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1. INTRODUCTION

Proofs of the prime number theorem, i.e., of the assertion

$$\pi(x) \sim \frac{x}{\log x}, \quad x \to \infty \quad \left(\text{or equivalently } \pi(x) \sim \int_2^x \frac{\mathrm{d}u}{\log u}, \ x \to \infty\right),$$

where $\pi(x)$ denotes the number of primes less than or equal to x, divide roughly into three groups. The first group, historically the oldest, is formed by the analytic proofs. In these proofs a wide aparatus of the theory of functions of one complex variable on the Riemann ζ-function is used. These proofs yield the strongest results, because they give the best estimation of the remainder term $\pi(x) - \int_2^x \mathrm{d}u/\log u$. Probably, the best known estimation of the remainder term is

$$\pi(x) - \int_2^x \frac{\mathrm{d}u}{\log u} = O(x \exp(-c \log^{3/5} x (\log\log x)^{-1/5})), \quad x \to \infty,$$

where c is a suitable positive constant (cf. e.g. [7]). The second group contains the so called elementary proofs; they are called elementary, for they avoid the use of the ζ-function, i.e., of the theory of functions of one complex variable, as well as the Fourier transform of functions. The estimations of the remainder term yielded by the elementary proofs are more complicated, but weaker than those yielded by the analytic proofs; e.g., in the paper [1] the estimation

$$\pi(x) - \int_2^x \frac{\mathrm{d}u}{\log u} = O(x \exp(-\log^{1/7} x (\log\log x)^{-2})), \quad x \to \infty,$$

is proved. The third group is formed by the proofs of intermediate type, which use the theory of functions of one complex variable in most cases only to prove that $\zeta(1 + it) \neq 0$ for $t \neq 0$. These proofs are based especially on the Fourier transform of functions and Tauberian theorems. They are relatively simple, but they have not yielded any estimation of the remainder term till now. The aim of this paper is to show that the known proofs of the prime number theorem without the remainder term, belonging to the third group, can be modified to obtain

$$(1) \qquad \pi(x) - \int_2^x \frac{\mathrm{d}u}{\log u} = O(x \log^{-n} x), \quad x \to \infty$$

for every natural number n. We shall use a simple lemma on the Fourier transform of functions (cf. Lemma 1) to obtain the relation (1). Our procedure has two advantages: firstly, our proofs are not much more complicated than the original ones and, secondly, there is no need to use the properties of the Riemann ζ-function in the halfplane $\operatorname{Re} s < 1$. On the other hand, the estimation (1) of the remainder term in the prime number theorem is weak in comparison with the quoted results. For the sake of clarity we mention all the properties of the Riemann ζ-function which we shall use in this work. They are the following ones: 1) the ζ-function is analytic and non-zero in the halfplane $\operatorname{Re} s \geqq 1$ except for the point $s = 1$ where it has a simple pole, 2) $\zeta^{(k)}(s) = O(\log^{k+1} t)$, $\sigma \geqq 1$, $t \geqq 3$, $s = \sigma + it$, for $k = 0, 1, 2, \ldots$, 3) the function $\zeta'(s)/\zeta(s) + 1/(s-1)$ is analytic in the halfplane $\operatorname{Re} s \geqq 1$, 4) $1/\zeta(s) = O(\log^7 t)$, $\sigma \geqq 1$, $t \geqq 3$, $s = \sigma + it$. (Cf. e.g. [2], [6].)

2. BASIC ASSERTIONS

We call a function f absolutely continuous on an open interval $(a; b)$, if it is absolutely continuous on every closed bounded subinterval of $(a; b)$. For $f \in L^1(E_1)$ we define $\hat{f}(x) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{itx}\, \mathrm{d}t$, $x \in E_1$. The function $\hat{f}$ is called the Fourier transform of a function f.

Lemma 1. *Let $m \in N$ and let a function $f \in L^1(E_1)$ have absolutely continuous and integrable derivatives up to the order $m - 1$. If $f^{(m)} \in L^1(E_1)$, then* $\hat{f}(x) = (-\mathrm{i}x)^{-m} \int_{-\infty}^{\infty} f^{(m)}(t)\, \mathrm{e}^{itx}\, \mathrm{d}t$, $x \neq 0$. *Especially* $\hat{f}(x) = O(|x|^{-m})$, $x \to \pm\infty$.

