Časopis pro pěstování matematiky

Jiří Jarník; Jaroslav Kurzweil

A non-absolutely convergent integral which admits C^1-transformations

Časopis pro pěstování matematiky, Vol. 109 (1984), No. 2, 157--167

Persistent URL: http://dml.cz/dmlcz/108497

Terms of use:

© Institute of Mathematics AS CR, 1984

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

Časopis pro pěstování matematiky, roč. 109 (1984), Praha

A NON-ABSOLUTELY CONVERGENT INTEGRAL WHICH ADMITS C^1-TRANSFORMATIONS

JIŘÍ JARNÍK, JAROSLAV KURZWEIL, Praha

(Received June 30, 1983)

1. INTRODUCTION

It has been known since 1957 – cf. [1] – that a formally inconspicuous modification of Riemann's definition of integral leads to a concept of integral that includes both Lebesgue and Perron integrals. Roughly speaking, the modification consists in replacing the (positive) constant that measures the "fineness" of a partition of the domain of integration by a (positive) function called a gauge. Analogously to Riemann's definition, each interval of a partition is associated with a "distinguished" point at which the function value is taken and the integral sum corresponding to the partition is evaluated. (The conditions imposed on the partition together with the "distinguished" points make it possible to obtain a definition equivalent either to Lebesgue's or to Perron's one.) If the integral sums converge (in a certain sense), then their limit is called the S-integral (summation integral).

J. Mawhin in [2] suggested a generalization that consisted in reducing the class of "admissible" partitions by introducing a "rate of stretching" of an interval as a measure of the "irregularity" of a partition. His aim was to obtain a concept of integral (in n-dimensional Euclidean space) for which the divergence theorem would hold under the possibly weakest assumptions.

Mawhin's paper stimulated further research in this direction. In [3] the present authors and Š. Schwabik gave some other modifications. The resulting concepts of integral preserved the "nice" features of Mawhin's integral and, moreover, possessed some other useful properties (additivity and continuity with respect to the integration domain, availability of a Lebesgue-type dominated convergence theorem). Another feature worth mentioning is that all these concepts cover some nonabsolutely convergent integrals. This suggested the possibility of obtaining a concept of a (nonabsolutely convergent) multiple integral that would admit transformation. The first step in this direction is made in the present paper, where such a concept of integral will be introduced and studied in the Euclidean plane; this restriction considerably diminishes the technical difficulties.

2. DEFINITION

Let us recall some notations used throughout the paper. $\mathbb{R}^2$ is the Euclidean plane with the usual Euclidean norm $\|\cdot\|$. If $M \subset \mathbb{R}^2$, then $m(M)$ stands for the Lebesgue measure of M and ∂M, $\operatorname{Int} M$, $\operatorname{Cl} M$ are the boundary, interior and closure of M, respectively. Moreover, if $x \in \mathbb{R}^2$, we denote by $\operatorname{dist}(x, M)$ the distance of x from M and by $B(x, r)$ the open ball with center x and radius $r > 0$.

We shall say that two sets $M_1, M_2 \subset \mathbb{R}^2$ are nonoverlapping if $\operatorname{Int} M_1 \cap \operatorname{Int} M_2 = \emptyset$.

Our definition follows the general scheme given in the conclusion of [3].

Let $\mathcal{I}$ be the family of compact domains $I \subset \mathbb{R}^2$, whose boundaries consist of a finite number of disjoint piecewise smooth simple closed curves.

Let $\mathcal{J}$ be the family of compact domains $J \subset \mathbb{R}^2$, whose boundaries are piecewise smooth simple closed curves.

Given $I \in \mathcal{I}$, then we denote by $CP_L(I)$ the collection of partitions Δ of I:

$$\Delta = \{(x^j, J^j) ; j = 1, \ldots, p\}$$

with $x^j \in I$, $J^j \in \mathcal{J}$, $I = \bigcup_{j=1}^{p} J^j$, J^j non-overlapping sets, and by $CP_P(I)$ the collection of all partitions $\Delta \in CP_L(I)$ satisfying $x^j \in J^j$, $j = 1, \ldots, p$. (In [3], indices 1, 2 were used instead of P, L, respectively. Here we wish to indicate that in the simplest case we obtain the Perron and Lebesgue integral, respectively — cf. [1].)

