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ON SOME BOUNDARY VALUE PROBLEMS FOR NONLINEAR FOURTH ORDER ORDINARY DIFFERENTIAL EQUATIONS

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The aim of this paper is to investigate the relationship between the existence of functions satisfying differential inequalities and the existence of a solution to the boundary value problems (1), (2), then (1), (3), and (1), (4), where

$$y^{(4)} = h(x, y, y', y'', y''') , \tag{1}$$

$$\begin{aligned} y(a) &= a_0 , \\ y'(a) &= a_1 , \\ y''(a) &= a_2 , \\ y''(d) &= d_2 , \end{aligned} \tag{2}$$

$$\begin{aligned} y''(a) &= a_2 , \\ y(d) &= d_0 , \\ y'(d) &= d_1 , \\ y''(d) &= d_2 , \end{aligned} \tag{3}$$

$$\begin{aligned} y''(a) &= a_2 , \\ y(b) &= b_0 , \\ y'(b) &= b_1 , \\ y''(d) &= d_2 . \end{aligned} \tag{4}$$

The method of G. A. KLAASEN from his article [1] will be used.

Throughout this paper it is assumed that R is the set of real numbers, $I = [a, d]$, $a < b < d$, $a_0, a_1, a_2, b_0, b_1, d_0, d_1, d_2$ are from R, $D = I \times R^4$, $h : D \to R$ is a continuous function.

Lemma 1. *If h is continuous and bounded, then for any numbers a_0, a_1, a_2, d_2 there exists a solution of the boundary value problem* (1), (2).

Proof. Since the corresponding homogeneous boundary value problem

$$y^{(4)} = 0, \tag{5}$$

$$y(a) = y'(a) = y''(a) = y''(d) = 0 \tag{6}$$

has only the trivial solution, there exists the Green function $G(x, t)$ of the non-homogeneous BVP (6), $y^{(4)} = r(x)$ so that the investigated BVP is equivalent to the integro-differential equation

$$y(x) = w(x) + \int_a^d G(x, t)\, h(t, y(t), y'(t), y''(t), y'''(t))\, dt,$$

where $w : I \to R$, $w(x) = c_1x^3 + c_2x^2 + c_3x + c_4$ is the solution of the BVP (5), (2).

Let us denote

$$\begin{aligned}
K_0 &= \sup\{|G(x, t)| : (x, t) \in I \times I\},\\
K_1 &= \sup\{|G_x(x, t)| : (x, t) \in I \times I\},\\
K_2 &= \sup\{|G_{xx}(x, t)| : (x, t) \in I \times I\},\\
K_3 &= \sup\{|G_{xxx}(x, t)| : (x, t) \in I \times I \setminus \{(t, t) : t \in I\}\},\\
K &= \max\{K_i : i \in \{0, 1, 2, 3\}\},\\
m &= \sup\{|h(x, y, z, u, v)| : (x, y, z, u, v) \in I \times R^4\},\\
M &= \sup\{|w^{(s)}(x)| : x \in I,\ s \in \{0, 1, 2, 3\}\}.
\end{aligned}$$

In the Banach space B of all functions defined on I which have continuous third derivative with the norm defined by $\|r\| = \max\{\{\max|r^{(s)}(t)| : t \in I\} : s \in \{0, 1, 2, 3\}\}$, the set $S = \{r \in B : \|r\| \leqq mK(d - a) + M\}$ is closed and convex. The mapping $T: S \to B$ defined by

$$(Tr)(x) = w(x) + \int_a^d G(x, t)\, h(t, r(t), r'(t), r''(t), r'''(t))\, dt$$

is continuous as well as compact and maps S into itself. It then follows from the Schauder Fixed-Point Theorem that T has a fixed point in S. The fixed point is a solution of the stated BVP.

Remark. Similar statements hold for the BVP (1), (3) and (1), (4). Applying any one of these statements, we shall refer always to Lemma 1.

