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EXTENSION OF MEASURE-LIKE SET FUNCTIONS

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In the classical measure theory the following extension theorem is well-known:

A non-negative, σ-additive real measure, defined on a ring of sets, has a unique extension to a measure, defined on the σ-ring generated by the original ring.

An examination of the extension procedure reveals the fact that a real-valued measure on a ring could be replaced by a set function with values in a more general space, having the linearity and order — or topological — properties of real numbers. With respect to the range space of the measure there are two main lines of generalization, one in the direction of linear topological spaces, the other using the order character of some linear spaces.

Recently, some authors (see [1], [2], [3] and [8]) have suggested another aspect of generalization. They are dealing with an extension of a real, finite, subadditive (not necessarily additive) measure.

The aim of this paper is the construction of extensions of certain non-negative measure-like set functions (namely: measure, submeasure, strong submeasure and strong supermeasure) from a ring $\mathscr{R}$ to the σ-ring (or δ-ring) generated by $\mathscr{R}$, where the range space of μ is a partially ordered group G.

For proving such an extension theorem one always needs some sort of completeness of the range space of the measure. (If, e.g., $\mathscr{R}$ is a ring of subsets which are finite disjoint unions of intervals with rational endpoints, G is the group of rational numbers, and $\mu : \mathscr{R} \to G$ is the restriction of the Borel measure to $\mathscr{R}$, then μ has no extension to the generated σ-ring.) Our completeness property will be the so called monotone σ-completeness (for definitions see below).

The second main tool we will use is the assumption that the family of additive, monotone, and order continuous functionals on G is "large enough", i.e., it separates points of G. If this is the case we say that G is separative.

Using this assumption we are able to transfer the convergence and related properties from the original group to the space of real numbers, where the measure extension is known.

Our assumption about separativeness of G seems to be too strong. As Floyd [4] pointed out, there exists a monotone complete linear space X (this is clearly a group) which is not Hausdorff in any vector topology compatible with the order. As we shall see (Theorem 3), a separative group has always a Hausdorff group topology, which is σ-compatible with order. So our result gives only a partial solution of the extension problem for measures and measure-like set functions with values in a partially ordered group. Our result about the extension of an additive measure is unrelated to [11] and [12]. In [11] G is assumed to be g-regular and o-separable, and in [12] the range space of μ is a vector lattice.

1. DEFINITIONS AND PRELIMINARY RESULTS

A *partially ordered group* is a set G endowed with a structure of a partially ordered space and a structure of a group satisfying the following compatibility condition:
If x, y and z are in G and $x \leqq y$, then

$$x + z \leqq y + z .$$

Θ will denote the neutral element of G. By G_+ we denote the set of all non-negative elements in G.

1. Lemma. *Let G be a partially ordered group. Let $x_n \nearrow x$ ($x_n \searrow x$) and $y_n \nearrow y$ ($y_n \searrow y$). Then*

(*i*) $x_n + y_n \nearrow x + y$ ($x_n + y_n \searrow x + y$),
(*ii*) $-x_n \searrow -x$ ($-x_n \nearrow -x$).

We say that a sequence $\{x_n\}$ of elements of a paratially ordered group G *converges in order* to x (in symbols $x_n \rightarrow x$ or $x_0 \xrightarrow{o} x$) iff there exist sequences $\{u_n\}$ and $\{v_n\}$ in G with $u_n \leqq x_n \leqq v_n$ and $u_n \nearrow x \swarrow v_n$.

An easy consequence of the above lemma and the definition of the order convergence is the following

2. Theorem. *Let G be a partially ordered group. Then G is a convergence group with respect to the order convergence (i.e., the map $(x, y) \mapsto x - y$ is order continuous).*

If G is a partially ordered group, then by the *order dual* of G we mean the set $G^<$ of all order continuous additive functionals on G which can be represented as a difference of two monotone additive functionals.

For our extension process one of the basic assumptions is that $G^<$ separates points of G, i.e., for $x \in G$, $x \neq \Theta$, there exists an $x^<$ in $G^<$ with $x^<(x) \neq 0$. If this is the case, we shall say that G is *separative*. It is easy to see that $G^<$ separates points of G

if and only if the set of all monotone elements from $G^<$ (denoted by $G_+^<$) separates points of G. The ordering on $G^<$ is defined as follows:

$$x^< \leqq y^< \quad \text{iff} \quad y^< - x^< \in G_+^< .$$

We say that a partially ordered group G is *upward filtering* if to any x and y in G there exists z in G with $x \leqq z$ and $y \leqq z$.

