cl} C_r$) if and only if*

$$\forall \varepsilon > 0 \quad \exists r > 0 \quad \forall s \in (0, r] \quad C_r \subset V_\varepsilon(C_s)\,. \tag{2.6}$$

Proof. Thanks to (2.1)–(2.3), (2.6) reads as follows:

$$\forall \varepsilon > 0 \quad \exists r > 0 \quad \forall s, t \in (0, r] \quad d(C_s \cap S,\ C_t \cap S) < \varepsilon\,.$$

Now it suffices to use [11, Ch. 2. § 33, IV] and Proposition 2.1.

Proposition 2.5. *Let $C_0 \subset Z$ be the conic limit of a nested net $\{C_r\}_{r>0} \subset \mathscr{C}(Z)$. Let W be another n.l.s. and suppose that there exist $L \in \mathscr{L}(Z, W)$, $\alpha > 0$ and $\delta > 0$ such that*

$$\forall z \in C_\delta \quad \alpha\|z\| \leqq \|Lz\|\,.$$

Then

$$\operatorname{cl}(L(C_0)) = \operatorname{c-lim}_{r \downarrow 0} L(C_r)\,.$$

Proof. By Proposition 1.2, $L(C_r)$, $r \in (0, \delta]$, are cones in W. Proposition 2.2 and the continuity of L imply that $\alpha\|z\| \leqq \|Lz\|$ for all $z \in C_0$. Hence $L(C_0)$ is a cone as well. Now, let $\Delta > 0$ be arbitrary and put

$$\varepsilon = \frac{\alpha\Delta}{\|L\| + \alpha\Delta + \|L\|\,\Delta}\,.$$

If we take $r > 0$ so that

$$\forall s \in (0, r] \quad C_0 \subset V_\varepsilon(C_s) \,\&\, C_s \subset V_\varepsilon(C_0),$$

Proposition 1.2 then asserts that

$$\forall s \in (0, r] \quad L(C_0) \subset V_\Delta(L(C_s)) \,\&\, L(C_s) \subset V_\Delta(L(C_0)).$$

Hence, by Proposition 1.1, the result follows.

3. FRECHET CONES OF SETS

Definition (Durdil [5]). Let M be a subset of a n.l.s. Z such that $M' \neq \emptyset$. Choose z_0 in M' and, for each $r > 0$, set

$$C_r(M, z_0) = \{\lambda(z - z_0) | \; z \in M, \; \|z - z_0\| < r, \; \lambda \geqq 0\}.$$

Then the conic limit (if it exists)

$$\operatorname*{c-lim}_{r \downarrow 0} C_r(M, z_0)$$

is called the *Fréchet cone of the set M at the point z_0* and is denoted by $C_0(M, z_0)$.

The Fréchet cone of M at z_0 gives us an information about the behaviour of M in the vicinity of z_0. This concept, with unsubstantial changes and without calling it „the Fréchet cone", is due to Durdil [5]. In this paper, he used it to characterize geometrically the Fréchet differentiability in infinite dimensional n.l.s., see Theorem 4.1.

It follows immediately from the definition that the Fréchet cone is a local concept. From Propositions 2.2–2.4, we can at once derive the following three propositions.

Proposition 3.1. *Let $M \subset Z$, choose $z_0 \in M'$ and set*

$$T(M, z_0) = \bigcap_{r > 0} \operatorname{cl} C_r(M, z_0).$$

Then M has a Fréchet cone at z_0 if and only if

$$(3.1) \quad T(M, z_0) \neq \{0\} \quad \text{and} \quad \forall \varepsilon > 0 \;\; \exists r > 0 \;\; C_r(M, z_0) \subset V_\varepsilon(T(M, z_0)).$$

In this case, $C_0(M, z_0) = T(M, z_0)$.

Let us recall that the set $T(M, z_0)$ is called the tangent cone of M at z_0, see for instance [1], [12], [13]. In view of the above proposition, we can say that the Fréchet cone is a tangent one with the additional property (3.1). It should be noted that there are sets with $T(M, z_0) \neq \{0\}$ which, however, have no Fréchet cone, see Example 4.2.

Proposition 3.2. *If Z is a finite dimensional n.l.s., then $M \subset Z$ has a Fréchet cone at each $z_0 \in M'$.*

Proposition 3.3. (*Cauchy condition*). *Let Z be a complete n.l.s., $M \subset Z$ and $z_0 \in M'$.*

Then M has a Fréchet cone at z_0 if and only if

$$\forall \varepsilon > 0 \quad \exists r > 0 \quad \forall s \in (0, r] \quad C_r(M, z_0) \subset V_\varepsilon(C_s(M, z_0)) .$$

Next we shall attempt to find a relation between the Fréchet cone of a set M and that of the image of M under a mapping from Z into another n.l.s. W. We shall first consider a linear continuous mapping and then a nonlinear one, Fréchet differentiable at z_0.

