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ON THE GENERALIZED LINEAR ORDINARY DIFFERENTIAL EQUATION

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We consider the generalized linear ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = D[A(t)\,x + f(t)] \tag{1}$$

on the closed interval $[a, b]$, where $-\infty < a < b < +\infty$ and A and f are matrix functions of bounded variation on $[a, b]$ of the type $n \times n$ and $n \times 1$, respectively. An n-vector function x defined on $[a, b]$ is said to be a solution to the equation (1) on the interval $[a, b]$ if there exists the Perron-Stieltjes integral

$$\mathrm{P}\int_a^b [\mathrm{d}A(s)]\, x(s)$$

and

$$x(t) = x(a) + \mathrm{P}\int_a^t [\mathrm{d}A(s)]\, x(s) + f(t) - f(a) \quad \text{for all} \quad t \in [a, b]\,. \tag{2}$$

The equation (1) is a special type of generalized ordinary differential equations introduced by J. KURZWEIL in [3]. Although the general nonlinear case has been studied hitherto by several authors ([4]–[9], [11]), relatively small attention was paid to the linear case. Only in [9] the equation (1) with A and f left continuous on $(a, b]$ was studied.

To the equation (1) the differentio-Stieltjes-integral equation

$$x(t) = x(a) + \mathrm{Y}\int_a^t [\mathrm{d}A(s)]\, x(s) + f(t) - f(a)\,, \tag{3}$$

where $\mathrm{Y}\int_a^t$ stands for the σ-Young integral, is related. (The definition and basic properties of the σ-Young integral can be found e.g. in the book of T. HILDEBRANDT [1].) For the equation (3) fundamental results (existence and uniqueness of a solution in the class of bounded functions, fundamental matrix solution to the corresponding homogeneous equation, variation of constants formula) were obtained by T. H. Hildebrandt in [2].

In [10] (cf. Theorem 3,2) the following assertion on the relation between the σ-Young and the Perron-Stieltjes integrals is proved.

Let g have bounded variation on $[a, b]$ and let f be bounded on $[a, b]$. Then the existence of the σ-Young integral

$$\mathrm{Y}\int_a^b f(t)\,\mathrm{d}g(t)$$

implies the existence of the Perron-Stieltjes integral

$$\mathrm{P}\int_a^b f(t)\,\mathrm{d}g(t)$$

and both integrals are equal to one another. Let us mention that the assumption on the boundedness of f can be weakened. Nevertheless some boundedness conditions on f are necessary and substantial for the existence of $\mathrm{P}\int_a^b f\,\mathrm{d}g$ (cf. Example 2,1 in [10]).

It follows that x being a solution to (3) on $[a, b]$, it is certainly a solution to (1) on $[a, b]$. Moreover, it is clear that all functions bounded and fulfilling (2) on $[a, b]$ are solutions to (3), as well. Hence for the generalized linear ordinary differential equation (1) we can adopt all the results of T. H. Hildebrandt from [2]. The assertions on the uniqueness has to be understood as "unique in the space of functions bounded on $[a, b]$", of course.

In this paper we prove that under the assumptions assuring the existence of a solution to (3) the equation (1) admits only solutions of bounded variation on $[a, b]$. In other words, the equations (1) and (3) are equivalent.

The open interval $a < t < b$ is denoted by (a, b) and the half-closed intervals $a < t \leqq b$ and $a \leqq t < b$ are denoted by $(a, b]$ and $[a, b)$, respectively. I denotes the identity $n \times n$-matrix. Given a matrix $M = (M_{i,j})_{i,j}$ its norm $\|M\|$ is defined by

$$\|M\| = \max_i \sum_j |M_{i,j}| .$$

Given a matrix function F of bounded variation on $[a, b]$ and $t \in (a, b)$, we design

$$\Delta^+F(t) = F(t+) - F(t),\quad \Delta^-F(t) = F(t) - F(t-);\quad \Delta^+F(a) = F(a+) - F(a),$$
$$\Delta^-F(b) = F(b) - F(b-)$$

and $\mathrm{var}_a^b F$ means the total variation of F on $[a, b]$ defined by

$$\mathrm{var}_a^b F = \sup \sum_j \|F(t_j) - F(t_{j-1})\| ,$$

where the least upper bound is taken with respect to all divisions $\{a = t_0 < t_1 < \dots \dots < t_m = b\}$ of $[a, b]$. Hereafter all integrals are considered as Perron-Stieltjes ones.