The proof of this lemma is easy and well known (it may be found e.g. in [5]). However, the proof of the relation (1) is based on Lemma 1.

For $n \in N$ we define the function $\Lambda(n)$ in this way: $\Lambda(n) = \log p$ if $n = p^r$, where p is a prime and r any natural number, $\Lambda(n) = 0$ otherwise. (Λ is called von Mangoldt's function.) In the rest of the paper we prove in two different ways the following assertion:

Theorem 1. *If the function g is defined by*

$$g(x) = \sum_{n \leqq x} \Lambda(n) \log \frac{x}{n}, \quad x \geqq 1,$$

then $g(x) = x + O(x \log^{-k} x)$, $x \to \infty$, for every natural k. The constant in O may depend on k.

The relation (1) follows from Theorem 1 in the usual way, e.g. using Lemmas V. 3. 14–16 from the book [7], pp. 172–174.

3. THE FIRST PROOF OF THEOREM 1

The proof of the prime number theorem without the remainder term, which we shall modify here, is taken from the book [3]. It is due to Titchmarsh.

According to Lemma V. 3. 5 from the book [7], p. 158, we have for $x > 0$

$$g(x) = -\frac{1}{2\pi i}\int_{2-\infty i}^{2+\infty i} \frac{x^s}{s^2} \cdot \frac{\zeta'(s)}{\zeta(s)}\,ds = I_1(x) + I_2(x)\,,$$

where

$$I_1(x) = \frac{1}{2\pi i}\int_{2-\infty i}^{2+\infty i} \frac{x^s\,ds}{s^2(s-1)}\,, \quad I_2(x) = -\frac{1}{2\pi i}\int_{2-\infty i}^{2+\infty i} \frac{x^s}{s^2}\,h(s)\,ds\,,$$

and $h(s) = \zeta'(s)/\zeta(s) + 1/(s-1)$ is an analytic function in the halfplane $\operatorname{Re} s \geqq 1$.

Analogously to the proof of Lemma V. 3. 4, [7], p. 158, we can prove by calculating the residues of the function $x^s/(s^2(s-1))$ at the points $s = 0$ and $s = 1$ that

$$(2) \qquad I_1(x) = 0 \quad \text{for} \quad x \in (0; 1)\,, \quad I_1(x) = x - \log x - 1 \quad \text{for} \quad x \geqq 1\,.$$

As soon as we prove that for every natural number k

$$(3) \qquad I_2(x) = O(x \log^{-k} x)\,, \quad x \to \infty\,,$$

holds, the proof of Theorem 1 will be complete.

Choose $A > 0$. Since the function $(x^s/s^2)\,h(s)$ is analytic in the halfplane $\operatorname{Re} s \geqq 1$, it follows from Cauchy's theorem that

$$(4) \qquad \frac{1}{2\pi i}\int_{2-Ai}^{2+Ai} \frac{x^s}{s^2}\,h(s)\,ds + \frac{1}{2\pi i}\int_{2+Ai}^{1+Ai} \frac{x^s}{s^2}\,h(s)\,ds + {} + \frac{1}{2\pi i}\int_{1+Ai}^{1-Ai} \frac{x^s}{s^2}\,h(s)\,ds + \frac{1}{2\pi i}\int_{1-Ai}^{2-Ai} \frac{x^s}{s^2}\,h(s)\,ds = 0\,.$$

We have $\zeta'(s)/\zeta(s) = O(\log^9 |t|)$ for $\sigma \geqq 1$, $|t| \geqq 3$, $s = \sigma + it$. It follows from this fact that $h(s) = O(\log^9 |t|)$ on the same set and, further, that for $A \geqq 3$ and $x \geqq 1$ the absolute value of the second and the fourth integral in the relation (4) is less than or equal to $(1/(2\pi))\,(x^2/A^2)\,C \log^9 A$, where $C > 0$ is a suitable constant. This means that

$$\lim_{A\to\infty} \frac{1}{2\pi i}\int_{2+Ai}^{1+Ai} \frac{x^s}{s^2}\,h(s)\,ds = \lim_{A\to\infty} \frac{1}{2\pi i}\int_{1-Ai}^{2-Ai} \frac{x^s}{s^2}\,h(s)\,ds = 0\,.$$