A gauge on I is a function $\delta : I \to (0, +\infty)$ and we say that Δ is δ-fine if $J^j \subset \operatorname{Cl} B(x^j, \delta(x^j))$. Finally, we define the function $\Sigma : CP_L(I) \to \mathbb{R}$ by

$$\Sigma(\Delta) = \sum_{j=1}^{p} \int_{\partial J^j} \|x - x^j\| \, ds ,$$

where s represents the length of arc on ∂J^j.

Definition. Let $I \in \mathcal{I}$, $f : I \to \mathbb{R}$, $i \in \{P, L\}$. Let $\gamma \in \mathbb{R}$. If for every $\varepsilon > 0$ and $K > 0$ there is a gauge δ on I such that for every δ-fine $\Delta \in CP_i(I)$ with $\Sigma(\Delta) \leqq K$ the inequality

$$|\gamma - S(I, f, \Delta)| \leqq \varepsilon$$

holds, then f is said to be *MT_i-integrable over I* and γ is called the *MT_i-integral of f over I* and we write $\gamma = (MT_i) \int_I f$.

Here, of course, $S(I, f, \Delta) = \sum_{j=1}^{p} f(x^j)\, m(J^j)$.

The letters MT stand for "Mawhin's Transformable". Notice that $CP_P(I) \subset CP_L(I)$ implies that an MT_L-integrable function is MT_P-integrable as well (over the same domain U), and both integrals coincide.

In order for our definition to make good sense, we have to prove

Proposition. *For every $I \in \mathscr{I}$ there is such a $K > 0$ that for every gauge δ on I there is a δ-fine partition $\Delta \in CP_P(I)$ that satisfies $\Sigma(\Delta) \leqq K$.* (In view of the inclusion $CP_P(I) \subset CP_L(I)$, Proposition justifies both the definitions, of MT_P- as well as MT_L-integral.)

Proof. Let $I \in \mathscr{I}$. Then the boundary ∂I consists of a finite number of disjoint piecewise smooth simple closed curves, say $\Gamma_0, \Gamma_1, \ldots, \Gamma_k$. Let us assume without loss of generality that the curves $\Gamma_1, \ldots, \Gamma_k$ lie in the inner domain of the curve Γ_0.

Each curve Γ_i, $i = 0, 1, \ldots, k$, can be partitioned into a finite number of smooth arcs γ_{ij}, $j = 1, \ldots, j_i$, with parametrizations

$$x = \varphi_{ij}(s), \quad s \in [a_{ij}, a_{i,j+1}], \quad j = 1, \ldots, j_i,$$

$\varphi(a_{i,j_i+1}) = \varphi(a_{i1})$, where the parameter s represents the length of arc on Γ_i, and the arcs γ_{ij} have the following property: if we write

$$\dot{x}(s) = u(s),$$

where $u(s)$ is a unit vector, $u(s) = (\cos \alpha(s), \sin \alpha(s))$, then for fixed i, j and all $s \in [a_{ij}, a_{i,j+1}]$ we have either

(i) $-2\omega \leqq \alpha \leqq 2\omega$ or $\pi - 2\omega \leqq \alpha \leqq \pi + 2\omega$

(a "horizontal" arc),

or

(ii) $\dfrac{\pi}{2} - 2\omega \leqq \alpha \leqq \dfrac{\pi}{2} + 2\omega$ or $\dfrac{3\pi}{2} - 2\omega \leqq \alpha \leqq \dfrac{3\pi}{2} + 2\omega$

(a "vertical" arc),

or

(iii) $\omega \leqq \alpha \leqq \dfrac{\pi}{2} - \omega$ or $\dfrac{\pi}{2} + \omega \leqq \alpha \leqq \pi - \omega$ or $\pi + \omega \leqq \alpha \leqq \dfrac{3\pi}{2} - \omega$

or $\dfrac{3\pi}{2} + \omega \leqq \alpha \leqq 2\pi - \omega$

(an "oblique" arc),

where ω is a fixed constant, say $0 < \omega < \dfrac{\pi}{16}$.

