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ON RANDOMISED SOLUTIONS OF LAPLACE'S EQUATION

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The questions studied in this paper are randomised solutions of the Dirichlet and Poisson problem for Laplace's equation with random boundary conditions and random right-hand side respectively.

The partial differential equations of mathematical physics always contain a set of coefficients and factors which, in general, serve to describe certain physical entities. Thus in the equation of heat diffusion there appear coefficients of heat conduction, of specific heat and of density; in the mathematical theory of elasticity, the module of elasticity plays an important part, etc. The magnitude of these coefficients is in every case determined by measurement — and subsequent calculations then use the mean value of these experimentally obtained values. In fact, of course, these physical constants are not really constant at all, but vary from place to place in the material, and often may well be considered as random functions. In actual cases the variance is not negligible; e. g. the rigidity of concrete in dams will often have a large coefficient of variance, reaching even 30% or more.

The situation with boundary and initial conditions is similar. For instance, in certain problems connected with large dams, it is important to determine the effect of (atmospheric) external temperature; this temperature will then appear as a boundary condition in the problem of heat conduction. Here also it is the mean temperature that is usually considered, while its variance is obviously not negligible.

Similarly for rounding-off errors in numerical solution, if considered statistically. If we study the effect of these errors on Ritz's (Richardson's) iterations in the relaxation solution of Dirichlet's problem, we have the problem of random heat sources in the equation of heat conduction, i. e. a random right-hand side.

There is a whole series of similar problems. For connected questions see [1], [2], [3].

In the present paper we will study some questions connected with random boundary conditions and random right-hand sides for the Dirichlet and the Poisson problems.

I. THE DIRICHLET PROBLEM FOR LAPLACE'S EQUATION WITH RANDOM BOUNDARY CONDITIONS

1. In this chapter we will study the problem of randomised solutions of Dirichlet's problem with random boundary conditions. Our starting point will be the so-called classical solution; by this we mean the classical formulation of Dirichlet's problem and its generalisation in Wiener's sense. We might also start with solutions determined on the basis of variational principles; but we will rather consider general regions, and solutions generalised in Wiener's sense.

In the sequel, E_n with $n = 1, 2, 3, \ldots$ will be the n-dimensional Euclidean space, K the open unit parallelepiped in E_n, and $\bar{K}$ its closure in E_n. $\mathfrak{E}$ denotes the set of all real-valued functions η defined on $\bar{K}$. There will also be given field of sets $\mathfrak{A}$ whose elements are subsets of $\mathfrak{E}$ and such that $\mathfrak{E} \in \mathfrak{A}$; and a σ-additive non-negative function μ defined on $\mathfrak{A}$, such that $\mu(\mathfrak{E}) = 1$. The elements $\eta \in \mathfrak{E}$ will be called elementary events, and elements of $\mathfrak{A}$ will be called random events. The set-function μ will be termed a probability measure.

Functions defined on $\mathfrak{E}$ will be denoted by f, g, etc.; and $f_x(\eta)$, $x \in \bar{K}$, will be the function defined on $\mathfrak{E} \times \bar{K}$ by $f_x(\eta) = \eta(x)$. We will always assume that for every $x \in \bar{K}$ the function $f_x(\eta)$ is μ-measurable and that

V,1 $$\int_{\mathfrak{E}} f_x^2(\eta)\, \mathrm{d}\mu \leqq C^2 < \infty$$

with C independent of x; and also that $x \in \bar{K}$, $y \in \bar{K}$

V,2 $$\lim_{y \to x} \int_{\mathfrak{E}} (f_x(\eta) - f_y(\eta))^2\, \mathrm{d}\mu = 0\,.$$

The set consisting of all functions $f_x(\eta)$ defined on $\mathfrak{E}$ for every $x \in \bar{K}$, and of the function $z_0(\eta) = 1$ will be denoted by L^{**}. L^* will then be the linear module over L^{**} with the reals as coefficient domain.