The following assertion follows readily from (2) and from properties of the Perron-Stieltjes integral as a special kind of the Kurzweil integral ([3], Theorem 1, 3, 6).

Proposition 1. *Let x be a solution of (1) on $[a, b]$. Then all the limits $x(a+)$, $x(b-)$, $x(t+)$, $x(t-)$ $(t \in (a, b))$ exist and it holds*

$$x(t+) = [I + \Delta^+ A(t)]\, x(t) + \Delta^+ f(t) \quad \textit{for all} \quad t \in [a, b)$$

and

$$x(t-) = [I - \Delta^- A(t)]\, x(t) + \Delta^- f(t) \quad \textit{for all} \quad t \in (a, b]\,.$$

The second proposition can be easily obtained from §7 in [2].

Proposition 2. *Let*

$$\det [I - \Delta^- A(t)] \neq 0 \quad \textit{for all} \quad t \in (a, b]\,. \tag{4}$$

Then given an arbitrary n-vector c, there exists at least one solution $\tilde{x}$ of (1) on $[a, b]$ with $\tilde{x}(a) = c$. This solution is of bounded variation on $[a, b]$ and given an arbitrary $t_0 \in [a, b]$ and an arbitrary function x bounded on $[a, t_0]$ fulfilling (2) on $[a, t_0]$ and such that $x(a) = c$, it holds $x(t) \equiv \tilde{x}(t)$ on $[a, t_0]$.

Remark 1. Let us notice that the assumption (4) is substantial for the existence of a solution to (1). In fact, if $n = 2$, $[a, b] \equiv [0, 1]$, $f(t) \equiv 0$ on $[a, b]$ and

$$A(t) = 0 \quad \text{for} \quad 0 \leqq t < \tfrac{1}{2}\,,$$

$$A(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad \tfrac{1}{2} \leqq t \leqq 1\,,$$

then for an arbitrary solution x of (1) on $[0, 1]$ we have by Proposition 1

$$x(t) = x(0) \quad \text{for} \quad 0 < t < \tfrac{1}{2}\,,$$

$$x(\tfrac{1}{2}-) = x(0) = [I - \Delta^- A(t)]\, x(\tfrac{1}{2}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x(\tfrac{1}{2}) = \begin{pmatrix} x_1(\frac{1}{2}) \\ 0 \end{pmatrix}$$

and consequently to a given n-vector c a solution x of (1) on $[0, 1]$ with $x(0) = c$ exists iff $c_2 = 0$.

Remark 2. It is easy to see that any solution x of (1) on $[a, b]$ which is bounded on $[a, b]$ is of bounded variation on $[a, b]$.

Theorem 1. *Let (4) hold. Then given an arbitrary n-vector c, there exist a unique solution x of (1) on $[a, b]$ such that $x(a) = c$.*

Proof. The only fact to prove is that given an arbitrary solution x of (1) on $[a, b]$ with $x(a) = c$ (which generally could be unbounded on $[a, b]$), it holds $x(t) \equiv \tilde{x}(t)$ on $[a, b]$, where $\tilde{x}$ is the solution of (1) on $[a, b]$ from Proposition 2.