Let us denote

$$V_i = \{x\,;\, x = \varphi_{ij}(a_{ij}),\ j = 1, \ldots, j_i\},$$
$$V = \bigcup_{i=0}^{k} V_i.$$

Let us denote by δ_0 a gauge on I with the following properties:

(1°) if $x \in I \setminus \partial I$ then $\delta_0(x) \leqq \frac{1}{2} \operatorname{dist}(x, \partial I)$;

(2°) if $x \in \partial I \setminus V$ then

(α) $\delta_0(x) \leqq \frac{1}{2} \operatorname{dist}(x, V)$;
(β) $B(x, 2\,\delta_0(x)) \cap \partial I$ is connected (in particular, the neighborhood $B(x, 2\,\delta_0(x))$ contains points of a single arc γ_{ij});

(3°) if $x \in V$, then
(α) $\delta_0(x) \leqq \frac{1}{2} \operatorname{dist}(x, V \setminus \{x\})$;
(β) $B(x, \delta_0(x)) \cap \partial I$ is connected (in particular, the neighborhood $B(x, \delta_0(x))$ contains points of only two adjacent arcs $\gamma_{ij}, \gamma_{i,j+1}$ for which x is the common endpoint).

Let δ be an arbitrary gauge on I. Without loss of generality we may assume that $\delta(x) \leqq \delta_0(x)$ for all $x \in I$. Our task is to construct a δ-fine CP_P-partition Π of I such that $\Sigma(\Pi) \leqq K$, where K is a constant independent of the gauge δ and the partition Π (K may depend on the parameters of the domain I, viz. on $m(I)$).

The construction of the partition Π will proceed in three parts. First we shall cover the "vertices" of I, i.e. the points from the set V, then the "edges", i.e. the rest of the boundary, and finally the rest of the interior of I.

Part 1. We start by covering I with a square net whose nodes are the points $(k_1 2^{-r}, k_2 2^{-r})$ with k_1, k_2 arbitrary integers, r a fixed positive integer. Denote

$$Q^r_{p,q} = [p\,2^{-r}, (p+1)\,2^{-r}] \times [q\,2^{-r}, (q+1)\,2^{-r}]$$

and choose r so that the inclusion

$$Q^r_{p(v),q(v)} \subset B(v, \tfrac{1}{4}\,\delta(v))$$

holds for all $v \in V$ (V is a finite set!), where $p(v), q(v)$ are such integers that $v \in Q^r_{p(v),q(v)}$. Let

$$I^*(v) = \bigcup_{\lambda,\mu=-1}^{1} Q^r_{p(v)+\lambda,q(v)+\mu} \,.$$

Then $I^*(v) \subset B(v, \delta(v))$ and we include $(v, I(v))$ with $I(v) = I^*(v) \cap I$ in the partition Π.

Let us estimate the value of the integral

$$\int_{\partial I(v)} \|v - x\| \,.$$

We evidently have $\|v - x\| \leqq 3\sqrt{(2)}\,.\,2^{-r}$ for $x \in \partial I(v)$. The integration path consists of part of the perimeter of the square with sides of length $3\,.\,2^{-r}$ and of the curved part (which is part of boundary of I); the latter is again estimated by const $.\,2^{-r}$. Thus the above integrals contribute to $\Sigma(\Pi)$ by const $.\,4^{-r}\,.\,|V|$, where $|V|$ is the number of elements of V. By choosing r sufficiently small we can make this contribution arbitrarily close to zero.

Part 2. Let us consider a square Q^r_{pq} such that no part of it was included in the partition Π in the previous step, but such that $Q^r_{pq} \cap \partial I \neq \emptyset$. Assume that

(*) there is a point $u \in Q^r_{pq}$ such that $Q^r_{pq} \subset B(u, \frac{1}{4}\delta(u))$.