We will define a scalar product and norm in L^* by

$$(g_1, g_2) = \int_{\mathfrak{E}} g_1(\eta)\, g_2(\eta)\, \mathrm{d}\mu\,, \quad g_1 \in L^*\,, \quad g_2 \in L^*\,, \quad \|g\|^2 = (g, g)\,.$$

This scalar product is obviously meaningful, since each $g_l \in L^*$, $e = 1, 2, \ldots$ is a linear combination of functions $f_{x_j}(\eta)$, $x_{lj} \in \bar{K}$, $j = 1, 2, \ldots, N$, and of the function $z_0(\eta) = 1$.

L will be the completion of the linear space L^* with respect to the norm just define; and $(g_1 g_2)$, $\|g\|$ will be the corresponding extensions of scalar product and norm, respectively.[1])

2. Let there be given a continuous function φ, defined on $\bar{K}$, and a region Ω with $\bar{\Omega} \subset K$. Denote by $W(\varphi, x)$ the function defined on $\bar{K}$ in the following manner,

[1]) Evidently every element $g \in L$ is a μ-measurable function defined on $\mathfrak{E}$.

1. $x \notin \Omega$ for $W(\varphi, x) = \varphi(x)$,

2. for $x \in \Omega$, $W(\varphi, x)$ is the generalised solution of the Dirichlet problem for Laplaces equation on Ω, with the boundary condition defined by $\varphi(x)$ on the frontier of Ω. The function $W(\varphi, x)$ will be called the solution of Dirichlet's problem for boundary function φ.

Note 1. The solution $W(\varphi, x)$ is dependent only on the values which φ assumes on $\Omega^{\cdot}$ ($\Omega^{\cdot}$ is the frontier of Ω). Thus it would have sufficed for the set of elementary events to consist of functions defined on Ω only. For formal reasons, however, we use the set of elementary events formed by functions on $\bar{K}$.

Note 2. It can be proved that, for every $x \in \Omega$, there exists a measure function $\sigma(x)$ on $\Omega^{\cdot}$ such that

$$W(\varphi, x) = \int_{\Omega^{\cdot}} \varphi(x) \, \mathrm{d}\sigma(x) .$$

Our next step will be to define solutions of the Dirichlet problem for random boundary conditions. This can be performed in diverse ways; and it becomes necessary to impose different conditions in the space of elementary events $\mathfrak{E}$. For instance, we might assume that almost all functions $\eta \in \mathfrak{E}$ are continuous. To every such realisation we can then determine the solution $W(\eta, x)$, and then consider it as a random function. Such a definition is analogous to Slutsky's definition of the integral of a random function (see [4]). E. SLUTSKY assumes first that almost all realisations are measurable, so that almost all possess an integral; the resulting random function is then studied (cf. J. L. DOOB [5]). Later it was realised that this definition is not satisfactory since the value of the integral need not be a random magnitude in the corresponding field of probabilities (cf. [6]). Since then the question of a definition of the integral of a random function has been studied intensively, and a number of definitions has been put forward. Since the solution of the Dirichlet problem is very closely connected with the notion of integral (see our Note 2), it is evident that these results are applicable. In this direction, M. J. KAMPÉ de FÉRIET [7], solves the problem for a very special region (the interior of a circle) and with rather strict conditions on the field of probabilities. Our method will be similar to that of K. KARHUNEN [6] in defining the integral of a random function.

Definition 1. A random function $w(\eta, x)$ defined on $\mathfrak{E} \times \bar{K}$ will be termed a random Wienerian solution of the Dirichlet problem on Ω, with the boundary condition defined by $f_x(\eta)$ if $w(\eta, x) \in L$ for every $x \in K$, and, for every $z \in L$,

$$(z, w) = W((z, f_x(\eta)), x) .$$

This definition evidently has sense, since in view of property V,2 $(z, f_x(\eta))$ is a continuous function of x on $\bar{K}$, so that $W((z, f_x(\eta)), x)$ is defined. Now, the following theorem holds.