Lemma 2. *Assume that* 1. *h is continuous;*

2. *all solutions of the initial value problems for the equation* (1) *extend to all I or extend to the interior I° of I and are unbounded in neighbourhoods of the points a and d;*
3. *functions $h_n : D \to R$ are continuous for $n = 1, 2, \ldots$;*

4. *on every compact set $K \subset D$ the sequence $(h_n \mid K)_1^\infty$ uniformly converges to $h \mid K$;*

5. *y_n are solutions of the equations $y^{(4)} = h_n(x, y, y', y'', y''')$ on I;*

6. *sequences $(y_n)_1^\infty, (y_n')_1^\infty, (y_n'')_1^\infty$ are uniformly bounded on I. Then there exists a solution $y : I \to R$ of the equation (1) such that there exists a subsequence $(y_{n_k})_{k=1}^\infty$ of the sequence $(y_n)_1^\infty$ with $y_{n_k}^{(i)} \to y^{(i)}$, $i = 0, 1, 2, 3$ and the convergence is uniform on I.*

Proof. There exists a number M such that for all positive integers n and all $x \in I$ it is $|y_n''(x)| \leqq M$ (assumption 6). Therefore there exist $x_n \in I^\circ$ such that

$$|y_n'''(x_n)| = \left|\frac{y_n''(d) - y_n''(a)}{d - a}\right| \leqq \frac{2M}{d - a}.$$

The sequences $(x_n), (y_n(x_n)), (y_n'(x_n)), (y_n''x_n)), (y_n'''(x_n))$ are bounded. Hence there exist subsequences $(x_{n_k}), (y_{n_k}(x_{n_k})), (y_{n_k}'(x_{n_k})), (y_{n_k}''(x_{n_k}))$ and $(y_{n_k}'''(x_{n_k}))$ which are all convergent. Let us denote the limits of these subsequences respectively by $x_0, y_0, y_0', y_0'', y_0'''$. For the sake of simplicity we denote the subsequences by $(x_n), (y_n(x_n)), (y_n'(x_n)), (y_n''(x_n)), (y_n'''(x_n))$. By applying the standard convergence theorem (see [2], page 15) to the vector differential equation

$$(y, y', y'', y''')' = (y', y'', y''', h(x, y, y', y'', y'''))$$

we get that there exists a subsequence $((y_{n_k}, y_{n_k}', y_{n_k}'', y_{n_k}'''))_1^\infty$ of $((y_n, y_n', y_n'', y_n'''))_1^\infty$ and a solution (y, y', y'', y''') of this equation satisfying the initial condition

$$(y, y', y'', y''')(x_0) = (y_0, y_0', y_0'', y_0''')$$

and for every compact part of I°

$$(y_{n_k}, y_{n_k}', y_{n_k}'', y_{n_k}''') \to (y, y', y'', y''') \quad \text{for} \quad k \to \infty$$

and this convergence is uniform. Since the sequence (y_{n_k}) is uniformly bounded, the function y is not unbounded, and thus extends to all I and the convergence is uniform on I.

Theorem 1. *Assume that* 1. *h is continuous and nonincreasing in the second and the third argument;*

2. *solutions of (1) extend to I or to I° and are unbounded in neighbourhoods of the points a and d;*

3. *there exists a function $u : I \to R$ satisfying the inequality*

$$u^{(4)} \geqq h(x, u, u', u'', u''')\,; \tag{7}$$

4. *there exists a function $v : I \to R$ satisfying the inequality*

$$v^{(4)} \leqq h(x, v, v', v'', v''')\,; \tag{8}$$

5. $u \leqq v,\ u' \leqq v',\ u'' \leqq v''$;

6. $u(a) \leqq a_0 \leqq v(a),\ u'(a) \leqq a_1 \leqq v'(a),\ u''(a) \leqq a_2 \leqq v''(a),\ u''(d) \leqq d_2 \leqq v''(d)$.

Then there exists a function $y : I \to R$ which satisfies the boundary value problem (1), (2) *and the inequalities*

$$u^{(i)} \leqq y^{(i)} \leqq v^{(i)}, \quad i = 0, 1, 2 .$$

Proof. For every positive integer $n \geqq N_0$, where

$$N_0 = \max\{\max\{|u'''(x)| : x \in I\}, \max\{|v'''(x)| : x \in I\}\},$$

we define functions $h_{in} : I \times R^4 \to R$ as follows:

$$h_{1n}(x, y, y', y'', y''') = \begin{cases} h(x, y, y', y'', n) & \text{for } y''' > n \\ h(x, y, y', y'', y''') & \text{for } |y'''| \leqq n \\ h(x, y, y', y'', -n) & \text{for } y''' < -n \end{cases} \tag{9}$$

$$h_{2n}(x, y, y', y'', y''') = \begin{cases} h_{1n}(x, y, y', v'', y''') + \dfrac{y'' - v''}{1 + y'' - v''} & \text{for } y'' > v'' \\ h_{1n}(x, y, y', y'', y''') & \text{for } u'' \leqq y'' \leqq v'' \\ h_{1n}(x, y, y', u'', y''') - \dfrac{u'' - y''}{1 + u'' - y''} & \text{for } y'' < u'' \end{cases} \tag{10}$$

$$h_{3n}(x, y, y', y'', y''') = \begin{cases} h_{2n}(x, y, v', y'', y''') & \text{for } y' > v' \\ h_{2n}(x, y, y', y'', y''') & \text{for } u' \leqq y' \leqq v' \\ h_{2n}(x, y, u', y'', y''') & \text{for } y' < u' \end{cases} \tag{11}$$

$$h_{4n}(x, y, y', y'', y''') = \begin{cases} h_{3n}(x, v, y', y'', y''') & \text{for } y > v \\ h_{3n}(x, y, y', y'', y''') & \text{for } u \leqq y \leqq v \\ h_{3n}(x, u, y', y'', y''') & \text{for } y < u \end{cases} \tag{12}$$

The functions h_{4n} are continuous and bounded. According to Lemma 1 there exist solutions of BVP (2) and

$$y_n^{(4)} = h_{4n}(x, y_n, y_n', y_n'', y_n'''), \quad x \in I . \tag{1_n}$$

Now we shall show that $u'' \leqq y_n'' \leqq v''$ for any solution y_n of (1_n), (2). Suppose that there exists $x \in I$ such that $y_n''(x) > v''(x)$. Since $y_n''(a) \leqq v''(a)$, $y_n''(d) \leqq v''(d)$, there must exist $x_0 \in I$ at which the function $y_n'' - v''$ has a positive relative maximum. Thus

$$(y_n'' - v'')(x_0) > 0, \quad (y_n''' - v''')(x_0) = 0, \quad (y_n^{(4)} - v^{(4)})(x_0) \leqq 0 .$$

However, according to the assumptions 1 and 4 and the definition of functions h_{in}, $i = 1, 2, 3, 4$ we have

$$
\begin{aligned}
y_n^{(4)}(x_0) - v^{(4)}(x_0) &\geqq h_{4n}(x_0, y_n(x_0), y_n'(x_0), y_n''(x_0), y_n'''(x_0)) - \\
&- h(x_0, v(x_0), v'(x_0), v''(x_0), v'''(x_0)) \geqq \\
&\geqq h_{3n}(x_0, v(x_0), y_n'(x_0), y_n''(x_0), y_n'''(x_0)) - h(x_0, v(x_0), v'(x_0), v''(x_0), \\
&v'''(x_0)) \geqq h_{2n}(x_0, v(x_0), v'(x_0), y_n''(x_0), y_n'''(x_0)) - h(x_0, v(x_0), v'(x_0), \\
&v''(x_0), v'''(x_0)) \geqq h_{1n}(x_0, v(x_0), v''(x_0), v''(x_0), y_n'''(x_0)) - \\
&- h(x_0, v(x_0), v'(x_0), v''(x_0), v'''(x_0)) + \frac{(y_n'' - v'')(x_0)}{1 + (y_n'' - v'')(x_0)} \geqq \\
&\geqq h(x_0, v(x_0), v'(x_0), v''(x_0), v'''(x_0)) - \\
&- h(x_0, v(x_0), v'(x_0), v''(x_0), v'''(x_0)) + \frac{(y_n'' - v'')(x_0)}{1 + (y_n'' - v'')(x_0)} > 0 .
\end{aligned}
$$

This contradiction says that our assumption that there exists $x \in I$ such that $y_n''(x) > v''(x)$ is false. By a similar argument, $y_n''(x) \geqq u''(x)$ on I can be shown. From $u^{(i)}(a) \leqq y_n^{(i)}(a) \leqq v^{(i)}(a)$, $i = 0, 1$ and $u'' \leqq y_n'' \leqq v''$ we obtain $u' \leqq y_n' \leqq v'$ and $u \leqq y_n \leqq v$ on I. Thus the function y_n is a solution of the equation

$$
(13) \qquad y^{(4)} = h_{1n}(x, y, y', y'', y''') .
$$

Taking into account that $h_{1n}|K_m \rightrightarrows h|\,K_m$, where

$$
\begin{gathered}
K_m = \{(x, y, y', y'', y''') \in D : x \in I\,,\quad u(x) \leqq y \leqq v(x)\,, \\
u'(x) \leqq y' \leqq v'(x)\,,\quad u''(x) \leqq y'' \leqq v''(x)\,,\quad |y'''(x)| \leqq m\}\,,\quad m = 1, 2, \ldots,
\end{gathered}
$$

we see that all conditions of Lemma 2 are satisfied and thus there exists a subsequence (y_{n_k}) and a function $y : I \to R$ such that $y_{n_k} \to y$ uniformly on I and

$$
y^{(4)}(x) = h(x, y(x), y'(x), y''(x), y'''(x))\,,\quad x \in I\,.
$$

For all k we have $y_{n_k}(a) = a_0$ and therefore $y_{n_k} \to y$ yields $y(a) = a_0$. In a similar way $y'(a) = a_1$, $y''(a) = a_2$, $y''(d) = d_2$ can be shown.

Theorem 2. *Assume that* 1. *h is continuous and nonincreasing in the second and nondecreasing in the third argument;*

2. *the solutions of* (1) *extend to I or to I° and are unbounded in neighbourhoods of the points a and d;*
3. *there exists a function $u : I \to R$ satisfying* (7);
4. *there exists a function $v : I \to R$ satisfying* (8);

5. $u \leqq v$, $u' \geqq v'$, $u'' \leqq v''$;

6. $u''(a) \leqq a_2 \leqq v''(a)$, $u(d) \leqq d_0 \leqq v(d)$, $u'(d) \geqq d_1 \geqq v'(d)$, $u''(d) \leqq d_2 \leqq v''(d)$.

Then there exists a function $y : I \to R$ which satisfies the boundary value problem (1), (3) *and the inequalities*

$$u \leqq y \leqq v\,, \quad u' \geqq y' \geqq v'\,, \quad u'' \leqq y'' \leqq v''\,.$$

Proof. Similarly as in the proof of Theorem 1, let us define the function h_{1n} by (9), h_{2n} by (10), h_{4n} by (12), and h_{3n} as follows:

$$h_{3n}(x, y, y', y'', y''') = \begin{cases} h_{2n}(x, y, u', y'', y''') & \text{for} \quad y' > u'\,, \\ h_{2n}(x, y, y', y'', y''') & \text{for} \quad v' \leqq y' \leqq u'\,, \\ h_{2n}(x, y, v', y'', y''') & \text{for} \quad y' < v'\,. \end{cases}$$

Then the functions h_{4n} are continuous and bounded. According to Lemma 1 there exist solutions y_n of BVB (3) and (1_n).

By the method used in the proof of the preceding theorem, $u'' \leqq y_n'' \leqq v''$ for any solution y_n can be shown. From these inequalities as well as from the inequalities

$$u'(d) \geqq y_n'(d) \geqq v'(d)$$

we obtain

$$u' \geqq y_n' \geqq v'\,.$$

Now the last inequality and the inequality

$$u(d) \leqq y_n(d) \leqq v(d)$$

yields the inequality

$$u \leqq y_n \leqq v\,.$$

Thus y_n is a solution of the equation (13).

Taking into account that $h_{1n} \mid K_m \rightrightarrows h \mid K_m$, where

$$K_m = \{(x, y, y', y'', y''') \in D : x \in I\,, \quad u(x) \leqq y \leqq v(x)\,,$$
$$u'(x) \geqq y' \geqq v'(x)\,, \quad u''(x) \leqq y'' \leqq v''(x)\,, \quad |y'''(x)| \leqq m\}\,, \quad m = 1, 2, \ldots,$$

we see that all conditions of Lemma 2 are satisfied and thus there exists a subsequence (y_{n_k}) and a function $y : I \to R$ such that $y_{n_k} \to y$ uniformly on I and

$$y^{(4)}(x) = h(x, y(x), y'(x), y''(x), y'''(x))\,, \quad x \in I\,.$$

For all k we have $y_{n_k}(d) = d_0$ and therefore $y_{n_k} \to y$ yields $y(d) = d_0$. In a similar way $y'(d) = d_1$, $y''(d) = d_2$, $y''(a) = a_2$ can be shown.

Theorem 3. *Suppose that* 1. *the function h is continuous, nonincreasing in the second argument and nondecreasing in the third argument for each $x \in [a, b]$ as well as nonincreasing for each $x \in [b, d]$;*

2. *the solutions of initial value problems for* (1) *extend to* I *or to its interior* I° *and are unbounded in neighbourhoods of the points* a *and* d;
3. *there exist functions* $u \in C^{(4)}(I)$, $v \in C^{(4)}(I)$ *satisfying* (7) *and* (8), *respectively.*
4. $u \leqq v$, $x \in [a, b] \Rightarrow v'(x) \leqq u'(x)$, $x \in [b, d] \Rightarrow u'(x) \leqq v'(x)$, $u'' \leqq v''$;
5. $u''(a) \leqq a_2 \leqq v''(a)$, $u(b) = b_0 = v(b)$, $u'(b) = b_1 = v'(b)$, $u''(d) \leqq d_2 \leqq v''(d)$.

Then there exists a function $y : I \to R$ *which satisfies* (1), (4) *and*

$$u \leqq y \leqq v, x \in [a, b] \Rightarrow v'(x) \leqq y'(x) \leqq u'(x), x \in [b, d] \Rightarrow u'(x) \leqq y'(x) \leqq v'(x),$$
$$u'' \leqq y'' \leqq v''.$$

Proof. Let us denote $N_0 = \max\{\max\{|u'''(x)| : x \in I\}, \max\{|v'''(x)| : x \in I\}\}$ and for all $n \geqq N_0$ define the functions h_{1n} by (9), h_{2n} by (10) and h_{3n}, h_{4n} as follows:

$$h_{3n}(x, y, y', y'', y''') = \begin{cases} h_{2n}(x, y, v', y'', y''') & \text{for} \quad a \leqq x \leqq b\,, \quad y' < v' \quad \text{and} \\ & \phantom{\text{for}} \quad b \leqq x \leqq d\,, \quad y' > v'\,, \\ h_{2n}(x, y, y', y'', y''') & \text{elsewhere}\,, \\ h_{2n}(x, y, u', y'', y''') & \text{for} \quad a \leqq x \leqq b\,, \quad y' > u' \quad \text{and} \\ & \phantom{\text{for}} \quad b \leqq x \leqq d\,, \quad y' < u'\,, \end{cases}$$

$$h_{4n}(x, y, y', y'', y''') = \begin{cases} h_{3n}(x, v, y', y'', y''') & \text{for} \quad y > v\,, \\ h_{3n}(x, y, y', y'', y''') & \text{for} \quad u \leqq y \leqq v\,, \\ h_{3n}(x, u, y', y'', y''') & \text{for} \quad y < u\,. \end{cases}$$

Every function h_{4n} is continuous and bounded and therefore (see Lemma 1) there exists a y_n which satisfies (1_n), (4).

We will now show that $u''(b) \leqq y_n''(b) \leqq v''(b)$. Suppose that $y_n''(b) < u''(b)$. Since $y_n''(a) \geqq u''(a)$, $y_n''(d) \geqq u''(d)$, there exists a subinterval containing b on which $y_n''(x) - u''(x) < 0$ and in that subinterval there exists an x_0 at which $y_n'' - u''$ has negative relative minimum (and either $x_0 < b$ and $y_n'(x_0) > u'(x_0)$ or $x_0 \geqq b$ and $y_n'(x_0) \leqq u'(x_0)$). And thus we have

$$y_n'''(x_0) = u'''(x_0)\,, \quad y_n^{(4)}(x_0) \geqq u^{(4)}(x_0)\,.$$

However,

$$\begin{aligned} y_n^{(4)}(x_0) - u^{(4)}(x_0) &\leqq h_{4n}(x_0, y_n(x_0), y_n'(x_0), y_n''(x_0), y_n'''(x_0)) - \\ &- h(x_0, u(x_0), u'(x_0), u''(x_0), u'''(x_0)) \leqq \\ &\leqq h_{3n}(x_0, u(x_0), y_n'(x_0), y_n''(x_0), y_n'''(x_0)) - h(x_0, u(x_0), u'(x_0), u''(x_0), u'''(x_0)) \leqq \\ &\leqq h_{2n}(x_0, u(x_0), u'(x_0), y_n''(x_0), y_n'''(x_0)) - h(x_0, u(x_0), u'(x_0), u''(x_0), u'''(x_0)) \leqq \\ &\leqq h_{1n}(x_0, u(x_0), u'(x_0), u''(x_0), y_n'''(x_0)) - h(x_0, u(x_0), u'(x_0), u''(x_0), \\ & u'''(x_0)) - \frac{u''(x_0) - y_n''(x_0)}{1 + u''(x_0) - y_n''(x_0)} = h(x_0, u(x_0), u'(x_0), u''(x_0), u'''(x_0)) - \\ &- h(x_0, u(x_0), u'(x_0), u''(x_0), u'''(x_0)) - \frac{u''(x_0) - y_n''(x_0)}{1 + u''(x_0) - y_n''(x_0)} < 0 \end{aligned}$$

which contradicts the previous inequality and thus our assumption that $y_n''(b) < u''(b)$ is not true. The invalidity of the assumption $y_n''(b) > v''(b)$ can be shown in the same way.

We will now show that $u'' \leqq y_n'' \leqq v''$. The validity of these inequalities and consequently of the inequalities $u^{(i)} \leqq y_n^{(i)} \leqq v^{(i)}$, $i = 0, 1$ on the interval $[b, d]$ follows from the proof of Theorem 1 in virtue of the inequalities

$$u(b) \leqq b_0 \leqq v(b), \quad u'(b) \leqq b_1 \leqq v'(b), \quad u''(b) \leqq y_n''(b) \leqq v''(b), \\ u''(d) \leqq d_2 \leqq v''(d).$$

The validity of the inequalities $u'' \leqq y_n'' \leqq v''$, $v' \leqq y_n' \leqq u'$, and $u \leqq y_n \leqq v$ on the interval $[a, b]$ follows in a similar way from the proof of Theorem 2.

The stated inequalities show that each y_n satisfies the equation (1_n).

The standard convergence theorem applied to the vector differential equation $(y, y', y'', y''')' = (y', y'', y''', h(x, y, y', y'', y'''))$ yields the existence of a subsequence (y_{n_k}) of (y_n) and the existence of a function $y : I \to R$ such that $(y_{n_k}, y'_{n_k}, y''_{n_k}, y'''_{n_k})$ converges to (y, y', y'', y''') uniformly on every compact subinterval of the interior of I and y satisfies (1).

Since (y_n), (y_n'), (y_n'') are uniformly bounded, the function y is bounded and thus y is defined on all I and the convergence is uniform on I.

Because y_n satisfy the boundary conditions (3), the same is true about y.

From the analogous inequalities for y_n we obtain by the limit process

$$u \leqq y \leqq v, \quad x \in [a, b] \Rightarrow v'(x) \leqq y'(x) \leqq u'(x), \\ x \in [b, d] \Rightarrow u'(x) \leqq y'(x) \leqq v'(x), \quad u'' \leqq y'' \leqq v''.$$

References

[1] *Gene A. Klaasen,* Differential Inequalities and Existence Theorems for Second and Third Order Boundary Value Problems, J. Differential Equations *10,* 529—537 (1971).

[2] *P. Hartman,* Ordinary Differential Equations, Russian translation, Izdavatelstvo Mir, Moscow 1970.

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