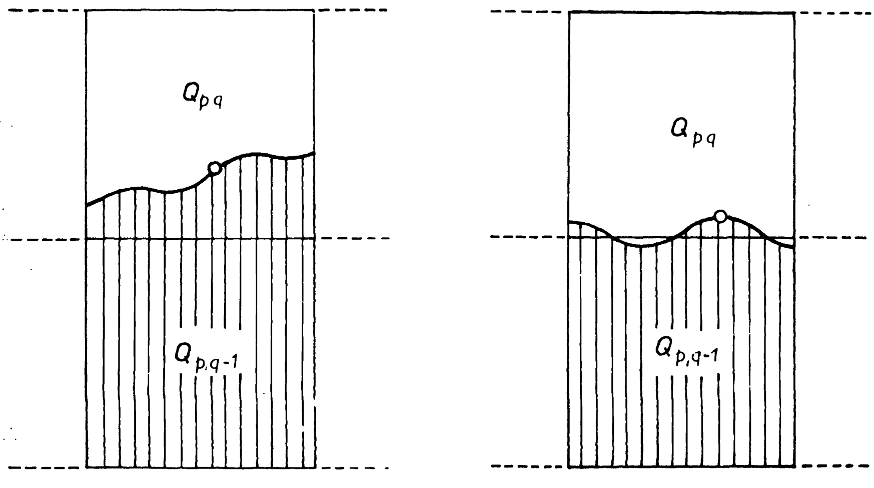
It is evident that in this case the intersection $Q^r_{pq} \cap \partial I$ contains only points of one arc γ_{ij} (cf. 2°(β)), and consequently, for all points $x \in Q^r_{pq} \cap \partial I \subset \gamma_{ij}$ one of the inequalities (i)–(iii) holds (the same for all such x).

Moreover, $u \in \gamma_{ij}$ (cf. 2°(α)). The are three possibilities according to which one of the inequalities (i)–(iii) holds. We shall deal with each case separately.

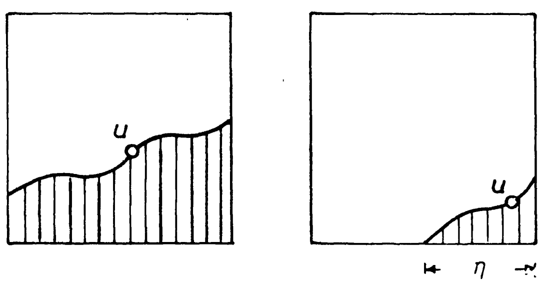
First, let γ_{ij} be "horizontal". Then either $\gamma_{ij} \cap \operatorname{Int} Q^r_{p,q+\lambda} = \emptyset$ for $\lambda = \mp 1$, or one (and only one) of these intersections is nonvoid. (See Figs. (a), (b)). In the former case, one of the squares $Q^r_{p,q+\lambda}$, $\lambda = \mp 1$, is in I (this follows from the fact that γ_{ij} is "horizontal"); we add this square to $Q^r_{p,q}$, set

(**) $$I(u) = (Q^r_{pq} \cup Q^r_{p,q+\lambda}) \cap I$$

and include the pair $(u, I(u))$ in the partition Π. In the latter case, we add the square $Q^r_{p,q+\lambda}$ ($\lambda = 1$ or -1) that has nonvoid intersection with γ_{ij}, set again (**) and include



(a) (b)



(c) (d)

$(u, I(u))$ in the partition Π. (Evidently $I(u) \subset B(u, \delta(u))$.) If γ_{ij} is "vertical", we proceed analogously, adding to $Q^r_{p,q}$ either the square $Q^r_{p-1,q}$ or $Q^r_{p+1,q}$. (Notice that the assumptions on δ and u guarantee that no square is required more than once: for example, if γ_{ij} is "horizontal" in $Q^r_{p,q}$, and if the square $Q^r_{p,q-1}$ is added to it, then γ_{ij} cannot be vertical in $Q^r_{p-1,q-1}$ nor in $Q^r_{p+1,q-1}$ and the square $Q^r_{p,q-1}$ is not added to any of them.) Finally, if γ_{ij} is "oblique", we put $I(u) = Q^r_{pq} \cap I$ and include $I(u)$ in Π again.

In this way we deal step by step with all intervals Q^r_{pq} that fulfil $Q^r_{pq} \cap \partial I \neq \emptyset$, the condition $(*)$, and that have not yet been included in Π. (It can be proved from 2° (β) that all sets $I(u)$ obtained in this way are non-overlapping.)

