Časopis pro pěstování matematiky

William A. Webb

On the Diophantine equation $k/n = a_1/x_1 + a_2/x_2 + a_3/x_3$

Časopis pro pěstování matematiky, Vol. 101 (1976), No. 4, 360--365

Persistent URL: `http://dml.cz/dmlcz/117932`

Časopis pro pěstování matematiky, roč. 101 (1976), Praha

ON THE DIOPHANTINE EQUATION $\frac{k}{n} = \frac{a_1}{x_1} + \frac{a_2}{x_2} + \frac{a_3}{x_3}$

WILLIAM A. WEBB, Pullman

(Received July 15, 1975)

Introduction. Given positive integers k and n, and nonzero integers $a_1, a_2, \ldots, a_r$; consider the equation

$$\frac{k}{n} = \frac{a_1}{x_1} + \frac{a_2}{x_2} + \ldots + \frac{a_r}{x_r} \tag{1}$$

where the x_i are positive integers such that $(a_i, x_i) = 1$. Let $(1')$ denote the same equation where the x_i can be any nonzero integer. In the special case $a_1 = a_2 = \ldots \ldots = a_r = 1$, the so-called Egyptian or unit fractions, these equations have been extensively studied.

Let $\lambda = \lambda(k; a_1, a_2, \ldots, a_r)$ be the largest integer n for which the equation (1) has no solution. If (1) is unsolvable for infinitely many values of n, set $\lambda = \infty$. If (1) is solvable for all positive n, set $\lambda = 0$. Also, define λ' similarly with respect to equation $(1')$. Very little is known about precise values of λ and λ', even in special cases.

In this paper we will consider solutions of equation (1) with particular attention to the cases $r = 2$ and 3. The principal result obtained is a lower bound for λ and λ' when $r = 3$, and k is large.

Preliminary Results and the Case $r = 2$. If p is a prime, and $p \mid (a_1, \ldots, a_r)$, then (1) is not solvable for $n = p^s$ and all s sufficiently large, since $(x_i, p) = 1$. Hence, if $(a_1, \ldots, a_r) \neq 1$, $\lambda = \infty$ and so we assume henceforth that $(a_1, \ldots, a_r) = 1$.

The case $r = 1$ is trivial and will not be mentioned again.

The following result gives necessary and sufficient conditions for (1) to be solvable in the case $r = 2$.

Theorem 1. *The equation*

$$\frac{k}{n} = \frac{a_1}{x_1} + \frac{a_2}{x_2} \tag{2}$$

is solvable in positive integers x_1, x_2 *such that* $(a_1, x_1) = 1 = (a_2, x_2)$, *if and* *only if there exist positive divisors d_1 and d_2 of n such that $a_1d_1 + a_2d_2 = kt$ for some positive integer t such that $(a_1a_2, t) = 1$, and $(n/d_1, a_1) = (n/d_2, a_2) = 1$.*

Proof. If the conditions of the theorem are satisfied, then let $x_i = tn/d_i$. We then have

$$\frac{a_1}{x_1} + \frac{a_2}{x_2} = \frac{a_1d_1}{tn} + \frac{a_2d_2}{tn} = \frac{kt}{tn} = \frac{k}{n}$$

and $(a_i, x_i) = (a_i, tn/d_i) = 1$ by the hypotheses.

Now suppose that (2) is satisfied by x_1 and x_2 such that $(a_1, x_1) = (a_2, x_2) = 1$. Also, assume $(k, n) = 1$. Let $d = (x_1, x_2)$, $x_i = dX_i$ and $t = (d, a_1X_2 + a_2X_1)$.

Then

$$(3) \qquad \frac{k}{n} = \frac{a_1x_2 + a_2x_1}{x_1x_2} = \frac{a_1X_2 + a_2X_1}{dX_1X_2} = \frac{(a_1X_2 + a_2X_1)/t}{(d/t)\,X_1X_2}.$$

Since $(X_1, X_2) = 1$ and $(a_i, X_i) = 1$, $(a_1X_2 + a_2X_1, X_1X_2) = 1$. This, together with $((a_1X_2 + a_2X_1)/t, d/t) = 1$, implies that the right hand fraction in (3) is reduced, and so $k = (a_1X_2 + a_2X_1)/t$ and $n = (d/t)\,X_1X_2$.