Theorem 1. *There exists precissely one random solution of the Dirichlet problem in the sense just defined.*

Proof. It is simple to show that, for x fixed, $W((z, f_x(\eta)), x)$ is a continuous linear functional on L. Indeed, for $\|z\| \leqq 1$ and all x, we have $(z, f_x(\eta)) \leqq C$. From the maximum principle it then follows that $W((z, f_x(\eta)), x) \leqq C$; linearity is obvious. By the Riesz-Fischer theorem, there exists precisely one $w(\eta, x)$ such that $(z, w) = W((z, f_x(\eta)), x)$. This proves theorem 2.

Note 1. If L has finite dimension (so that essentially it is a Euclidean space), evidently definition [1] is equivalent to the definition mentioned above, eg. to the definition using realisations.

Note 2. When defining the space L^{**}, we added the function $z_0(\eta) = 1$. It can be shown easily that this has no effect on $w(\eta, x)$.

3. Now define $\varrho(x, y) = (f_x(\eta), f_y(\eta))$; this function $\varrho(x, y)$ will be called the covariantive function. From V,2 it follows that $\varrho(x, y)$ is a continuous function of x, y on $\bar{K} \times \bar{K}$.

According to theorem 1, there is precisely one random solution of the Dirichlet problem in the sense of definition 1; this solution we denoted by $w(\eta, x)$. Now $w(\eta, x) \in L$ for every x, so that we may define the function $R(x, y) = (w(\eta, x), w(\eta, y))$. This function $R(x, y)$ will be called the covariantive function of the random solution of the Dirichlet problem on Ω.

Theorem 2. *Let* $\kappa_x(x, y) = W(\varrho(x, y), x)$, $\kappa_y(x, y) = W(\varrho(x, y), y)$. *Then* $R(x, y) = W(\kappa_x(x, y), y) = W(\kappa_y(x, y), x)$.

Proof. By definition 1, $(z, w(\eta, x)) = W((z, f_x(\eta)), x)$. Set therefore $z = w(\eta, y)$; we obtain

$$R(x, y) = (w(\eta, y), w(\eta, x)) = W((w(\eta, y), f_x(\eta)), x) .$$

But

$$(w(\eta, y), f_x(\eta)) = (f_x(\eta), w(\eta, y)) = W((f_x(\eta), f_y(\eta)), y) = W(\varrho(x, y), y) = \kappa_y(x, y) .$$

Thus $R(x, y) = W(\kappa_y(x, y), x)$. The remainder of our theorem is obtained by symmetry.

Corollary to theorem 2. *Let* $y \notin \Omega$. *Then* $R(x, y) = \kappa_x(x, y)$.

Proof. By theorem 2, we have $R(x, y) = W(\kappa_x(x, y), y)$. Since $y \notin \Omega$ by assumption, $W(\kappa_x(x, y), y) = \kappa_x(x, y)$. But this is our statement.

Theorem 3. *Let* $\varrho(x, y) \leqq \varepsilon$; *then* $R(x, y) \leqq \varepsilon$. This is an immediate consequence of theorem 2 and the maximum principle.

The function $\bar{w}(x) = \int_{\mathfrak{E}} w(\eta, x) \, d\mu$ will be termed the mean value of the random solution. The function $\bar{f}(x) = \int_{\mathfrak{E}} f_x(\eta) \, d\mu$ will be termed the mean value of the random conditions.

We then have the following theorem:

Theorem 4. *Let $w(\eta, x)$ be the random solution in the sense of definition* 1. *Then* $\overline{w}(x) = W(\overline{f}(x), x)$.

Proof. This theorem follows from definition 1 immediately. Indeed, $z_0(\eta) \equiv 1 \in L$, so that $(z_0, w) = W((z_0, f(\eta, x)), x)$. But in addition we have that $(z_0, w) = \overline{w}(x)$, $(z_0, f(\eta, x)) = \overline{f}(x)$.

Note 1. Theorem 2 makes possible an effective computation of the covariantive function $R(x, y)$.

Note 2. When constructing the space L we had assumed that the probability field had certain properties. In the actual construction, we start with the experimental data and construct the covariantive function $\varrho(x, y)$. This will usually not be determined quite precisely, i. e. it will somewhat differ from the true covariantive function describing the probability field according to all the assumptions. Theorem 3 then states that a small error in the determination of $\varrho(x, y)$ will lead to an error also small in the determination of $R(x, y)$.

Note 3. Theorem 4 confirms the intuitive conclusion that calculations starting with mean values determine the mean value of solutions.

Note 4. In our theorems we used the function $\varrho(x, y)$ defined for all $x \in \overline{K}$ and $y \in \overline{K}$. However, it is obvious that to determine $R(x, y)$ with $x \in \overline{\Omega}$, $y \in \overline{\Omega}$, it suffices to know $\varrho(x, y)$ for $x \in \Omega^{\cdot}$, $y \in \Omega^{\cdot}$ only ($\Omega^{\cdot}$ is the frontier of Ω). In other words, two correlation functions identical on $\Omega^{\cdot} \times \Omega^{\cdot}$ define the same function $R(x, y)$.

Note 5. Essentially, theorem 2 is a formulation of a special problem on differential equations or their systems. The boundary conditions are not given on the frontier of the range of definition, i. e. on $(\Omega \times \Omega)^{\cdot}$, but merely on its "edges" $\Omega^{\cdot} \times \Omega^{\cdot}$.

Note 6. From theorem 2 it follows that all the properties of the Laplace equation are, in essence, preserved. E. g. the region remains stable for the problem of determining $R(x, y)$ if it was originally stable for the Laplace equation.

Note 7. Theorem 2 makes possible an effective numerical computation of the function $R(x, y)$.

II. THE POISSON PROBLEM FOR LAPLACE'S EQUATION WITH RANDOM RIGHT-HAND SIDE

In this chapter we will use the notation of the preceding chapter; this mostly concern Ω, K, $\mathfrak{E}$, L, (u, v), $\|u\|$. We will study the problem $\Delta u = f$ with a random functions f. The Dirichlet boundary conditions are assumed homogeneous.

1. Let there be given a continuous function φ on $\overline{K}$ and a region $\Omega \subset \overline{\Omega} \subset K$. Define a function $P(\varphi, x)$ on $\overline{K}$ thus

$$P(\varphi, x) = Q(\varphi, x) - W(Q(x), x)$$

where we have put

$$Q(\varphi, x) = -\frac{1}{\omega}\int_K \frac{1}{r^{n-2}}\,\varphi(\xi)\,\mathrm{d}K \quad \text{for} \quad n > 2\,,$$

$$Q(\varphi, x) = -\frac{1}{\omega}\int_K \lg\frac{1}{r}\,\varphi(\xi)\,\mathrm{d}K \quad \text{for } n = 2\,,$$

$$Q(\varphi, x) = \frac{1}{2}\int_K r\varphi(\xi)\,\mathrm{d}K \quad \text{for } n = 1$$

and

$$r = |x - \xi| = \sqrt{\sum_{r=1}^{n} (x_i - \xi_i)^2}\,;$$

ω is the surface of the unit sphere in E_n, $\xi = (\xi_1, \ldots, \xi_n) \in K$ $x = (x_1, \ldots, x_n) \in K$. The function $P(\varphi, x)$ will be called the solution of the Poisson problem for the function φ (and for region Ω).

Note 1. The function $P(\varphi, x)$ is the solution of the Poisson problem for homogenous (i. e. zero) boundary conditions.

Note 2. The definition evidently has sense; if φ is continuous, then $Q(\varphi, x)$ is continuous on K, so that we can consider the function $W(Q(x), x)$.

Note 3. If the function φ is sufficiently regular – e. g. if its first partial are continuous – then $P(\varphi, u)$ has continuous second partials on Ω and $\Delta P(\varphi, x) = \varphi$. However, if φ is merely continuous, the function $P(\varphi, x)$ need not have second partials at all, and $\Delta P(\varphi, x) = \varphi$ must be considered in the generalised sense.

Note 4. If $x \notin \Omega$ then obviously $P(\varphi, x) = 0$.

After these introductory remarks we proceed to the definition of the random solution of the Poisson problem.

Definition 2. A random function $q(\eta, x)$ defined on $\mathfrak{E} \times \bar{K}$ will be termed a random solution of the Poisson problem on Ω, if $q(\eta, x) \in L$ for every $x \in \bar{K}$ and $(z, q) = P((z, f_x(\eta)), x)$ for every $z \in L$.

This definition evidently has sense, since by V,2 $(z, f_x(\eta)$ is a continuous function of x on $\bar{K}$.

Theorem 5. *There exists precisely one random solution of the Poisson problem in the sense of definition* 2.

The proof is similar to that of theorem 1.

2. As in the preceding chapter, we denote

$$R(x, y) = (q(\eta, x), q(\eta, y)) \quad \text{and} \quad \varrho(x, y) = (f_x(\eta), f_y(\eta))\,.$$

Then the following theorem holds: –

Theorem 6. *Let* $\kappa_x(x, y) = P(\varrho(x, y), x)$, $\kappa_y(x, y) = P(\varrho(x, y), y)$. *Then* $R(x, y) = P(\kappa_x(x, y), y) = P(\kappa_y(x, y), x)$.

The proof is just as in theorem 2.

Theorem 7. *Let* $|\varrho(x, y)| < \varepsilon$. *Then* $R(x, y) \leqq \frac{1}{2}\varepsilon$.

The proof follows easily from theorem 6. Indeed, since K is the unit parallelepiped, we have $\kappa_x(x, y) \leqq \frac{1}{2}\varepsilon$ and thus $R(x, y)$, etc.

The function $\bar{q} = \int_{\mathfrak{E}} q(\eta, x)\,\mathrm{d}\mu$ will be termed the mean value of the random solution. The function $\bar{f}(x) = \int_{\mathfrak{E}} f(\eta, x)\,\mathrm{d}\mu$ will be termed the mean value of the random right-hand sides. Then the following thorem holds:

Theorem 8. *Let* $q(\eta, x)$ *be the random solution of the Poisson problem in the sense of definition* 2. *Then* $\bar{q}(x) = P(\bar{f}(x), x)$.

The proof is similar to that of theorem 4.

Note. As to the significance of these theorems, remarks similar to those of the preceding chapter might be made.

3. We have been concerned with the randomised differential Poisson problem. Completely analogous theorems can be proved for the relaxation method solution of the Poisson problem, i. e. the solution of the difference-form Poisson problem.

4. On the basis of the preceding results the correlation function $R(x, y)$ of the randomised solution may be found. Obviously in the general case nothing can be stated about the distribution function of solutions at a given point. In actual cases, however, it is usually possible to assume that the process is Gaussian, so that the solution will be a random variable of Gaussian type also, and is completely characterised by the covariantive function.

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Výtah

O ZNÁHODNĚLÉM ŘEŠENÍ LAPLACEOVY DIFERENCIÁLNÍ ROVNICE

Ivo BABUŠKA, Praha

V práci se studuje problém znáhodnělého řešení Dirichletova resp. Poissonova problému na obecných omezených oblastech pro náhodové okrajové podmínky resp. pravou stranu.

Vychází se přitom ze slabého řešení problému, podobně jako zavádí znáhodnělý integrál K. KARHUNEN [6].

Je konstruována kovarianční funkce $R(x, y)$ hledaného řešení z kovarianční funkce okrajových podmínek resp. pravé strany.

Резюме

О СЛУЧАЙНОМ РЕШЕНИИ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ЛАПЛАСА

Иво Бабушка (Ivo Babuška), Прага

В работе исследуется проблема случайного решения задач Дирихле и Пуассона на органиченных областях общего вида при случайных краевых условиях, соотв. при случайной правой части.

При этом автор исходит из слабого решения проблемы, подобно тому, как К. Каргунен вводит случайный интеграл [6].

Построена ковариантная функция $R(x, y)$ искомого решения из ковариантной функции краевых условий, соотв. правой части.