Denoting $y(t) = x(t) - \tilde{x}(t)$ for $t \in [a, b]$, we get $y(a) = 0$ and

$$y(t) = \int_a^t [\mathrm{d}A(s)]\, y(s) \quad \text{for} \quad t \in [a, b]. \tag{5}$$

Since by Proposition 1 $y(a+) = 0$, there exists a $\delta_0 > 0$ such that y is bounded on $[a, a + \delta_0]$ and thus by Proposition 2 $y(t) \equiv 0$ on $[a, a + \delta_0]$. Let t_0 be the least upper bound of the set of all $t \in [a, b]$ with the property $y(\tau) = 0$ for all $\tau \in [a, t]$. Clearly $y(t) \equiv 0$ on $[a, t_0)$ and therefore

$$y(t_0) = [I - \Delta^- A(t_0)]^{-1}\, y(t_0-) = 0$$

owing to (4) and Proposition 1. Let $t_0 < b$, then Proposition 1 yields

$$y(t_0+) = [I + \Delta^+ A(t_0)]\, y(t_0) = 0\,.$$

Consequently there exists a $\delta > 0$ such that y is bounded on $[a, t_0 + \delta]$. Applying again Proposition 2 we get $y(t) \equiv 0$ on $[a, t_0 + \delta]$, which contradicts the definition of t_0. Hence $t_0 = b$ and $y(t) \equiv 0$ on $[a, b]$.

Theorem 1 establishes the equivalence between the generalized linear ordinary differential equation (1) and the differentio-Stieltjes-integral equation (2). For the further investigations of generalized linear ordinary differential equations it is convenient to give here a survey of fundamental theorems for these equations. All the proofs follow from the results of [2] by the similar reasoning as Theorem 1.

Theorem 2. *Let (4) hold. There there exists just one $n \times n$-matrix function $U(t, s)$ defined for $a \leqq s \leqq t \leqq b$ and such that*

$$U(t, s) = I + \int_s^t [\mathrm{d}A(\sigma)]\, U(\sigma, s) \quad \text{for all} \quad s \in [a, b]\,, \quad t \in [s, b]\,. \tag{6}$$

The function U has the following properties.

(i) *There exists $K < \infty$ such that*

$$\operatorname{var}_a^t U(t, .) \leqq K\,, \quad \operatorname{var}_s^b U(., s) \leqq K \quad \textit{for all} \quad t, s \in [a, b]$$

and

$$\|U(t, s)\| \leqq K \quad \textit{for all} \quad t, s \in [a, b]\,, \quad t \geqq s\,.$$

(ii)
$$\begin{aligned}
U(t+, s) &= [I + \Delta^+ A(t)]\, U(t, s) && \textit{if} \quad a \leqq s \leqq t < b\,,\\
U(t-, s) &= [I - \Delta^- A(t)]\, U(t, s) && \textit{if} \quad a \leqq s < t \leqq b\,,\\
U(t, s) &= U(t, s+)\, [I + \Delta^+ A(s)] && \textit{if} \quad a \leqq s < t \leqq b\,,\\
U(t, s) &= U(t, s-)\, [I - \Delta^- A(s)] && \textit{if} \quad a < s \leqq t \leqq b\,.
\end{aligned}$$

(iii) *Given t, s, r ∈ [a, b] such that s ≦ r ≦ t, it holds*

$$U(t, s) = U(t, r)\, U(r, s) \quad and \quad U(t, t) = I\,.$$

(iv) *Given an arbitrary n-vector c, the unique solution x of* (1) *on* [a, b] *with* $x(a) = c$ *is given by*

$$x(t) = U(t, a)\, c + f(t) - f(a) + \int_a^t [\mathrm{d}_\sigma U(t, \sigma)]\,(f(\sigma) - f(a))\,, \quad t \in [a, b]\,.$$

(v) *Let* $a \leqq s > t \leqq b$. *Then the matrix* $U(t, s)$ *possesses an inverse* $U^{-1}(t, s)$ *iff*

$$\det\,[I + \Delta^+ A(\tau)] \neq 0 \quad for\ all \quad \tau \in [s, t]\,.$$

The last assertion can be proved similarly as Theorem 4,3 of [9].