Therefore, letting $d_1 = X_2$ and $d_2 = X_1$, we have immediately that $d_i \mid n$ and $a_1d_1 + a_2d_2 = kt$. Also, $(n/d_1, a_1) = (dX_1/t, a_1) = (x_1/t, a_1) = 1$. Similarly $(n/d_2, a_2) = 1$. Finally, $(a_i, x_i) = 1$ which implies $(a_i, d) = 1$ and hence $(a_i, t) = 1$, which gives us $(a_1a_2, t) = 1$.

If $(k, n) = b > 1$, apply the above argument to $K = k/b$ and $N = n/b$ and then use divisors $D_i = bd_i$.

We are now ready to consider $\lambda(k; a_1, a_2)$ in more detail. We have already noted that we must have $(a_1, a_2) = 1$. It is also obvious that $(k, a_1a_2) \neq 1$ implies $\lambda = \infty$, and so we will also assume $(k, a_1a_2) = 1$ for the rest of this section. Finally, $\lambda = \infty$ if both a_1 and a_2 are negative, so without loss of generality $a_1 > 0$.

Theorem 2. *Let $(a_1, a_2) = (k, a_1a_2) = 1$ and $a_1 > 0$. Then $\lambda(k; a_1, a_2) = \infty$ unless*

(i) *$k = 1$ or 2 and $a_2 \geqq -1$*

or

(ii) *$k > 2$, $a_2 = -1$ and $a_1 \neq 1$ has the property that all primes dividing a_1 are $\equiv 1 \pmod{k}$.*

In these cases $\lambda = 0$, except that $\lambda(1; 1, -1) = 1$ and $\lambda(2; 1, -1) = 2$.

Proof. Write $n = A_1A_2m$ where A_i is the largest divisor of n containing only primes which divide a_i. The property mentioned in (ii) above will be called property P.

If $a_2 < -1$, then there is a prime p which divides a_2. Then by Theorem 1, equation (2) is not solvable if $n = p^s$ for s sufficiently large. The conditions $(n/d_2, a_2) = 1$ and $((a_1d_1 + a_2d_2)/k, a_1a_2) = 1$ imply $d_2 = p^s$ and $d_1 = 1$, respectively. Thus $t = (a_1 + a_2p^s)/k < 0$ for s sufficiently large. Therefore $\lambda = \infty$ if $a_2 < -1$.

If $k > 2$ and either a_1 or a_2 does not have property P, let $p \not\equiv 1 \pmod{k}$ be a divisor of a_1. (We may suppose a_1 is divisible by p and not a_2 since $a_2 = -1$ is only remaining case where $a_2 < 0$.) There are now infinitely many values of s for which (2) is not solvable with $n = p^s$. Applying Theorem 1, just as above we find $d_1 = p^s$ and $d_2 = 1$, and k does not divide $a_1 p^s + a_2$ for infinitely many values of s. Thus $\lambda = \infty$ in this case also.

If $k > 2$, $a_2 > 0$ and both a_1 and a_2 have property P, then again $\lambda = \infty$ since (2) is not solvable for all primes $q \equiv 1 \pmod{k}$. In applying Theorem 1 we find $a_1 d_1 + a_2 d_2 \equiv 1 + 1 \not\equiv 0 \pmod{k}$.

If $k > 2$, $a_1 = 1$ and $a_2 = -1$ we apply Theorem 1 to $n = p$, a prime $\not\equiv 1 \pmod{k}$. Clearly none of the cases for $a_1 d_1 + a_2 d_2 = d_1 - d_2$ yield a positive integer divisible by k. Hence $\lambda = \infty$.

The only remaining case for $k > 2$ is $a_2 = -1$ and $a_1 \neq 1$ having property P. Write $n = A_1 m$ and apply Theorem 1 with $d_1 = A_1$ and $d_2 = 1$. Then $a_1 d_1 + a_2 d_2 = a_1 A_1 - 1 \equiv 1 - 1 \equiv 0 \pmod{k}$ and $a_1 A_1 - 1 > 0$. The other conditions of the theorem are satisfied since $(a_1, a_1 A_1 - 1) = 1$, $(m, a_1) = 1$ and $(n, -1) = 1$. Therefore equation (2) is solvable for all n, and so $\lambda = 0$. The case $k = 1$ or 2, $a_2 = -1$ and a_1 any positive integer > 1 uses exactly the same argument. The special cases $\lambda(1; 1, -1) = 1$ and $\lambda(2; 1, -1) = 2$ are easily checked.