Letting $A \to \infty$ in the relation (4), we obtain

$$I_2(x) = -\frac{1}{2\pi i}\int_{2-\infty i}^{2+\infty i} \frac{x^s}{s^2} h(s)\,ds = -\frac{1}{2\pi i}\int_{1-\infty i}^{1+\infty i} \frac{x^s}{s^2} h(s)\,ds =$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{x^{1+it}}{(1+it)^2} h(1+it)\,dt = -\frac{x}{2\pi}\int_{-\infty}^{\infty} \frac{h(1+it)}{(1+it)^2} e^{it\log x}\,dt = -\frac{x}{2\pi} I_3(\log x),$$

where $I_3(y) = \int_{-\infty}^{\infty} (h(1+it)/(1+it)^2)\,e^{ity}\,dt$. To prove the relation (3) it is now sufficient to show that $I_3(y) = O(y^{-k})$, $y \to \infty$, for every $k \in N$. To this aim, according to Lemma 1, we must check that the derivatives of all orders of the function $h(1+it)/(1+it)^2$ are absolutely continuous (this is clear) and integrable on E_1. We evidently have $h(1+it)/(1+it)^2 = \overline{[h(1+it)/(1+it)^2]}$. It now suffices to estimate the growth of the k-th derivative of this function for $t \to +\infty$. By the definition of the function h we have

$$\frac{h(1+it)}{(1+it)^2} = \frac{\zeta'(1+it)}{(1+it)^2\,\zeta(1+it)} + \frac{1}{it(1+it)^2} = h_1(t) + h_2(t),$$

say. It is easy to show that $h_2^{(k)}(t) = O(t^{-k-3})$, $t \to +\infty$, $k = 0, 1, 2, \dots$. Since $[\zeta'(s)/\zeta(s)]^{(k)} = O(\log^{9(k+1)} t)$, $\sigma \geqq 1$, $t \geqq 3$, $s = \sigma + it$, as follows from Leibniz's rule, the theorem on differentiation of a composite function and the properties 2) and 4) of the ζ-function mentioned in the first section, we obtain for the function h_1 again by Leibniz's rule that $h_1^{(k)}(t) = O(t^{-2}\log^{9(k+1)} t)$, $t \to \infty$, $k = 0, 1, \dots$. Altogether we have

$$\left[\frac{h(1+it)}{(1+it)^2}\right]^{(k)} = O(t^{-2}\log^{9(k+1)} t) + O(t^{-k-3}) = O(t^{-2}\log^{9(k+1)} t), \quad t \to \infty,$$

for $k = 0, 1, \dots$; this means that the derivatives of all orders of the function $h(1+it)/(1+it)^2$ are integrable on E_1. The assumptions of Lemma 1 are satisfied for every $k \in N$ and, accordingly, $I_3(y) = O(y^{-k})$, $y \to \infty$, for every $k \in N$. This implies the relation (3) for every $k \in N$.

4. THE SECOND PROOF OF THEOREM 1

We use a modification of the Wiener-Ikehara theorem from the book [4] (Theorem 2, p. 124) for the second proof of Theorem 1. This modification – our Theorem 2 – may be interesting in itself.

Theorem 2. *Let $n \in N$. Let further $A(x)$ be a nonnegative nondecreasing function defined for $x \in \langle 0; +\infty)$ and let the integral*

$$f(s) = \int_0^{\infty} A(x)\,e^{-xs}\,dx, \quad s = \sigma + it,$$

converge for $\sigma > 1$. *Let the function* $g(s) = f(s) - 1/(s-1)$ *be continuous in the halfplane* $\sigma \geqq 1$. *If there exists an absolutely continuous derivative* $(\mathrm{d}^{n-1}/\mathrm{d}t^{n-1})\, g(1+it)$ *on* E_1 *and* $(\mathrm{d}^n g/\mathrm{d}t^n) \in L^1(E_1)$, *then*

$$e^{-x} A(x) = 1 + O(x^{-n}), \quad x \to \infty .$$

For the sake of brevity we mention only the points of difference between the proof of this theorem and the proof of Theorem 2 in [4]. Let us observe first of all the course of the original proof. The author, denoting $B(x) = e^{-x} A(x)$, chooses the function u such that $u(t) = 1 - |t|$ for $|t| \leqq 1$, $u(t) = 0$ otherwise. Now, he first proves that $\lim_{y\to\infty} \int_{-\infty}^{2\lambda y} B(y - v/(2\lambda))\, \hat{u}(v)\, \mathrm{d}v = \int_{-\infty}^{\infty} \hat{u}(v)\, \mathrm{d}v$ for every $\lambda > 0$ and then that this identity implies $\lim_{x\to\infty} B(x) = 1$ (cf. [4], p. 124, relations (9) and (10)).