Put $I_{r+1} = I \setminus \bigcup_u I(u)$. If $I_{r+1} \cap \partial I \neq \emptyset$, we cover I_{r+1} with squares Q^{r+1}_{pq} of the form

$$[p\, 2^{-(r+1)}, (p + 1)\, 2^{-(r+1)}] \times [q\, 2^{-(r+1)}, (q + 1)\, 2^{-(r+1)}]$$

(i.e., $I_{r+1} \cap Q^{r+1}_{pq} \neq \emptyset$ for every couple p, q).

Let $Q^{r+1}_{pq} \cap \partial I \neq \emptyset$ and assume that there exists such a $u \in Q^{r+1}_{pq}$ that

$$Q^{r+1}_{pq} \subset B(u, \tfrac{1}{4}\, \delta(u)) .$$

Then obviously $u \in \gamma_{ij}$. For such a square Q^{r+1}_{pq} construct $I(u)$ in a way analogous to that used above. After dealing with all such squares Q^{r+1}_{pq}, put $I_{r+2} = I \setminus \bigcup_u I(u)$ (the union is taken over all sets $I(u)$ constructed till now and corresponding to the squares Q^r_{pq} and Q^{r+1}_{pq}). If $I_{r+2} \cap \partial I \neq \emptyset$, cover I_{r+2} with squares Q^{r+2}_{pq}, etc. After a finite number of steps we obtain such a set I_l that $I_l \cap \partial I = \emptyset$, i.e. $\partial I \subset \bigcup_u I(u)$, where the sets $I(u)$ are all the sets resulting in the described way from the squares of "orders" $r, r + 1, \ldots, l$.

Indeed, if there were z satisfying $z \in \partial I \cap I_m$ for $m = r + 1, r + 2, \ldots$, then there would exist such a square Q^v_{pq} that $z \in Q^v_{pq} \subset B(z, \frac{1}{4}\, \delta(z))$ and consequently, $z \in \partial I \cap I_{v+1}$, a contradiction.

Let us again estimate the contribution of all the sets just constructed to the value of $\Sigma(\Pi)$, i.e. the sum

$$\sum_{u \in \partial I \setminus V} \int_{\partial I(u)} \|u - x\| .$$

Denote $\tan 2\omega = k$. If γ_{ij} is a "horizontal" arc, then for the case sketched in Fig. (a) we have estimates

$$\|u - x\| \leqq 2^{-r} \sqrt{5} \quad \text{for} \quad x \in \partial I(u) ,$$
$$\ell(\partial I(u)) \leqq 5 \,.\, 2^{-r} + (1 + k^2)^{1/2}\, 2^{-r} ,$$
$$m(I(u)) \geqq 2^{-2r} ,$$

where ℓ denotes the length of a curve. Hence

$$\int_{\partial I(u)} \|u - x\| \leqq 2^{-2r} \sqrt{5} \, [5 + (1 + k^2)^{1/2}] \leqq C(k) \, m(I(u)) .$$

Similarly, the situation sketched in Fig. (b) allows estimates

$$\|u - x\| \leqq 2^{-r}(\sqrt{(2)} + k) \quad \text{for} \quad x \in \partial I(u) ,$$
$$\ell(\partial I(u)) \leqq 5 \, . \, 2^{-r} + (1 + k^2)^{1/2} \, 2^{-r} ,$$
$$m(I(u)) \geqq 2^{-2r} - (1 + k^2)^{1/2} \, 2^{-2r} ,$$

which yields

$$\int_{\partial I(u)} \|u - x\| \leqq 2^{-2r}(\sqrt{(2)} + k) \, [5 + (1 + k^2)^{1/2}] \leqq C m(I(u)) ,$$

C a constant.

The case of "vertical" arcs is symmetrical, so that we obtain analogous estimates.