Proposition 3.4. *Let Z, W be n.l.s., $M \subset Z$, $z_0 \in M'$ and suppose there exists $A \in \mathscr{L}(Z, W)$ such that*

$$(3.2) \quad \exists \alpha > 0 \quad \exists \delta > 0 \quad \forall z \in M \quad \|Az - Az_0\| < \delta \Rightarrow \alpha\|z - z_0\| \leqq \|Az - Az_0\| .$$

Then, if M has a Fréchet cone at z_0, then $A(M)$ has it at Az_0 and

$$(3.3) \qquad C_0(A(M), Az_0) = \mathrm{cl}(A(C_0(M, z_0))) .$$

Proof. Let M have a Fréchet cone at z_0. As A is continuous, (3.2) yields that $Az_0 \in A(M)'$. Also, (3.2) implies that

$$(3.4) \qquad \forall z \in M \quad \|z - z_0\| < \frac{\delta}{\alpha} \Rightarrow \alpha\|z - z_0\| \leqq \|Az - Az_0\| .$$

In fact, if there existed $z \in M$ fulfilling $\|z - z_0\| < \delta/\alpha$ and $\alpha\|z - z_0\| > \|Az - Az_0\|$, then it would imply $\|Az - Az_0\| < \delta$, and so, by (3.2), $\alpha\|z - z_0\| \leqq \|Az - Az_0\|$, which is a contradiction. (3.4) and the continuity of A imply that $\alpha\|z\| \leqq \|Az\|$ for all $z \in \mathrm{cl}\, C_r(M, z_0)$, where $r \in (0, \delta/\alpha)$. We can thus apply Proposition 2.8, which yields

$$(3.5) \qquad \mathrm{cl}(A(C_0(M, z_0))) = \operatorname*{c-lim}_{r \downarrow 0} A(C_r(M, z_0)) .$$

Denote $\beta = \|A\|$. Thanks to (3.2), $\beta > 0$. For each $r > 0$, we have

$$A(C_r(M, z_0)) = A\{\lambda(z - z_0) |\; z \in M,\ \|z - z_0\| < r,\ \lambda \geqq 0\} \subset$$
$$\subset \{\lambda(Az - Az_0) |\; Az \in A(M),\ \|Az - Az_0\| < \beta r,\ \lambda \geqq 0\} =$$
$$= C_{\beta r}(A(M), Az_0) ,$$

$$(3.6) \qquad \forall r > 0 \quad A(C_r(M, z_0)) \subset C_{\beta r}(A(M), Az_0) .$$

For each $r \in (0, \delta/\alpha)$, (3.2) yields

$$C_{\alpha r}(A(M), Az_0) = \{\lambda(w - Az_0) |\; w \in A(M),\ \|w - Az_0\| < \alpha r,\ \lambda \geqq 0\} \subset$$
$$\subset \{\lambda(Az - Az_0) |\; z \in M,\ \|z - z_0\| < r,\ \lambda \geqq 0\} = A(C_r(M, z_0)) ,$$

$$(3.7) \qquad \forall r \in (0, \delta/\alpha) \;\; C_{\alpha r}(A(M), Az_0) \subset A(C_r(M, z_0)).$$

Now, put $D = \text{cl}(A(C_0(M, z_0)))$ and let $\varepsilon > 0$ be arbitrary. (3.5) says that there exists $\delta_1 \in (0, \delta/\alpha)$ such that

$$\forall r \in (0, \delta_1) \;\; D \subset V_\varepsilon(A(C_r(M, z_0))) \;\&\; A(C_r(M, z_0)) \subset V_\varepsilon(D),$$

which, together with (3.6) and (3.7), yields that

$$\forall r \in (0, \delta_1) \qquad D \subset V_\varepsilon(C_{\beta r}(A(M), Az_0)) \;\&\; C_{\alpha r}(A(M), Az_0) \subset V_\varepsilon(D).$$

Hence, putting $\delta_2 = \delta_1 \min(\alpha, \beta)$, we have

$$\forall r \in (0, \delta_2) \;\; D \subset V_\varepsilon(C_r(A(M), Az_0)) \;\&\; C_r(A(M), Az_0) \subset V_\varepsilon(D).$$

That is,

$$\text{cl}(A(C_0(M, z_0))) = D = \operatorname*{c-lim}_{r \downarrow 0} C_r(A(M), Az_0) = C_0(A(M), Az_0).$$

The proof is thus completed.

If (3.2) is not fulfilled, (3.3) need not hold:

Example 3.1. Let $Z = W = \mathbb{R}^2$, $M = \{(t, \sqrt{|t|}) \mid t \in \mathbb{R}\}$, and define $A \in \mathscr{L}(Z, W)$ by

$$A(x, y) = (x, 0), \quad (x, y) \in Z.$$

Then

$$C_0(M, (0, 0)) = \{0\} \times [0, +\infty), \quad A(C_0(M, (0, 0))) = \{(0, 0)\}.$$

But $A(M) = \mathbb{R} \times \{0\}$ and so

$$C_0(A(M), (0,0)) = \mathbb{R} \times \{0\} \neq \{(0, 0)\}.$$

Corollary 3.1. *Let Z, W be n.l.s., $M \subset Z$, $z_0 \in M'$ and suppose there exists $A \in \text{Isom}(Z, W)$.*

Then M has a Fréchet cone at z_0 if and only if $A(M)$ has it at Az_0. In this case,

$$C_0(A(M), Az_0) = A(C_0(M, z_0)).$$

Corollary 3.2. *The Fréchet cone is independent of which equivalent norm in Z is taken.*

Theorem 3.1. *Let Z, W be n.l.s., $M \subset Z$, $M' \neq \emptyset$ and $z_0 \in M'$. Assume there exists a mapping $B : Z \to 2^W$, Fréchet differentiable at z_0, and such that the following two conditions are satisfied:*

$$(3.8) \qquad (z \in M,\; w \in Bz,\; w \to Bz_0) \Rightarrow z \to z_0,$$

$$(3.9) \qquad \begin{cases} \exists \alpha > 0 \;\; \exists \delta > 0 \;\; \forall z \in M \\ \|dB(z_0)(z - z_0)\| < \delta \Rightarrow \alpha\|z - z_0\| \leqq \|dB(z_0)(z - z_0)\|. \end{cases}$$

Then, if M has a Fréchet cone at z_0, then $B(M)$ has it at Bz_0 and

$$C_0(B(M), Bz_0) = \mathrm{cl}(\mathrm{d}B(z_0)(C_0(M, z_0))) .$$

Proof. As B is singlevalued and u.s.c. at z_0 (since it is Fréchet differentiable at z_0), it readily follows from (3.8) that $Bz_0 \in B(M)'$. Let $C_0(M, z_0)$ exist. For brevity denote $A = \mathrm{d}B(z_0)$. In virtue of Proposition 3.4, we know that $A(M)$ has a Fréchet cone at Az_0 and that

$$C_0(A(M), Az_0) = \mathrm{cl}(A(C_0(M, z_0))) .$$

So the proof will be complete when we show that

$$C_0(A(M), Az_0) = \operatorname*{c-lim}_{r \downarrow 0} C_r(B(M), Bz_0) . \tag{3.10}$$

For $r > 0$, define

$$\varphi(r) = \sup\left\{\frac{\|w - Bz_0 - (Az - Az_0)\|}{\|z - z_0\|} \,\middle|\, z_0 \neq z \in Z,\ \|z - z_0\| < r,\ w \in Bz\right\},$$

$$\psi(r) = \sup\{\|z - z_0\| \mid z \in M,\ w \in Bz,\ \|w - Bz_0\| < r\} .$$

Obviously, both φ and ψ are nondecreasing. The Fréchet differentiability of B at z_0 and (3.8) imply that $\psi(r) > 0$ for all $r > 0$ and

$$\lim_{r \downarrow 0} \varphi(r) = 0 , \quad \lim_{r \downarrow 0} \psi(r) = 0 . \tag{3.11}$$

Hence, there exists $\delta_1 \in (0, \delta)$ so that

$$\forall r \in (0, \delta_1) \quad \varphi(r/\alpha) < \alpha \ \&\ \psi(r) < \delta/\alpha . \tag{3.12}$$

Next, let $\varepsilon > 0$ be aribitrary. From (3.11), we can find $\delta_2 \in (0, \delta_1)$ so that

$$\forall r \in (0, \delta_2) \quad \frac{\varphi(r/\alpha)}{\alpha} < \varepsilon/2 \ \&\ 0 \leqq \frac{\varphi(\psi(r))}{\alpha - \varphi(\psi(r))} < \varepsilon/2 . \tag{3.13}$$

Now, fix $r \in (0, \delta_2)$ and choose $\lambda(Az - Az_0) \neq 0$ arbitrarily in $C_r(A(M), Az_0)$. Then $\|Az - Az_0\| < r < \delta$, and therefore, by (3.9), $\|z - z_0\| < r/\alpha$. Thus, using (3.12), (3.13) and the definition of φ, we can estimate

$$\begin{aligned}\|w - Bz_0\| &\leqq \|Az - Az_0\| + \|w - Bz_0 - (Az - Az_0)\| < \\ &< r + \varphi(\|z - z_0\|)\,\|z - z_0\| \leqq r + \varphi(r/\alpha)\,r/\alpha < 2r ,\\ \|Az - Az_0 - (w - Bz_0)\| &\leqq \varphi(\|z - z_0\|)\,\|z - z_0\| \leqq \\ &\leqq \frac{\varphi(r/\alpha)}{\alpha}\,\|Az - Az_0\| < \frac{\varepsilon}{2}\,\|Az - Az_0\|\end{aligned}$$

for each $w \in Bz$. Hence

$$\lambda(Az - Az_0) \in V_{\varepsilon/2}(\{\mu(w - Bz_0) | \ \mu \geqq 0\}) \subset V_{\varepsilon/2}(C_{2r}(B(M), Bz_0)),$$

i.e.,

$$(3.14) \qquad \forall r \in (0, \delta_2) \quad C_r(A(M), Az_0) \subset V_{\varepsilon/2}(C_{2r}(B(M), Bz_0)).$$

Further, for fixed $r \in (0, \delta_2)$, take $\lambda(w - Bz_0) \neq 0$ arbitrarily in $C_r(B(M), Bz_0)$. It follows that $\|w - Bz_0\| < r$. Choosing z in $B^{-1}w$, we have $\|z - z_0\| \leqq \psi(r) < \delta/\alpha$ by (3.12). Hence, by (3.4) (which follows from (3.9)), $\|Az - Az_0\| \geqq \alpha\|z - z_0\| > 0$. Thus, by (3.13), we can estimate

$$\|Az - Az_0\| \leqq \|A\| \, \|z - z_0\| \leqq \|A\| \, \psi(r),$$

$$\|w - Bz_0\| \geqq \|Az - Az_0\| - \|w - Bz_0 - (Az - Az_0)\| \geqq$$

$$\geqq \alpha\|z - z_0\| - \varphi(\|z - z_0\|) \, \|z - z_0\| \geqq (\alpha - \varphi(\psi(r))) \, \|z - z_0\| > 0,$$

$$\|w - Bz_0 - (Az - Az_0)\| \leqq \varphi(\|z - z_0\|) \, \|z - z_0\| \leqq$$

$$\leqq \frac{\varphi(\psi(r))}{\alpha - \varphi(\psi(r))} \|w - Bz_0\| < \frac{\varepsilon}{2} \|w - Bz_0\|.$$