We shall proceed similarly with the only difference. Four u we take any function for which $u \in C^\infty(E_1)$, $\operatorname{supp} u = \langle -1; 1\rangle$ and $\hat{u}(x) \geqq 0$ for all $x \in E_1$. Such a function u exists. It suffices to define $u(t) = \int_{-\infty}^{\infty} v(x)\, v(t-x)\, \mathrm{d}x$, $t \in E_1$, where $v(x) = \exp[(4x^2 - 1)^{-1}]$ for $|x| < \frac{1}{2}$, $v(x) = 0$ otherwise. Since $u \not\equiv 0$, we have $\int_{-\infty}^{\infty} \hat{u}(v)\, \mathrm{d}v > 0$. Choosing the function u in this way we first show that for $\lambda > 1$

$$\int_{-\infty}^{2\lambda y} B\left(y - \frac{v}{2\lambda}\right) \hat{u}(v)\, \mathrm{d}v = \int_{-\infty}^{\infty} \hat{u}(v)\, \mathrm{d}v + O(y^{-n}), \quad y \to \infty, \tag{5}$$

(the constant in O does not depend on λ!) and then we derive from (5) that

$$B(x) = 1 + O(x^{-n}), \quad x \to \infty . \tag{6}$$

The proof now proceeds like in [4]; generally we write $u(t/(2\lambda))$ instead of $1 - |t|/(2\lambda)$. In this way we obtain the relation

$$\int_{-\infty}^{2\lambda y} B\left(y - \frac{v}{2\lambda}\right) \hat{u}(v)\, \mathrm{d}v = \int_{-\infty}^{2\lambda y} \hat{u}(v)\, \mathrm{d}v + \int_{-2\lambda}^{2\lambda} g(1 + it)\, u\left(\frac{t}{2\lambda}\right) e^{iyt}\, \mathrm{d}t . \tag{7}$$

Since $u \in C^\infty(E_1)$ and u has a compact support, the relation $\hat{u}(x) = O(x^{-k})$, $x \to \infty$, holds for any $k \in N$. Consequently

$$\int_{-\infty}^{2\lambda y} \hat{u}(v)\, \mathrm{d}v = \int_{-\infty}^{\infty} \hat{u}(v)\, \mathrm{d}v + O(y^{-n}), \quad y \to \infty, \tag{8}$$

where the constant in O does not depend on λ for $\lambda > 1$. Further,

$$\int_{-2\lambda}^{2\lambda} g(1 + it)\, u\left(\frac{t}{2\lambda}\right) e^{iyt}\, \mathrm{d}t = \left[g(1 + it)\, u\left(\frac{t}{2\lambda}\right)\right]^{\wedge}(y)$$

and the function $g(1 + it)\, u(t/(2\lambda))$ satisfies the assumptions of Lemma 1 in virtue of the assumptions of Theorem 2. According to Lemma 1 we have for $y \neq 0$

(9) $$\int_{-2\lambda}^{2\lambda} g(1 + it)\, u\left(\frac{t}{2\lambda}\right) e^{iyt}\, dt = \left(-\frac{1}{iy}\right)^n \int_{-2\lambda}^{2\lambda} \frac{d^n}{dt^n}\left[g(1 + it)\, u\left(\frac{t}{2\lambda}\right)\right] e^{iyt}\, dt .$$

If we show that there exists a constant $K > 0$ such that for all $\lambda > 1$ the inequality

(10) $$\int_{-2\lambda}^{2\lambda} \left|\frac{d^n}{dt^n}\left[g(1 + it)\, u\left(\frac{t}{2\lambda}\right)\right]\right| dt < K$$

holds, then it follows from the inequality and the relation (9) that

(11) $$\int_{-2\lambda}^{2\lambda} g(1 + it)\, u\left(\frac{t}{2\lambda}\right) e^{iyt}\, dt = O(y^{-n}) , \quad y \to \infty .$$

The relation (5) is a consequence of (7), (8) and (11). For the proof of the inequality (10) it is sufficient to realize that if $\int_0^{2\lambda} |g^{(n)}(1 + it)|\, dt = O(1)$ on the set of all $\lambda > 1$, then $1/(2\lambda)^k \int_0^{2\lambda} |g^{(n-k)}(1 + it)|\, dt = O(1)$ on the same set for $k = 1, 2, \ldots, n$. Further, we have for $\lambda > 1$

$$\int_{-2\lambda}^{2\lambda} \left|\frac{d^n}{dt^n}\left[g(1 + it)\, u\left(\frac{t}{2\lambda}\right)\right]\right| dt \leqq \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(2\lambda)^k} \int_{-2\lambda}^{2\lambda} \left|g^{(n-k)}(1 + it)\, u^{(k)}\left(\frac{t}{2\lambda}\right)\right| dt \leqq$$
$$\leqq \sum_{k=0}^{n} \frac{C_{nk}}{(2\lambda)^k} \int_0^{2\lambda} |g^{(n-k)}(1 + it)| dt = O(1)$$

for $(d^n/dt^n)\, g(1 + it) \in L^1(E_1)$ and $g(1 - it) = \overline{g(1 + it)}$.

We shall now prove the relation (6). Let numbers $a > 1, \lambda > 1$ be given. According to (5) there exist numbers $K > 0$ and $y_0 > a/2\lambda$ such that

$$\int_{-a}^{a} B\left(y - \frac{v}{2\lambda}\right) \hat{u}(v)\, dv \leqq \int_{-\infty}^{\infty} \hat{u}(v)\, dv + K y^{-n}$$

holds for all $y > y_0$. Hence we deduce similarly as in the proof in [4] that for all $y > y_0$

$$\lim_{x \to y-} B(x) \int_{-\infty}^{\infty} \hat{u}(v)\, dv \leqq \int_{-\infty}^{\infty} \hat{u}(v)\, dv + K y^{-n} ,$$

i.e.,

$$\lim_{x \to y-} B(x) \leqq 1 + K' y^{-n} \quad \text{for all } y > y_0, \text{ where } K' = \left(\int_{-\infty}^{\infty} \hat{u}(v)\, dv\right)^{-1} K .$$

Since $B(x) = e^{-x} A(x)$ and the function A is nondecreasing, the function B is continuous on the interval $(y_0; +\infty)$ except for a countable set of points. The inequality $B(y) \leqq 1 + K' y^{-n}$ holds, however, at the points of discontinuity of the function B. We can verify this fact letting $y \to y_1 +$, where y_1 is a point of discontinuity and y are points of continuity of the function B. The inequality $B(y) \geqq 1 - K' y^{-n}$ for

$y > y_0$ can be proved similarly as in [4], p. 126, (14), if we consider the continuity properties of the function B. We have proved the relation (6), and Theorem 2 as well.

If we want to deduce Theorem 1 from Theorem 2, i.e., to verify that $g(x) = \sum_{n \leqq x} \Lambda(n) \log (x/n) = x + O(x \log^{-k} x)$, $x \to \infty$, for any $k \in N$, we must put $A(x) = g(e^x)$ in Theorem 2 and show that in the halfplane $\operatorname{Re} s > 1$ the relation

$$- \frac{\zeta'(s)}{s^2 \zeta(s)} = \int_1^\infty \frac{g(x)}{x^{s+1}} \, dx = \int_0^\infty g(e^x) \, e^{-xs} \, dx$$

holds. It can be verified without any difficulties that the function $f(s) = - (\zeta'(s)/s^2 \zeta(s))$ satisfies all the assumptions of Theorem 2 for any $n \in N$; the integrability of the n-th derivative of the function $f(s) - 1/(s - 1)$ was verified in the third section. According to Theorem 2 we have $e^{-x} g(e^x) = 1 + O(x^{-n})$, $x \to \infty$, i.e. $g(x) = x + O(x \log^{-n} x)$, $x \to \infty$, q.e.d.

It seems quite probable that no better estimation of the remainder term in the prime number theorem can be proved by the method used in this paper than $O(x \log^{-k} x)$, $x \to \infty$.

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