If γ_{ij} is an "oblique" arc, there are four possible intervals for the angle α (cf. (iii)). Suppose, for instance, that the inequality

$$\omega \leqq \alpha \leqq \frac{\pi}{2} - \omega$$

holds (the other cases are quite analogous). Denote $\tan \omega = h$, then $\tan [(\pi/2) - \omega] = h^{-1}$ and

$$h \leqq \tan \alpha \leqq h^{-1} .$$

Figs. c, d show the two essentially different situations. For (c) we obtain the estimates

$$\|u - x\| \leqq \sqrt{(2)} \, . \, 2^{-r} \quad \text{for} \quad x \in \partial I(u) ,$$
$$\ell(\partial I(u)) \leqq 4 \, . \, 2^{-r} ,$$
$$m(I(u)) \geqq \tfrac{1}{2} \, 2^{-r} \, . \, 2^{-r} h .$$

Hence

$$\int_{\partial I(u)} \|u - x\| \leqq \sqrt{(2)} \, . \, 2^{2-2r} \leqq 8 \sqrt{(2)} \, m(I(u)) \, h^{-1} .$$

For (d) we have

$$\|u - x\| \leqq \eta(1 + h^{-2})^{1/2} \quad \text{for} \quad x \in \partial I(u) ,$$
$$\ell(\partial I(u)) \leqq \eta(1 + h^{-1}) + \eta(1 + h^{-2})^{1/2} ,$$
$$m(I(u)) \geqq \tfrac{1}{2} \eta^2 h ,$$

which yields

$$\int_{\partial I(u)} \|u - x\| \leqq (1 + h^{-2})^{1/2} \, [1 + h^{-1} + (1 + h^{-2})^{1/2}] \, \eta^2 \leqq$$
$$\leqq 2h^{-1}(1 + h^{-2})^{1/2} \, [1 + h^{-1} + (1 + h^{-2})^{1/2}] \, m(I(u)) .$$

Combining the above estimates and summing over all "distinguished" points $u \in \in \partial I \setminus V$ we obtain an estimate

$$\sum_{u \in \partial I \setminus V} \int_{\partial I(u)} \|u - x\| \leqq C \sum_{u \in \partial I \setminus V} m(I(u)),$$

where C denotes a constant.

Part 3. The closure of the set I_l is the union of squares Q^l_{pq}; all squares included in this union are part of the interior of I. Thus, let

$$Q^l_{pq} \subset \operatorname{Int} I .$$

We proceed similarly as in the preceding case. If there is $w \in Q^l_{pq}$ such that $Q^l_{pq} \subset \subset B(w, \delta(w))$, we include the square in Π; if not, we halve both sides and try the same with the resulting squares; after a finite number of steps we arrive at squares for which the above condition is fulfilled (this follows by the same argument as above), so that the partition Π of I (which of course is δ-fine) is fully constructed. The estimate of the contribution of these "inner" squares to $\Sigma(\Pi)$ is easy: we have

$$\int_{\partial Q^l_{pq}} \|w - x\| \leqq 4\sqrt{(2)} \,.\, 2^{-2r} = 4\sqrt{(2)}\, m(Q^l_{pq}),$$

and

$$\sum_{Q^l_{pq} \subset \operatorname{Int} I} \int_{\partial Q^l_{pq}} \|w - x\| \leqq 4\sqrt{2} \sum_{Q^l_{pq} \subset \operatorname{Int} I} m(Q^l_{pq}).$$

Putting together all three estimates corresponding to points from V, from $\partial I \setminus V$ and from $\operatorname{Int} I$, we eventually obtain

$$\sum_{(u^j, I^j) \in \Pi} \int_{\partial I^j} \|u^j - x\| \leqq \text{const.}\, m(I),$$

where the constant depends only on the angle ω. This completes the proof of Proposition.

3. TRANSFORMATION THEOREM

Theorem 1. *Let $U, V \in \mathcal{I}$, let $\Phi : U \to V$ be a regular diffeomorphism of U onto V. Let $i \in \{P, L\}$. Then*

$$(1) \qquad (MT_i)\int_U f \,\mathrm{d}x = (MT_i)\int_V (f \circ \Phi^{-1}) \left|\det D\Phi^{-1}\right| \mathrm{d}y$$

provided one of the integrals exists.