Thus, denoting $\chi(r) = \|A\| \, \psi(r)$, $r > 0$, we obtain

$$\lambda(w - Bz_0) \in V_{\varepsilon/2}(\{\mu(Az - Az_0) | \ \mu \geqq 0\}) \subset V_{\varepsilon/2}(C_{\chi(r)}(A(M), Az_0)).$$

i.e.,

$$(3.15) \qquad \forall r \in (0, \delta_2) \quad C_r(B(M), Bz_0) \subset V_{\varepsilon/2}(C_{\chi(r)}(A(M), Az_0)).$$

Finally, denote $D = C_0(A(M), Az_0)$ and find $\delta_3 \in (0, \delta_2)$ so that

$$(3.16) \quad \forall r \in (0, \delta_3) \quad D \subset V_{\varepsilon/2}(C_r(A(M), Az_0)) \; \& \; C_r(A(M), Az_0) \subset V_{\varepsilon/2}(D).$$

(3.11) implies that there exists $\delta_4 \in (0, \delta_3)$ such that $\chi(r) < \delta_3$ for $r \in (0, \delta_4)$. Combining (3.14) and (3.15) with (3.16), we thus obtain, by Proposition 1.1 (iii), that

$$\forall r \in (0, \delta_4) \begin{cases} D \subset V_{\varepsilon/2}(C_r(A(M), Az_0)) \subset V_\varepsilon(C_{2r}(B(M), Bz_0)) \\ C_r(B(M), Bz_0) \subset V_{\varepsilon/2}(C_{\chi(r)}(A(M), Az_0)) \subset V_\varepsilon(D). \end{cases}$$

According to Proposition 2.4, this means that (3.10) holds and the proof is complete.

Corollary 3.3. *Let Z, W be n.l.s., $M \subset Z$, $z_0 \in M'$. Let $B : Z \to W$ map a neighbourhood of z_0 onto a neighbourhood of Bz_0 homeomorphically. Moreover, assume that B is Fréchet differentiable at z_0 and that $dB(z_0) \in$ Isom (Z, W).*

Then M has a Fréchet cone at z_0 if and only if $B(M)$ has it at Bz_0. In this case,

$$C_0(B(M), Bz_0) = dB(z_0)\,(C_0(M, z_0)).$$

It should be noted that, for tangent and other similar kinds of cones, results like Theorem 3.1 can be found in [12], [13].

Let X, Y be n.l.s. and put $Z = X \times Y$. Then each nonempty subset of Z is a (multivalued) mapping from X into Y. It leads us to the following

Definition. Let $F : X \to 2^Y$ be a mapping with $\operatorname{int} D(F) \neq \emptyset$ and choose $x_0 \in \operatorname{int} D(F)$. Let F be singlevalued and u.s.c. at x_0 and suppose that (the set) $F \subset X \times Y$ has a Fréchet cone at the point (x_0, Fx_0).

Then we call it the *Fréchet cone of the mapping F at the point x_0* and write $C_0(F, x_0)$ instead of $C_0(F, (x_0, Fx_0))$.

As immediate consequences of Propositions 3.1–3.3 and the above definition, we get

Proposition 4.1. *Let $F : X \to 2^Y$ be singlevalued and u.s.c. at some $x_0 \in \operatorname{int} D(F)$ and denote*

$$T(F, x_0) = \bigcap_{r>0} \operatorname{cl} C_r(F, (x_0, Fx_0)) . \tag{4.1}$$

Then the mapping F has a Fréchet cone at x_0 if and only if

$$T(F, x_0) \neq \{0\} \quad \text{and} \quad \forall \varepsilon > 0 \;\; \exists r > 0 \;\; C_r(F, (x_0, Fx_0)) \subset V_\varepsilon(T(F, x_0)) . \tag{4.2}$$

In this case, $C_0(F, x_0) = T(F, x_0)$.

The set $T(F, x_0)$ is called the tangent cone of the mapping F at x_0, see, for instance, [1], [9]. Hence, the Fréchet cone of a mapping is the tangent one with the property (4.2).

Proposition 4.2. *Let X, Y be finite dimensional n.l.s. and let $F : X \to 2^Y$ be singlevalued and u.s.c. at $x_0 \in \operatorname{int} D(F)$.*

Then F has a Fréchet cone at x_0.

Proposition 4.3 (*Cauchy condition*). *Let X, Y be complete n.l.s. and let $F : X \to 2^Y$ be singlevalued and u.s.c. at $x_0 \in \operatorname{int} D(F)$.*

Then F has a Fréchet cone at x_0 if and only if

$$\forall \varepsilon > 0 \;\; \exists r > 0 \;\; \forall s \in (0, r] \;\; C_r(F, (x_0, Fx_0)) \subset V_\varepsilon(C_s(F, (x_0, Fx_0))) .$$

In the following two examples, mappings which have no Fréchet cone at some points are constructed. The second one is a slight modification of [4, Example (2.2)]; see also [10].

Example 4.1. Let Y be a real separable Hilbert space with a total orthonormal system $\{f_n\}_1^\infty$. The mapping $F : \mathbb{R} \to Y$ is defined as follows:

$$F0 = 0, \quad Fx = f_1 \;\text{ for }\; |x| > 1, \quad Fx = |x| f_n \;\text{ for }\; |x| \in \left(\frac{1}{n+1}, \frac{1}{n} \right], \quad n \in \omega .$$

Clearly F is continuous at 0. It is easy to check that

$$C_{1/n}(F, (0, F0)) = \bigcup_{i=n}^{\infty} \{(t, tf_i) \mid t \in R\},$$

the last cone being closed. If $z \in T(F, 0)$, then, by (4.1),

$$z \in C_{1/n}(F, (0, F0)) \subset \operatorname{sp} \{(1, f_n), (1, f_{n+1}), \ldots\} \quad \text{for all} \quad n \in \omega .$$

It follows that $z = (0, 0)$ and so $T(F, 0) = \{(0, 0)\}$. Thus, owing to Proposition 4.1, F cannot have a Fréchet cone at 0.

Example 4.2. Let Y and $\{f_n\}_1^\infty$ be as in the above example and define the mapping $G : R \to Y$ by

$$G\left(\pm \frac{1}{n}\right) = \frac{1}{n} f_n, \quad n \in \omega, \quad Gx = 0 \quad \text{otherwise}.$$

Obviously, G is continuous at 0 and we have

$$C_{1/n}(G, (0, G0)) = C_{1/(n+1)}(F, (0, F0)) \cup \{(t, 0) \mid t \in R\}$$

for all $n \in \omega$ and thus $T(G, 0) = \{(t, 0) \mid t \in R\}$. But $V_1(T(G, 0))$ does not contain $C_{1/n}(G, (0, G0))$ for any $n \in \omega$ and so, by Proposition 4.1, G has no Fréchet cone at 0.

In order to clarify the relation between the Fréchet cones and the Fréchet differentiability, let us recall

Theorem 4.1 (*Durdil* [5]). *A mapping $F : X \to 2^Y$ is Fréchet differentiable at $x_0 \in \operatorname{int} D(F)$ if and only if $C_0(F, x_0)$ exists and (considered as a mapping from X into Y) is an element of $\mathscr{L}(X, Y)$. In this case*

$$\mathrm{d}F(x_0) = C_0(F, x_0).$$

In [5], this theorem was proved for singlevalued mappings. It can be checked, however, that the proof, after small alterations, applies to multivalued mappings as well. For another proof of this theorem we refer the reader to [6].

Theorem of Durdil suggests that the Fréchet cone is a natural generalization of the Fréchet differential. Example 4.2 shows that, in Theorem 4.1, we must not replace $C_0(F, x_0)$ by the tangent cone $T(F, x_0)$. Here $T(G, 0) = \{(t, 0) \mid t \in R\}$ is an element of $\mathscr{L}(R, Y)$ but G is not Fréchet differentiable at 0 (which follows from Theorem 4.1 and from the fact that G has no Fréchet cone at 0). It means that tangent cones, although lying in $\mathscr{L}(X, Y)$, need not always be Fréchet differentials in the case of infinite dimensional spaces. And since our aim is to develop a theory that would include differential calculus, we should prefer the Fréchet cones.

Now the question whether the Fréchet cones can be handled like the Fréchet differentials arises. The answer is often affirmative as is shown in the following two sections.

5. CALCULUS WITH FRECHET CONES

We shall now prove statements concerning the Fréchet cones of the inverses, linear combinations, and compositions of mappings. They will be derived from Corollary 3.1 and Theorem 3.1 by a suitable choice of the spaces Z, W and the mappings $A \in \text{Isom}(X, Y)$ and $B : Z \to 2^W$. As corrollaries, we shall then obtain, with help of Theorem 4.1, some classical rules from differential calculus in n.l.s. It should be noted that the intended program is similar to that of [6, §3].

Let X, Y, U be n.l.s. and let $F : X \to 2^Y$ be a mapping. We shall consider the following four cases:

Case 1. $Z = X \times Y$, $W = Y \times X$. The mapping $A : Z \to W$ is defined by

$$A(x, y) = (y, x), \quad (x, y) \in X \times Y.$$

Then $A(F) = F^{-1}$, $A \in \text{Isom}(Z, W)$ and Corollary 3.1 yields

Theorem 5.1. *Let $F : X \to 2^Y$ have a Fréchet cone at some $x_0 \in X$. Suppose that the inverse mapping $F^{-1} : Y \to 2^X$ is singlevalued and u.s.c. at Fx_0 and let $Fx_0 \in$ $\in \text{int}\, R(F)$.*

Then the mapping F^{-1} has a Fréchet cone at Fx_0 and

$$C_0(F^{-1}, Fx_0) = (C_0(F, x_0))^{-1}.$$

Corollary 5.1. *Let $F : X \to 2^Y$ be Fréchet differentiable at $x_0 \in X$. Suppose that $Fx_0 \in \text{int}\, R(F)$, $\mathrm{d}F(x_0) \in \text{Isom}(X, Y)$ and that $F^{-1} : Y \to 2^X$ is singlevalued and u.s.c. at Fx_0.*

Then F^{-1} is Fréchet differentiable at Fx_0 and

$$\mathrm{d}F^{-1}(Fx_0) = (\mathrm{d}F(x_0))^{-1}.$$

Case 2. $Z = X \times Y$, $W = X \times Y$. Let $0 \neq \lambda \in \mathbb{R}$ be given and define $A : Z \to W$ by

$$A(x, y) = (x, \lambda y), \quad (x, y) \in X \times Y.$$

Then $A(F) = \lambda F$ (see (0.1)) and $A \in \text{Isom}(Z, W)$. It should be noted that λF is different from the λ-multiple of the set F in the space Z! Corollary 3.1 implies

Theorem 5.2. *If $F : X \to 2^Y$ has a Fréchet cone at some $x_0 \in X$, then $F : X \to 2^Y$ has it as well and*

$$C_0(\lambda F, x_0) = \lambda\, C_0(F, x_0) \quad [= \{(x, \lambda y) \mid (x, y) \in C_0(F, x_0)\}].$$

Corollary 5.2. *If $F : X \to 2^Y$ is Fréchet differentiable at $x_0 \in X$, then so is λF and*

$$\mathrm{d}(\lambda F)(x_0) = \lambda\, \mathrm{d}F(x_0).$$

Case 3. $Z = X \times Y$, $W = X \times Y$. Let a mapping $G : X \to 2^Y$ be given and define $B : Z \to 2^W$ as follows;

$$B(x, y) = \{(x, y + v) |\ v \in Gx\}\,, \quad (x, y) \in D(G) \times Y\,.$$

Then $B(F) = F + G$ (see (0.2)). Let us note that $F + G$ is different from the sum of the sets F and G in the space Z! Further, suppose that G is Fréchet differentiable at some $x_0 \in X$. A simple computation yields that B is Fréchet differentiable at each point (x_0, y), $y \in Y$, and that

$$\text{(5.1)} \qquad \mathrm{d}B(x_0, y)(h, k) = (h, k + \mathrm{d}G(x_0)\,h)\,, \quad (h, k) \in X \times Y\,.$$

Theorem 5.3. *Let $F : X \to 2^Y$ have a Fréchet cone at some $x_0 \in X$ and let $G : X \to 2^Y$ be Fréchet differentiable at x_0.*

Then the mapping $F + G : X \to 2^Y$ has a Fréchet cone at x_0 and

$$\text{(5.2)} \qquad C_0(F + G, x_0) = C_0(F, x_0) + \mathrm{d}G(x_0)\,.$$
$$.\,[= \{(x, y + \mathrm{d}G(x_0)\,x) |\ (x, y) \in C_0(F, x_0)\}]\,.$$

Proof. In order to be able to apply Theorem 3.1, we have to verify (3.8) and (3.9) for $z_0 = (x_0, Fx_0)$. Let

$$z \in F\,, \quad w \in Bz\,, \quad w \to Bz_0 = (x_0, Fx_0 + Gx_0)\,,$$

where $z = (x, y)$, $w = (x, y + v)$, $(x, v) \in G$. Hence $x \to x_0$ and, from the single-valuedness and u.s.c. of F at x_0, it follows that $y \to Fx_0$. Thus, $z = (x, y) \to (x_0, Fx_0) = z_0$ as (3.8) asserts. (3.9) will be proved by contradiction. Let it be false. Then we can find a sequence $\{z_n\} = \{(x_n, y_n)\} \subset F$, $z_n \neq z_0$, $n \in \omega$, such that

$$\mathrm{d}B(z_0)(z_n - z_0) \to 0 \;\&\; \frac{\|\mathrm{d}B(z_0)(z_n - z_0)\|}{\|z_n - z_0\|} \to 0\,.$$

Hence, by (5.1), $x_0 \neq x_n \to x_0$ and

$$\frac{\max(\|x_n - x_0\|, \|y_n - Fx_0 + \mathrm{d}G(x_0)(x_n - x_0)\|)}{\max(\|x_n - x_0\|, \|y_n - Fx_0\|)} \to 0\,,$$

which implies

$$\frac{\|x_n - x_0\|}{\|y_n - Fx_0\|} \to 0\,, \quad \frac{\|y_n - Fx_0 + \mathrm{d}G(x_0)(x_n - x_0)\|}{\|y_n - Fx_0\|} \to 0\,.$$

Hence

$$0 = \lim_{n\to\infty} \frac{\|y_n - Fx_0 + \mathrm{d}G(x_0)(x_n - x_n)\|}{\|y_n - Fx_0\|} \geq \lim_{n\to\infty}\left(1 - \|\mathrm{d}G(x_0)\| \frac{\|x_n - x_0\|}{\|y_n - Fx_0\|}\right) = 1\,;$$

which is impossible. Thus (3.9) must hold.

Now, applying Theorem 3.1, we obtain that at the point $Bz_0 = (x_0, Fx_0 + Gx_0)$ the set $B(F) = F + G \subset X \times Y$ has the following Fréchet cone:

$$C_0(F + G, Bz_0) = \mathrm{cl}(\mathrm{d}B(z_0)(C_0(F, z_0))).$$

But this set is equal to $C_0(F, z_0) + \mathrm{d}G(x_0)$. Indeed, (5.1) and (0.2) yield

$$\mathrm{d}B(z_0)(C_0(F, z_0)) = C_0(F, z_0) + \mathrm{d}G(x_0),$$

and this cone can be shown to be closed. In fact, let

$$\{(x_n, y_n + \mathrm{d}G(x_0)\, x_n)\} \subset C_0(F, z_0) + \mathrm{d}G(x_0)$$

be a sequence converging to some $(x, y) \in X \times Y$. It follows that

$$\mathrm{d}G(x_0)\, x_n \to \mathrm{d}G(x_0)\, x, \quad y_n \to y - \mathrm{d}G(x_0)\, x.$$

Since $C_0(F, z_0)$ is closed by definition, $(x, y - \mathrm{d}G(x_0)\, x)$ lies in $C_0(F, z_0)$, i.e., $(x, y) \in C_0(F, z_0) + \mathrm{d}G(x_0)$. Therefore,

$$C_0(F + G, (x_0, Fx_0 + Gx_0)) = C_0(F, (x_0, Fx_0)) + \mathrm{d}G(x_0).$$

Finally, $x_0 \in \mathrm{int}\, D(F + G)$ and $F + G$ is singlevalued and u.s.c. at x_0. Thus the mapping $F + G$ has a Fréchet cone at x_0 and (5.2) holds.

In the above theorem, the Fréchet differentiability of the mapping G cannot be replaced by the existence of its Fréchet cone. Indeed, taking $X = Y = \mathbb{R}$, $x_0 = 0$, $Fx = x \sin(x^{-1})$, $x \neq 0$, $F0 = 0$, $G = -F$, we can see that

$$C_0(F, 0) = C_0(G, 0) = \{(x, y) \in \mathbb{R}^2 \mid |y| \leqq |x|\},$$

$$C_0(F, 0) + C_0(G, 0) = \{(x, y) \in \mathbb{R}^2 \mid |y| \leqq 2|x|\}.$$

But $C_0(F + G, 0) = \mathbb{R} \times \{0\}$.

Corollary 5.3. *If $F, G : X \to 2^Y$ are Fréchet differentiable at $x_0 \in X$, then so is $F + G : X \to 2^Y$ and*

$$\mathrm{d}(F + G)(x_0) = \mathrm{d}F(x_0) + \mathrm{d}G(x_0).$$

Case 4. $Z = X \times Y$, $W = Y \times U$. Let $G : Y \to 2^U$ be a given mapping and define $B : Z \to 2^W$ by

$$B(x, y) = \{(x, u) \mid (y, u) \in G\}, \quad (x, y) \in X \times D(G).$$

Then $B(F) = G \circ F$ (see (0.3)) is a mapping from X into U. Assume that G is Fréchet differentiable at some $y_0 \in Y$. It is easy to check that B is also Fréchet differentiable at each (x, y_0), $x \in X$, and that

$$\mathrm{d}B(x, y_0)(h, k) = (h, \mathrm{d}G(y_0)\, k), \quad (h, k) \in X \times Y. \tag{5.3}$$

Theorem 5.4 (*Chain rule*). *Let $F : X \to 2^Y$ have a Fréchet cone at $x_0 \in X$ and let $G : Y \to 2^U$ be Fréchet differentiable at $y_0 = Fx_0$. Moreover, for each sequence $\{(x_n, y_n)\} \subset F$ with $x_0 \neq x_n \to x_0$, let the following implication hold:*

$$\frac{\|y_n - y_0\|}{\|x_n - x_0\|} \to +\infty \Rightarrow \liminf_{n \to \infty} \frac{\|dG(y_0)(y_n - y_0)\|}{\|y_n - y_0\|} > 0 . \tag{5.4}$$

Then the mapping $G \circ F : X \to 2^U$ has a Fréchet cone at x_0 and

$$C_0(G \circ F, x_0) = \mathrm{cl}\,(dG(y_0) \circ C_0(F, x_0)) . \tag{5.5}$$

Proof. The validity of (3.8) and (3.9) can be verified from (5.4) in a similar way as in the proof of Theorem 5.3. Thus, we can apply Theorem 3.1, which asserts that at the point $B(x_0, y_0) = (x_0, Gy_0)$, the set $G \circ F \subset X \times U$ has the Fréchet cone

$$C_0(G \circ F, (x_0, Gy_0)) = \mathrm{cl}(dB(x_0, y_0)(C_0(F, (x_0, y_0)))) .$$

But according to (5.3) and (0.3),

$$C_0(G \circ F, (x_0, Gy_0)) = \mathrm{cl}(dG(y_0) \circ C_0(F, (x_0, y_0))) .$$

Hence, bearing in mind that $G \circ F$ is singlevalued and u.s.c. at $x_0 \in \mathrm{int}\, D(G \circ F)$, we get the result.

If (5.4) is not satisfied, (5.5) need not hold. We shall demonstrate this on the situation which corresponds to Example 3.1. Take $X = Y = U = \mathbb{R}$, $Fx = \sqrt{|x|}$, $x \in X$, $Gy = 0$, $y \in Y$. Then (5.4) is violated. Further,

$$C_0(F, 0) = \{0\} \times [0, +\infty) , \quad C_0(G \circ F, 0) = \mathbb{R} \times \{0\} ,$$
$$G \circ C_0(F, 0) = \{(0, 0)\} .$$

Hence, (5.5) is false.

Moreover, the Fréchet differentiability of G in the above theorem cannot be replaced by the existence of the Fréchet cone. To check it, put $X = Y = U = \mathbb{R}$ and let F be a one-to-one mapping of X onto Y, with $F0 = 0$, not Fréchet differentiable at 0. Now it suffices to take $G = F^{-1}$.

Corollary 5.4 (*Chain rule*). *Let $F : X \to 2^Y$ be Fréchet differentiable at $x_0 \in X$ and let $G : Y \to 2^U$ be Fréchet differentiable at Fx_0.*

Then $G \circ F : X \to 2^U$ is Fréchet differentiable at x_0 and

$$d(G \circ F)(x_0) = dG(Fx_0) \circ dF(x_0) .$$

We shall use the following

Lemma 6.1 ([14, II.3.5]). *Let Y be a n.l.s., $\beta \geqq 0$ and $f : \mathbb{R} \to Y$ a singlevalued mapping with $[0, 1] \subset \operatorname{int} D(f)$, such that*

$$\forall t \in [0, 1] \quad \exists \Delta(t) > 0 \quad |s - t| < \Delta(t) \Rightarrow \|f(s) - f(t)\| \leqq \beta |s - t| .$$

Then

$$\|f(1) - f(0)\| \leqq \beta .$$

Theorem 6.1 (*Mean value theorem*). *Let X, Y be n.l.s. and $F : X \to Y$ a singlevalued mapping with a convex open domain $D(F)$, having a Fréchet cone at each point of $D(F)$.*

Then F is Lipschitzian with a constant $\gamma \geqq 0$ if and only if (*see* (1.3))

$$(6.1) \qquad \forall x \in D(F) \quad \|C_0(F, x)\|_X \leqq \gamma .$$

Proof. If F is Lipschitzian with a constant γ, then

$$\|\mathrm{cl}\, C_r(F, (x, Fx))\|_Y \leqq \gamma$$

for all $r > 0$ and so Proposition 4.1 yields (6.1).

Conversely, let (6.1) be satisfied. Fix arbitrary $x_0, x_1 \in D(F)$, $x_0 \neq x_1$. We have to show that

$$(6.2) \qquad \|Fx_1 - Fx_0\| \leqq \gamma \|x_1 - x_0\| .$$

Let $\varepsilon \in (0, 1/(1 + \gamma))$ be given. To every $t \in [0, 1]$, we can find $\delta(t) > 0$ such that

$$(6.3) \qquad C_{\delta(t)}(F, (x_t, Fx_t)) \subset V_\varepsilon(C_0(F, x_t)) ,$$

where

$$x_t = x_0 + t(x_1 - x_0) .$$

Since, by definition, F is continuous at x_t, there exists $\Delta(t) > 0$ so that $|s - t| < \Delta(t)$ implies $x_s \in D(F)$ and

$$\|(x_s - x_t, Fx_s - Fx_t)\| < \delta(t) .$$

Hence, by (6.3), $(x_s - x_t, Fx_s - Fx_t)$ belongs to $V_\varepsilon(C_0(F, x_t))$. Thus we get from (1.3), (6.1) and Proposition 1.4 that

$$\forall t \in [0, 1] \quad \exists \Delta(t) > 0 \quad |s - t| < \Delta(t) \Rightarrow$$
$$\Rightarrow \|Fx_s - Fx_t\| \leqq \gamma(\varepsilon) \|x_s - x_t\| = \gamma(\varepsilon) \|x_1 - x_0\| |s - t| ,$$

where

$$\gamma(\varepsilon) = \max\left(\gamma + \gamma\varepsilon + \varepsilon\,, \frac{1}{1 - \varepsilon - \gamma\varepsilon}\right). \tag{6.4}$$

Hence, if we take $f(t) = Fx_t$ and $\beta = \gamma(\varepsilon)\,\|x_1 - x_0\|$ in Lemma 6.1, we get

$$\|Fx_1 - Fx_0\| \leqq \gamma(\varepsilon)\,\|x_1 - x_0\|\,. \tag{6.5}$$

Finally, we can see from (6.4) that, for $\varepsilon \downarrow 0$, $\gamma(\varepsilon)$ converge to γ and so (6.5) reduces to (6.2), which was to prove.

Corollary 6.1. *Let $F : X \to Y$ have a Fréchet cone at each point of a neighbourhood of some $x_0 \in X$ and suppose there exists $L \in \mathscr{L}(X, Y)$ such that (see (1.3))*

$$\|C_0(F, x) - L\|_X \to 0 \quad \text{as} \quad x \to x_0\,. \tag{6.6}$$

Then F is strongly Fréchet differentiable at x_0 and $\mathrm{d}F(x_0) = L$, that is,

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall u \in X \quad \forall v \in X$$

$$(\|u - x_0\| < \delta,\ \|v - x_0\| < \delta) \Rightarrow \|Fu - Fv - L(u - v)\| \leqq \varepsilon\|u - v\|\,.$$

Proof. Let $\varepsilon > 0$ be given. By (6.6), there is $\delta > 0$ such that

$$\|x - x_0\| < \delta \Rightarrow \|C_0(F, x) - L\|_X < \varepsilon\,.$$

But, according to Theorem 5.3, $C_0(F, x) - L = C_0(F - L, x)$. Hence Theorem 6.1 yields the result.

Corollary 6.2. If *$F : X \to Y$ is continuously Fréchet differentiable at some $x_0 \in X$, then it is strongly Fréchet differentiable at x_0.*

Proof. Use Corollary 6.1 and Theorem 4.1.

CONCLUDING REMARKS

The Fréchet cones are useful in generalizing such theorems in which the Fréchet differentiability plays an active rôle; that is to say, theorems of the following type: "If F is Fréchet differentiable at x and $\mathrm{d}F(x)$ has some properties, then … ." As Examples of such a generalization, let us mention the so called mapping theorems [7, Section 7] and the exponential stability of differential inclusions [8].

It should be remarked that, as a generalization of the Gâteaux differential, a Gâteaux cone can be introduced and a theory analogous to that involving the Fréchet cones can be developed, see [7].

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Author's address: 128 00 Praha 2, Horská 3 (Ústav výpočtové techniky ČVUT).