The final case is $k = 1$ or 2, $a_2 > 0$. Write $n = A_1 A_2 m$ and apply Theorem 1 with $d_i = A_i$. Then $a_1 d_1 + a_2 d_2 \equiv 0 \pmod{k}$ and is clearly positive. The other conditions of the theorem are satisfied since $(a_1 A_1 + a_2 A_2, a_1 a_2) = 1$, $(A_2 m, a_1) = 1$ and $(A_1 m, a_2) = 1$.

The following results apply to the case where the x_i may be positive or negative, and can be proved similarly.

Theorem 1′. *The equation*

$$\frac{k}{n} = \frac{a_1}{x_1} + \frac{a_2}{x_2} \tag{4}$$

is solvable in integers x_1, x_2 such that $(a_1, x_1) = 1 = (a_2, x_2)$, if and only if there exist divisors (positive or negative) d_1 and d_2 of n such that $a_1 d_1 + a_2 d_2 = kt$ for some positive integer t such that $(a_1 a_2, t) = 1$ and $(n/d_1, a_1) = (n/d_2, a_2) = 1$.

Theorem 2′. *Let $(a_1, a_2) = (k, a_1 a_2) = 1$, then $\lambda(k; a_1, a_2) = \infty$ unless*

(i) *$|a_1 a_2| = 1$ and $k = 1, 2, 3, 4$ or 6*

or

(ii) *both a_1 and a_2 are $\neq \pm 1$ and have the property that all primes dividing a_1 or a_2 $\equiv \pm 1 \pmod{k}$.*

In these cases $\lambda = 0$ except that if $|a_1 a_2| = 1$, $\lambda(3; a_1, a_2) = \lambda(4; a_1, a_2) = 1$ and $\lambda(6; a_1, a_2) = 2$.

Case (i) in the above theorem was previously mentioned in [6].

The Case $r = 3$. The solution of equation (1) with $r = 3$ and all of the $a_i = 1$ has received considerable attention. The finiteness of $\lambda(4; 1, 1, 1)$, $\lambda(5; 1, 1, 1)$, $\lambda(k; 1, 1, 1)$ and $\lambda'(k; 1, 1, 1)$ has been conjectured by ERDÖS and STRAUSS, SIERPINSKI and SCHINZEL. Although many people have considered the problem, it is not known if $\lambda(k; 1, 1, 1)$ is finite for any $k > 3$. A fairly complete list of references can be found in [1].

Efforts on the problem for λ' have been a little more successful as SIERPINSKI [5], SEDLÁČEK [4], PALAMÀ [3], and STEWART and WEBB [6] have established that λ' is finite for $k < 36$.

Although the conjectured values of λ for small k are small ($\lambda(4; 1, 1, 1) = \lambda(5; 1, 1, 1) = \lambda(6; 1, 1, 1) = 1$, $\lambda(7; 1, 1, 1) = 2$), some numerical evidence obtained by Webb [7] indicates that λ increases rapidly with k. For example $\lambda(12; 1, 1, 1) \geqq 12241$. In a private communication, Erdös noted that $\lambda(k; 1, 1, 1) > ck^{1+\varepsilon}$ for $c > 0$ and any $\varepsilon < \frac{1}{2}$, and conjectured that $\lambda(k; 1, 1, 1) > k^s$ for every positive integer s and all k sufficiently large.

In this section we prove this conjecture by establishing a slightly stronger inequality which holds for any $\lambda(k; a_1, a_2, a_3)$ and $\lambda'(k; a_1, a_2, a_3)$.

Theorem 3. *There is a constant* $c > 0$ *such that*

$$\lambda(k; a_1, a_2, a_3) > \exp(c \log k \log\log k)$$

for all k sufficiently large.

Proof. Let $E = \exp(c \log k \log\log k)$. We will show that there exist primes p in the interval $E \leqq p \leqq 2E$ for which the equation

$$(5) \qquad \frac{k}{p} = \frac{a_1}{x_1} + \frac{a_2}{x_2} + \frac{a_3}{x_3} \qquad (a_i, x_i) = 1$$

has no solutions in positive x_i.

Without loss of generality we may suppose that a_3/x_3 is the largest of the three fractions a_i/x_i. This implies $k/3p \leqq a_3/x_3$ and so $x_3 \leqq 6a_3E/k$. Hence, there are at most $O(E/k)$ values of x_3 for which (5) is solvable for any $E \leqq p \leqq 2E$.

We now fix x_3 and bound the number of p for which (5) is solvable with the given x_3.

$$(6) \qquad \frac{k}{p} - \frac{a_3}{x_3} = \frac{kx_3 - pa_3}{px_3} = \frac{a_1}{x_1} + \frac{a_2}{x_2}.$$

We note that $p > k$ and $p > x_3$ so $(p, k) = (p, x_3) = 1$. Also, $(kx_3 - a_3p, px_3) = 1$.

From (6) we see that $p \mid x_1x_2$.

Case I. Suppose $p \mid x_1$ and $p \mid x_2$. Then by Theorem 1 there exist d_1 and d_2 which divide x_3 such that $kx_3 - pa_3 \mid a_1d_1 + a_2d_2$. We know $d_i \mid px_3$, but the condition $p \mid x_i$ implies $p \nmid d_i$. There are $d^2(x_3)$ choices for d_1 and d_2 and at most $d(a_1d_1 + a_2d_2)$ choices for p given a particular d_1 and d_2. ($d(m)$ denotes the number of divisors of m.) Thus, there are at most $d^2(x_3)\, d(a_1 d_1 + a_2 d_2) \ll f^3(E)$ values of p for which (6) is solvable, where $f(n)$ is the maximum value of $d(k)$ for all $k \leqq n$.

Case II. Suppose p divides only one of the integers x_1 and x_2. Say $x_1 = py_1$ and $(x_2, p) = 1$. Then

$$\frac{kx_3 - pa_3}{px_3} - \frac{a_1}{py_1} = \frac{y_1(kx_3 - pa_3) - x_3 a_1}{px_3 y_1} = \frac{a_2}{x_2},$$

which implies $p \mid y_1 k x_3 - x_3 a_1$ and so $p \mid y_1 k - a_1$. By Theorem 1, $x_1 = px_3(a_1 d_1 + a_2 d_2)/kd_1$ where $d_i \mid x_3 p$ which implies $y_1 < E^3/k^2$ and so $y_1 k - a_1 < E^3$. Hence, there are at most $d(y_1 k - a_1) < f(E^3)$ values of p for which (6) solvable.

Now by [2, Theorem 317] $f(n) = O(\exp(\log n/\log\log n))$, and so both $f^3(E)$ and $f(E^3)$ are $O(\exp(3c \log k))$. Therefore, the total number of primes p, $E \leqq p \leqq 2E$ for which (5) is solvable, is $O(\exp(3c\log k)\, E/k)$. However, there are at least $E/\log^2 k$ primes between E and $2E$, and hence picking $c < 1/3$ we see that there must be some primes $> E$ for which (5) is unsolvable.

Corollary. *There is a constant $c > 0$ such that*

$$\lambda'(k; a_1, a_2, a_3) > \exp(c \log k \log\log k)$$

for all k sufficiently large.

Proof. By the above argument, it is clear that there exist primes p, $E \leqq p \leqq 2E$ such that all eight equations

$$\frac{k}{p} = \frac{\pm a_1}{x_1} + \frac{\pm a_2}{x_2} + \frac{\pm a_3}{x_3}$$

are unsolvable.

There are a number of related questions which are still open and require further study. Some obvious examples are:

1. Can the bound on $\lambda(k; a_1, a_2, a_3)$ be improved?

2. Can similar bounds be obtained for $\lambda(k; a_1, a_2, a_3, a_4)$ or more generally for $\lambda(k; a_1, \ldots, a_r)$? (One result along these lines is that $k = o(\lambda(k))$. This is obvious from Lemma 1 of [6].)

References

[1] *M. N. Bleicher:* A new algorithm for the expansion of Egyptian fractions, J. of Number Theory, *4* (1972), 342—382.

[2] *G. H. Hardy* and *E. M. Wright:* An Introduction to the Theory of Numbers, London, (1960).

[3] *G. Palamà:* Su di una congettura di Schinzel, Boll. Un. Mat. Ital. *14* (1959), 82—94.
[4] *J. Sedláček:* Über die Stammbrüche, Časopis Pěst. Mat. *84* (1959), 188—197.
[5] *W. Sierpiński:* Sur les décomponitions des nombres rationnels en fractions primaries, Mathesis, *65* (1956), 16—32.
[6] *B. M. Stewart* and *W. A. Webb:* Sums of fractions with bounded numerators, Canadian J. Math., *18* (1969), 999—1003.
[7] *W. A. Webb:* Rationals not expressible as a sum of three unit fractions, Elemente der Math., *29* (1974), 1—6.

Author's address: Washington State University, Pullman, Washington, U.S.A.