Let now G be a separative upward filtering group. Similarly as in [10] one can prove that G can be embedded into its second dual $G^{<<}$ (the dual of $G^<$) in the following way: Let x be in G. We define a real valued map ξ_x on $G^<$ by

$$\xi_x(x) = x^<(x) .$$

The map $x \mapsto \xi_x$ embeds the group G into its second dual.

It is easy to see that a separative group is always a Hausdorff topological group with respect to the $G^<$ – weak topology (the coarsest topology on G in which every $x^< \in G^<$ is continuous). Moreover, the following theorem holds:

3. Theorem. *If G is a separative group, then there exists on G a Hausdorff group topology σ-compatible with the ordering (namely, the $G^<$ – weak topology), such that every additive continuous functional on G is order continuous.*

Our completeness property is the following: We say that a partially ordered group G is *monotone σ-complete* if every monotone increasing bounded sequence $\{x_n\}$ has a limit in G, i.e.,

$$\lim_n x_n = \bigvee_n x_n$$

exists in G.

By $\mathcal{S}(\mathcal{R})$ ($\mathcal{D}(\mathcal{R})$) we denote the σ-ring (δ-ring) generated by $\mathcal{R}$.

Let G be a partially ordered group. By $\bar{G}$ we denote the semigroup $G \cup \{\infty\}$, where $g + \infty = \infty + g = \infty$ and $g < \infty$ for every g in G.

We say that $\mu : \mathcal{R} \to \bar{G}$ is

(i) additive if

$$\mu(A \cap B) + \mu(A \cup B) = \mu(A) + \mu(B) ,$$

(ii) subadditive if

$$\mu(A \cup B) \leqq \mu(A) + \mu(B) ,$$

(iii) strongly subadditive if

$$\mu(A \cap B) + \mu(A \cup B) \leqq \mu(A) + \mu(B) ,$$

(iv) superadditive if

$$\mu(A \cup B) \geqq \mu(A) + \mu(B) ,$$

(v) strongly superadditive if

$$\mu(A \cap B) + \mu(A \cup B) \geqq \mu(A) + \mu(B),$$

(vi) monotone if

$$\mu(A \cap B) \leqq \mu(A),$$

for every A and B in $\mathscr{R}$, and

(vii) monotone continuous if $A_n \nearrow A$ or $A_n \searrow A$ (A_n, A in $\mathscr{R}$) and $\mu(A_n) < \infty$ implies $\mu(A_n) \to \mu(A)$ in G.

A *pre-measure* is a set function defined on a ring with values in an extended partially ordered group which is monotone and vanishes at empty set.

An additive (subadditive, strongly subadditive, superadditive, strongly superadditive) pre-measure is called a *measure* (*submeasure, strong submeasure, supermeasure, strong supermeasure*, respectively).

2. EXAMPLES

Since considerations concerning group valued set functions will occupy us during most of the remainder of this paper, we shall now have a look at a few of them.

4. Example. Let G be any group ordered by the discrete order, i.e., $x \leqq y$ iff $x = y$ (no distinct elements are comparable).

5. Example. Let P_m be the set of all polynomials of the form

$$a_0 + a_1 x + \ldots + a_{m-1} x^{m-1} + a_m x^m$$

(m fixed) with the pointwise ordering. Then P_m is a separative, monotone complete group (see Example 7 in [10]). If m is even then P_m is upward filtering, if m is odd then P_m is not upward filtering.

For the terminology and dotation in the next example see [10].

6. Example. Let $(\Omega, \mathscr{S}, \nu)$ be a real measure space. Let X be a monotone σ-complete separative linear space. Let $f : \Omega \to X$ be a non-negative integrable function. Then

$$\mu(E) = \int_E f \, d\nu$$

for E in $\mathscr{S}$ is an X − valued measure on $\mathscr{S}$.

For the terminology and proofs in the following examples see [5], [6], and [7].

7. Example. If Z is a locally convex topological space with a lattice ordering given by a closed cone, such that every linear continuous functional on Z is order continuous, then Z is separative.

8. Example. If Y is a vector lattice regularly ordered by a cone such that every order bounded linear functional is order continuous, then Y is separative.

Subadditive and superadditive set functions, via outer and inner measures, have occurred in the theory of measure from its beginning.

9. Example. Let G be a monotone complete group, $\mathscr{R}$ a ring of subsets of Ω, and let $\mu : \mathscr{R} \to G_+$ be a measure. Define μ_1 on $\mathscr{R}_\sigma \cup \mathscr{R}_\delta$ by $\mu_1(A) = \lim_k \mu(A_k)$ if $A_k \nearrow A$ or $A_k \searrow A$, $A_k \in \mathscr{R}$, and for $E \subset \Omega$ put

$$\mu^*(E) = \inf \{\mu_1(A);\ E \subset A \in \mathscr{R}_\sigma\} ,$$
$$\mu_*(E) = \sup \{\mu_1(A);\ A \subset E \text{ 1bd } A \in \mathscr{R}_\delta\} .$$

Then μ^* is a strongly subadditive and μ_* is a strongly superadditive set function on 2^Ω.

3. THE EXTENSION

Throughout this section we shall assume that G is a partially ordered, separative, monotone σ-complete group, $\mathscr{R}$ is a ring of subsets of the set Ω, and $\mu : \mathscr{R} \to \bar{G}_+$ is a continuous pre-measure on $\mathscr{R}$.

We shall extend the pre-measure μ from $\mathscr{R}$ to $\mathscr{S}(\mathscr{R})$ ($\mathscr{D}(\mathscr{R})$). Our extension process is related to the classical Borel method of extending a measure. The system $\mathscr{S}(\mathscr{R})$ is the minimal system on $\mathscr{R}$ containing the limit of every monotone sequence of its members.

Throughout this section k and l will denote natural numbers and n and m countable ordinals.

Define $\mathscr{R}_n$ and μ_n inductively as follows. $\mathscr{R}_0 = \mathscr{R}$. $\mathscr{R}_1$ is the system of all sets expressible as the union of an increasing sequence $\{A_k\}$ of elements in $\mathscr{R}_0$, $\mu_1(\bigcup_k A_k) = \lim_k \mu(A_k)$. $\mathscr{R}_2$ is the family of all sets expressible as the intersection of a decreasing sequences $\{A_k\}$ in $\mathscr{R}_1$,

$\mu_2(\bigcap_k A_k) = \lim_k \mu_1(A_k)$, and so on. Further,

$$\mathscr{R}_n = \begin{cases} (\mathscr{R}_{n-1})_\sigma & \text{iff } n - 1 \text{ is "odd"} , \\ (\mathscr{R}_{n-1})_\delta & \text{iff } n - 1 \text{ exists and is "even"} , \\ \bigcup_{m<n} \mathscr{R}_m & \text{iff } n = 1 \text{ does not exist} . \end{cases}$$

It is well-known that

$$\mathscr{S}(\mathscr{R}) = \bigcup \{\mathscr{R}_n;\ n \text{ is a countable ordinal}\} .$$

If $A \in \mathscr{R}_n$, we define

$$\mu_n(A) = \lim_k \mu_{n-1}(A_k)$$

if $A_k \nearrow A$ and $n - 1$ is odd ($A_k \searrow A$ and $n - 1$ is even), and we put

$$\mu_n(A) = \mu_m(A)$$

if $n - 1$ does not exist and $A \in \mathscr{R}_m$.

Finally, we define $\bar{\mu}$ on $\mathscr{S}(\mathscr{R})$ by putting

$$\bar{\mu}(A) = \mu_n(A) \text{ whenever } A \text{ is in } \mathscr{R}_n .$$

10. Lemma. *μ_1 is well-defined.*

Proof. Let $A \in \mathscr{R}_1$, $A_k \nearrow A$ and $B_l \nearrow A$, where A_k and B_l are in $\mathscr{R}$. Then $A_k \cap B_l \nearrow$ $\nearrow A_k$ $(l \to \infty)$; so

$$\mu(A_k) = \lim_l \mu(A_k \cap B_l) \leqq \lim_l \mu(B_l) .$$

Hence

$$\lim_k \mu(A_k) \leqq \lim_l \mu(B_l) .$$

The converse inequality can be proved in the same way. Thus

$$\lim_k \mu(A_k) = \lim_l \mu(B_l)$$

and μ_1 is well-defined.