Proof. Let us prove the theorem for the MT_L-integral; the proof for the other integral is analogous. Assume, for instance, that the lefthand side integral exists. Let $\varepsilon > 0$, K be given. Set

(2) $$K' = K \sup_{y \in V} (\|D\,\Phi^{-1}(y)\| + 1)^2$$

and find a gauge δ' on U such that for every δ'-fine partition $\Delta' \in CP_L(U)$ with $\Sigma(\Delta') \leqq K'$ we have

(3) $$\left|(MT_L)\int_U f\,\mathrm{d}x - S(U, f, \Delta')\right| < \tfrac{1}{2}\varepsilon\,.$$

Further, find a gauge δ on V such that

(4) $$\Phi^{-1}(B(y, \delta(y))) \subset B(x, \delta'(x)) \quad \text{for} \quad y = \Phi(x)\,, \quad x \in U\,,$$

(5) $$|\det D\,\Phi^{-1}(y) - \det D\,\Phi^{-1}(\eta)| \leqq \frac{\varepsilon}{2m(V)\,[1 + |f(\Phi^{-1}(\eta))|]}$$

$$\text{for} \quad y, \eta \in V\,, \quad y \in B(\eta, \delta(\eta))\,,$$

(6) $$\|\Phi^{-1}(y) - \Phi^{-1}(\eta)\| \leqq (\|D\,\Phi^{-1}(\eta)\| + 1)\,.\,\|y - \eta\|$$

$$\text{again for} \quad y, \eta \in V\,, \quad y \in B(\eta, \delta(\eta))\,.$$

Let $\Delta \in CP_L(V)$ be a δ-fine partition of V with $\Sigma(\Delta) \leqq K$,

$$\Delta = \{(y^j, G^j)\,;\; j = 1, \ldots, p\}\,.$$

Put $x^j = \Phi^{-1}(y^j)$, $H^j = \Phi^{-1}(G^j)$, $j = 1, \ldots, p$. Then

$$\Delta' = \{(x^j, H^j)\,;\, j = 1, \ldots, p\} \in CP_L(U)$$

is a partition of U and, in view of (4), it is δ'-fine. Further, we have

$$\Sigma(\Delta') = \sum_{j=1}^{p} \int_{\partial H^j} \|x - x^j\|\,\mathrm{d}s \leqq \sum_{j=1}^{p} \int_{\partial G^j} \|\Phi^{-1}(y) - \Phi^{-1}(y^j)\|\,\|D\,\Phi^{-1}(y)\|\,\mathrm{d}\sigma \leqq$$

$$\leqq \sum_{j=1}^{p} \sup_{y \in V} (\|D\,\Phi^{-1}(y)\| + 1)\,\|D\,\Phi^{-1}(y)\| \int_{\partial G^j} \|y - y^j\|\,\mathrm{d}\sigma$$

in view of (6), so that $\Sigma(\Delta') \leqq K \sup_{y \in V} (\|D\,\Phi^{-1}(y)\| + 1)^2$.

Let us estimate the difference

$$|S(U, f, \Delta') - S(V, (f \circ \Phi^{-1})|\det D\,\Phi^{-1}|, \Delta)| =$$

$$= \Big|\sum_{j=1}^{p} \{f(x^j)\,m(H^j) - f(\Phi^{-1}(y^j))|\det D\,\Phi^{-1}(y^j)|\,m(G^j)\}\Big| \leqq$$

$$\leqq \sum_{j=1}^{p} |f(x^j)|\,.\,|m(H^j) - |\det D\,\Phi^{-1}(y^j)|\,m(G^j)|\,.$$

Since $m(H^j) = \int_{H^j} dx = \int_{G^j} |\det D\,\Phi^{-1}(y)|\, dy$, we have by (4) and (5)

$$|m(H^j) - |\det D\,\Phi^{-1}(y^j)|\, m(G^j)| \leqq \int_{G^j} |\det D\,\Phi^{-1}(y) - \det D\,\Phi^{-1}(y^j)|\, dy \leqq$$

$$\leqq \frac{\varepsilon\, m(G^j)}{2\, m(V)\,(1 + |f(\Phi^{-1}(y^j))|)}\,; \quad \text{consequently}\,,$$

$$|S(U, f, \Delta') - S(V, (f \circ \Phi^{-1})|\det D\Phi^{-1}|, \Delta)| \leqq$$

$$\leqq \sum_{j=1}^{p} \frac{|f(x^j)|}{1 + |f(x^j)|} \cdot \frac{\varepsilon}{2\, m(V)}\, m(G^j) \leqq \frac{\varepsilon}{2}\,.$$