Remark 3. Let (4) hold. Further, let us assume that $\det\,[I + \Delta^+ A(t)] \neq 0$ for all $t \in [a, b)$. By Theorem 2 (v) it is reasonable to define $U(t, s) = U^{-1}(s, t)$ for $t, s \in [a, b]$, $t < s$. It is easy to verify that then $U(t, s)$ fulfils (6) for all $t, s \in [a, b]$. Moreover, $U(t, s) = U(t, r)\, U(r, s)$ for all $t, s, r \in [a, b]$. In particular, $U(t, s) = U(t, a)\, U(a, s)$ for all $t, s \in [a, b]$. It follows immediately that the Vitali two-dimensional variation of U on $[a, b] \times [a, b]$ is finite (cf. [1], pp. 106–107). Even the following assertion is true.

Proposition 3. *Let us put*

$$\tilde{U}(t, s) = \begin{cases} U(t, s) & for \quad t \in [a, b] \quad and \quad s \in [a, t]\,, \\ U(t, t) = I & for \quad t \in [a, b] \quad and \quad s \in [t, b]\,. \end{cases}$$

Then the Vitali two-dimensional variation of $\tilde{U}$ *on* $[a, b] \times [a, b]$ *is finite.*

Proof. Let $\sigma = \{a = t_0 < t_1 < \ldots < t_m = b\}$ be an arbitrary division of $[a, b]$. Let us put for $j, k = 1, 2, \ldots, m$

$$\Delta\Delta_{j,k}\tilde{U} = \tilde{U}(t_j, t_k) - \tilde{U}(t_{j-1}, t_k) - \tilde{U}(t_j, t_{k-1}) + \tilde{U}(t_{j-1}, t_{k-1})\,.$$

Then

$$\Delta\Delta_{j,k}\tilde{U} = U(t_j, t_j) - U(t_{j-1}, t_{j-1}) - U(t_j, t_j) + U(t_{j-1}, t_{j-1}) = 0 \ \text{ for } \ k \geqq j + 1\,,$$
$$\Delta\Delta_{j,j}\tilde{U} = I - U(t_j, t_{j-1})$$

and

$$w(\tilde{U}; \sigma) = \sum_{j=1}^{m} \sum_{k=1}^{m} \|\Delta\Delta_{j,k}\tilde{U}\| = \sum_{j=1}^{m} \Big(\sum_{k=1}^{j-1} \|\Delta\Delta_{j,k}U\|\Big) + \sum_{j=1}^{m} \|I - U(t_j, t_{j-1})\|\,.$$

Applying the assertions (i) and (iii) of Theorem 2 and (6) we get

$$w(\tilde{U}; \sigma) = \sum_{j=1}^{m} \sum_{k=1}^{j-1} \|[U(t_j, t_{j-1}) - I]\,[U(t_{j-1}, t_k) - U(t_{j-1}, t_{k-1})]\| +$$

$$+\sum_{j=1}^{m}\|I - U(t_j, t_{j-1})\| \leqq \sum_{j=1}^{m}(1 + K \operatorname{var}_a^{t_{j-1}} U(t_{j-1}, .)) \left\| \int_{t_{j-1}}^{t_j} [dA(\sigma)]\, U(\sigma, t_{j-1}) \right\| \leqq$$
$$\leqq (1 + K^2)\, K(\operatorname{var}_a^b A) < \infty .$$

This completes the proof.

Remark 4. Let us assume

$$\det [I + \Delta^+ A(t)] \neq 0 \quad \text{for all} \quad t \in [a, b)$$

instead of (4). Then the assertion of Proposition 2 has to be modified as follows.

Given an arbitrary n-vector c there exists at least one solution $\tilde{x}$ of (1) on $[a, b]$ such that $x(b) = c$. If $t_0 \in [a, b]$ and x is an arbitrary solution of (1) on $[t_0, b]$ which is bounded on $[t_0, b]$, then $x(t) \equiv \tilde{x}(t)$ on $[t_0, b]$.

The formulation and the proof of the statements analogous to Theorems 1 and 2 and Proposition 3 is evident. (The corresponding fundamental matrix solution $V(t, s)$ is defined for $a \leqq t \leqq s \leqq b$ and fulfils the relation

$$V(t, s) = I - \int_t^s [dA(\sigma)]\, V(\sigma, s) .)$$

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