11. Lemma. *The set function μ_1 is continuous iff the real-valued pre-measure $x^< \circ \mu$ has a continuous extension $\mathscr{R}_\sigma$ for every $x^<$ in $G_+^<$.*

Proof. The necessity of the condition is easy to see. Let now $x^<$ be in $G_+^<$ and let v be a continuous extension of $x^< \circ \mu$. Then clearly $x^< \circ \mu_1(A) = v(A)$ for every A in $\mathscr{R}_1$. Hence $x^< \circ \mu_1(A) = v(A)$.

Let now A_k, $A \in \mathscr{R}_1$, $\mu_1(A_k) < \infty$ and let $A_k \searrow A$. Denote

$$\lim_k \mu_1(A_k) = z .$$

Clearly $v(A_k) \searrow v(A)$ and so

$$x^<(z) = x^<(\lim_k \mu_1(A_k)) = \lim_k x^< \circ \mu_1(A_k)$$
$$= \lim_k v(A_k) = v(A) = x^< \circ \mu_1(A) .$$

Since $x^<$ was arbitrary and $G_+^<$ separates points of G, we have $z = \mu_1(A)$ and so

$$\lim_k \mu_1(A_k) = \mu_1(A).$$

The case $A_k \nearrow A$ is similar.

We shall assume in this section that every real pre-measure $x^< \circ \mu$ $(x^< \in G_+^<)$ has a continuous extension from $\mathscr{R}$ to $\mathscr{S}(\mathscr{R})$.

Using this assumption and a similar technique as in the last two lemmas we get

12. Theorem. *Every μ_n is a well-defined continuous set function defined on $\mathscr{R}_n$.*

13. Lemma. (i) *$\mathscr{R}_n$ is a lattice.*
(ii) *$\mathscr{R}_m \subset \mathscr{R}_n$ whenever $m \leqq n$.*
(iii) *$\mu_{n/\mathscr{R}_m} = \mu_m$ whenever $m \leqq n$.*

14. Lemma. (i) *$\mu_n(A) \geqq \Theta$ for all A in $\mathscr{R}_n$.*
(ii) *μ_n is monotone.*
(iii) *If μ is additive (subadditive, strongly subadditive, superadditive, strongly superadditive) on $\mathscr{R}$, then μ_n is additive (subadditive, strongly subadditive, superadditive, strongly superadditive, respectively) on $\mathscr{R}_n$.*

Proof. (i) and (ii) are obvious. Let A_k, B_k be in $\mathscr{R}_0$ and let A, B be in $\mathscr{R}_1$ with $A_k \nearrow A$, $B_k \nearrow B$. Then

$$A_k \cap B_k \nearrow A \cap B \quad \text{and} \quad A_k \cup B_k \nearrow A \cup B.$$

Since $A_k \cap B_k$ and $A_k \cup B_k$ are in $\mathscr{R}_0$ and $A \cap B$ and $A \cup B$ are in $\mathscr{R}_1$ and since μ is additive on $\mathscr{R}_0$ we have

$$\mu(A_k \cap B_k) + \mu(A_k \cup B_k) = \mu(A_k) + \mu(B_k).$$

Passing to limits and using Theorem 2 we get

$$\mu_1(A \cap B) + \mu_1(A \cup B) = \mu_1(A) + \mu_1(B).$$

The proof for a general n proceeds by transfinite induction. The proof for non-additive pre-measures is similar.

We can formulate the results of this section as follows:

15. Theorem. *Let G be a separative, monotone σ-complete group, and let μ be a G – valued pre-measure defined on a ring $\mathscr{R}$. Let every pre-measure $x^< \circ \mu$ $(x^< \in G_+^<)$ have a continuous extension form $\mathscr{R}$ to $\mathscr{S}(\mathscr{R})$. Then there exists a unique pre-measure $\bar{\mu} : \mathscr{S}(\mathscr{R}) \to \bar{G}_+$ such that*

(i) *$\bar{\mu}$ is the extension of μ,*

(ii) *If μ is additive (subadditive, strongly subadditive, superadditive, strongly superadditive), then $\bar{\mu}$ is additive (subadditive, strongly subadditive, superadditive or strongly superadditive, respectively).*

4. SPECIAL CASES

In this section $\mathscr{R}$ will denote a ring of subsets of Ω and G a partially ordered group. We will apply the theory developed in Section 3 to special cases.

First, we give some notes about our terminology. Let $\mu : \mathscr{R} \to \bar{G}_+$. We say that μ is continuous if $A_n, A \in \mathscr{R}$, $A_n \to A$ (i.e. $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = A$), and $\mu(A_n) < \infty$ imply $\mu(A_n) \to \mu(A)$. It is easy to see that μ is a real measure, then the concepts of the monotone continuity and continuity are equivalent. It is also obvious that if $\mathscr{R}$ is a σ-ring and μ is monotone, then μ is monotone continuous iff μ is continuous. However, as the next example shows, this is not true in general (i.e., when $\mathscr{R}$ is not a σ-ring).

16. Example. Let N be the set of all natural members. Let $\mathscr{P}$ be the family of all finite subsets of N. Clearly $\mathscr{P}$ is a ring which is not a σ-ring. We put $\mu(A) = 1$ if $\emptyset \neq A \in \mathscr{P}$ and $\mu(\emptyset) = 0$. Then μ is a strong submeasure. Obviously μ is monotone continuous. We shall show that μ is not continuous. Put

$$A_n = \{n, n + 1, \ldots, 2n\} ;$$

then $A_n \to \emptyset$ but $1 = \mu(A_n) \nrightarrow \mu(\emptyset) = 0$.

17. Theorem. *Let G be a separative, monotone σ-complete group, and let μ be a continuous G-valued measure defined on a ring $\mathscr{R}$. Then there exists a unique extension $\bar{\mu}$ of μ such that $\bar{\mu} : \mathscr{S}(\mathscr{R}) \to \bar{G}_+$ is a continuous measure. If μ is bounded on $\mathscr{R}$ then $\bar{\mu}$ is G – valued on $\mathscr{S}(\mathscr{R})$.*

Proof. Let $x^< \in G_+^<$. Then $x^< \circ \mu$ is a continuous real measure on $\mathscr{R}$ and so it has a continuous extension on $\mathscr{S}(\mathscr{R})$. The proof now follows by Theorem 15.

We note that in the last theorem the continuity can be replaced by order continuity.

For our next consideration we shall need the following result of B. Riečan.

18. Lemma. ([8] p. p. 217–218.) *Let v be a real-valued monotone continuous submeasure on $\mathscr{R}$. Then v can be extended to $\mathscr{S}(\mathscr{R})$ iff v is exhausting (i.e., if $A_k \subset A_{k+1}$ are in $\mathscr{R}$ and $\{v(A_k)\}$ is a sequence bounded from above, then $\lim_k v(A_{k+1} - A_k) = 0$). It is obvious that a continuous pre-measure is exhausting. Combining this by Theorem 15 and by the last lemma we get:*

19. Theorem. *Let G be a separative, monotone complete group, and let μ be a continuous G — valued submeasure (strong submeasure) defined on a ring $\mathscr{R}$. Then there exists a unique extension $\bar{\mu}$ of μ such that $\bar{\mu} : \mathscr{S}(\mathscr{R}) \to \bar{G}_+$ is a continuous submeasure (strong submeasure), and $\bar{\mu} : \mathscr{D}(\mathscr{R}) \to G_+$ is a G — valued continuous submeasure (strong submeasure). If μ is bounded on $\mathscr{R}$, then μ is G — valued on $\mathscr{S}(\mathscr{R})$.*

The central role in our extension theorems is played by functionals $x^<$ from $G_+^<$ and so our main assumption throughout this paper has been that the range space of the pre-measure is separative. It is possible to replace this assumption by a weaker one. We say that a functional $\eta : G \to \langle 0, \infty)$ is subadditive if it is monotone, $\eta(\Theta) = 0$, and $\Theta \leqq x, y \in G$ implies

$$\eta(x + y) \leqq \eta(x) + \eta(y) .$$

We say that a partially group is subseparative if the family of all subadditive order continuous functionals on G separates points of G.

It follows from Lemma 18 and from the reasoning following lemma 18 that in Theorem 17 and Theorem 19 it suffices to assume that G is subseparative (instead of being separative). It is also clear that the results of Section 3 remain valid provided G is only subseparative.

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