Hence in view of (3),

$$\left|(MT_L)\int_U f\, dx - S(V, (f \circ \Phi^{-1})|\det D\Phi^{-1}|, \Delta)\right| < \varepsilon$$

which proves (1).

4. DIVERGENCE THEOREM

Theorem 2. *Let $\Omega \subset \mathbb{R}^2$ be an open domain, let $f : \Omega \to \mathbb{R}^2$ be differentiable. Let $H \subset \Omega$, $H \in \mathcal{I}$, $i \in \{P, L\}$. Then* $\operatorname{div} f$ *is (MT_i)-integrable over H, and*

$$(7) \qquad (MT_i)\int_H \operatorname{div} f = \int_{\partial H} \omega_f = \int_{\partial H} f_1\, dx_2 - f_2\, dx_1\,.$$

Proof. Let us again give the proof for $i = L$. Given $\varepsilon > 0$, K we find for every $x \in H$ a $\delta = \delta(x) > 0$ such that

$$(8) \qquad \|f(y) - f(x) - Df_x(y - x)\| \leqq \varepsilon \|y - x\|\,,$$

where Df_x denotes the differential of f at x.

The function $\delta : x \mapsto \delta(x)$ defines a gauge on H. Let

$$\Delta = \{(x^j, H^j);\ j = 1, \ldots, m\} \in CP_L(H)$$

be a δ-fine partition of H with $\Sigma(\Delta) \leqq K$. Then

$$\int_{\partial H} \omega_f = \sum_{j=1}^{m} \int_{\partial H^j} \omega_f\,.$$

We estimate the difference

$$(9) \qquad \left|S(H, \operatorname{div} f, \Delta) - \int_{\partial H} \omega_f\right| = \left|\sum_{j=1}^{m}\left[\operatorname{div} f(x^j)\, m(H^j) - \int_{\partial H^j} \omega_f\right]\right|.$$

Following Mawhin's proof of Theorem 1 [2] we set

$$g^j(y) = f(x^j) + Df_{x^j}(y - x^j), \quad h^j(y) = f(y) - g^j(y),$$

$j = 1, \ldots, m,\ y \in H^j$. Then

$$\int_{\partial H^j} g_1^j \, dx_2 - g_2^j \, dx_1 = \int_{H^j} \operatorname{div} f(x^j) \, dx_1 \, dx_2 = \operatorname{div} f(x^j) \, m(H^j).$$

Further, by (8) we have

$$\left| \int_{\partial H^j} h_1^j \, dx_2 - h_2^j \, dx_1 \right| \leqq \varepsilon \int_{\partial H^j} \|y - x^j\| \, ds.$$

Since $f(y) = g^j(y) + h^j(y)$ for $y \in H^j$, $j = 1, \ldots, m$, we conclude from (9)

$$\left| S(H, \operatorname{div} f, \Delta) - \int_{\partial H} \omega_f \right| \leqq \sum_{j=1}^{m} \left| \int_{\partial H^j} \omega_{h^j} \right| \leqq \sum_{j=1}^{m} \int_{H^j} \|y - x^j\| \leqq \varepsilon K.$$

This proves (7).

References

[1] *J. Kurzweil:* Nichtabsolut konvergente Integrale. Teubner-Texte zur Mathematik 26, Teubner, Leipzig 1980.

[2] *J. Mawhin:* Generalized multiple Perron integrals and the Green-Goursat theorem for differentiable vector fields. Czechoslov. Math. Journal *31* (106) 1981, 614—632.

[3] *J. Jarník, J. Kurzweil, Š. Schwabik:* On J. Mawhin's approach to multiple nonabsolutely convergent integral. Časopis pěst. mat. *108* (1983), 356—380.

Authors' address: 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV).