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FINITE PLANAR VERTEX-TRANSITIVE GRAPHS OF THE REGULARITY DEGREE THREE

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In [1] V. G. VIZING poses some questions concerning vertex-transitive graphs. Among other problems he mentions that no complete characterization of vertex-transitive graphs of the regularity degree three have been demonstrated. Here we shall study this question in the particular case of finite planar graphs. We admit multiple edges, but not loops.

A vertex-transitive graph is a graph G such that to any two vertices x and y of G there exists an automorphism φ of G such that $\varphi(x) = y$.

Vertex-transitive graphs have been studied by various authors. Some of them, for example L. LOVÁSZ, use another terminology; they call these graphs symmetric. Here we use the term "vertex-transitive", because the term "symmetric" is often used in other senses.

Evidently every vertex-transitive graph is regular, i.e. all of its vertices have the same degree which is called the regularity degree of this graph.

Let us study finite planar vertex-transitive graphs of the regularity degree three. As we study planar graphs, we may speak about faces of a graph. First we prove a lemma.

Lemma. *A finite planar graph G can be drawn in the plane in such a way that the image of each boundary of a face in an arbitrary automorphism of G is again a boundary of a face.*

Proof. Each circuit of G which is not a boundary of any face in a drawing of G has the property that its vertex set is a separating set in G. A circuit which is a boundary of a face can have this property or not. (In 3-connected graphs it has never this property.) If it has not this property, then nor its image in any automorphism of G has this property and thus it is necessarily a boundary of some face. If a circuit C which is a boundary of a face has this property, let G' be the graph obtained from G by deleting all vertices of C and all edges incident to these vertices. It is easy to see that at drawing G we may decide freely whether some connected component of G'

will be drawn in the region bounded by C or outside of it. This holds also for each circuit which is an image of C in an automorphism of G. Thus if in a drawing of G such a circuit C is a boundary of a face and the image of C in an automorphism of G is not, we may change the drawing so that this image becomes a boundary of a face (all components mentioned are drawn outside of the region bounded by this circuit).

A connected vertex-transitive graph with at least 3 vertices cannot have the vertex connectivity degree equal to one; it would have at least one cut-vertex and, as a cut-vertex is mapped in each automorphism again onto a cut-vertex, all vertices of this graph would be cut-vertices, which is impossible in a finite graph. In any regular planar graph of the regularity degree three and of the vertex connectivity degree at least two each vertex belongs exactly to three faces f_1, f_2, f_3. Let the degree of the face f_i for $i = 1, 2, 3$ be d_i. (The degree of a face is the number of edges belonging to the boundary of this face.) If such a graph G is vertex-transitive, then, according to Lemma, it can be drawn in the plane so that at each vertex of this graph the degrees of the faces containing this vertex are d_1, d_2, d_3. Let us distinguish three cases (without loss of generality):

(α) $d_1 = d_2 = d_3$;

(β) $d_1 = d_2 \neq d_3$;

(γ) $d_1 \neq d_2 \neq d_3 \neq d_1$.

In the case (α) each vertex belongs to three faces of the same degree. It is easy to prove that then all faces of such a graph have equal degrees. The list of such finite planar graphs is well-known. It consists of the following graphs: the graph consisting of two vertices joined by three edges; the complete graph with four vertices (the graph of the regular tetrahedron); the graph of the (three-dimensional) cube; the graph of the regular dodecahedron. All of these graphs are vertex-transitive. Thus we have found all required graphs in the case (α).

Now consider the cases (β) and (γ). In both these cases d_3 is different from d_1 and d_2. This means that any two faces of degree d_3 are vertex-disjoint (and obviously also edge-disjoint). We decompose the edge set of G into two disjoint sets E_1 and E_2. The set E_1 consists of all edges which belong to the boundary of a face of degree d_3 and $E_2 = E - E_1$, where E is the edge set of G. Now we construct the graph $F(G)$ whose vertex set is the set of all faces of G of degree d_3 and in which two distinct vertices are joined by as many edges, as many are the edges in G joining a vertex of the face corresponding to one vertex with a vertex of the face corresponding to the other vertex. The transformation of G into $F(G)$ can be shown using the drawing of G in the plane. In this drawing the edges of G are represented by arcs of simple curves, faces of G are simply connected regions. In each face of G of degree d_3 we choose an inner point and we continue each edge of E_2 to these points in the faces to which its end vertices belong. Then we omit all edges of E_1. The resulting drawing is the drawing of $F(G)$. We see that $F(G)$ is a planar regular graph of the regularity degree d_3. If $d_1 = d_2$, then all faces of $F(G)$ have the degree $\frac{1}{2}d_1$. If $d_1 \neq d_2$, then each face of $F(G)$ has the degree either $\frac{1}{2}d_1$ or $\frac{1}{2}d_2$. In this case each edge of $F(G)$ belongs exactly to one face of degree $\frac{1}{2}d_1$ and exactly to one face of degree $\frac{1}{2}d_2$. In the first case (which is the case (β)) the graph $F(G)$ is a regular planar graph in which all faces have the same degree. The list of such graphs is well-known; it consists of the graphs listed when investigating the case (α) and, moreover, of the graphs consisting of two vertices joined by an arbitrary number of edges greater than or equal to two, of all circuits, of the graph of the regular octahedron and of the graph of the regular icosahedron. Now we shall make the list of all planar regular graphs in which the degrees of faces can have exactly two values and each edge belongs to two faces of different degrees. Let the possible values of degrees be denoted by d'_1 and d'_2. We see that the number of faces of degree d'_1 containing a vertex v of such a graph must be equal to the number of faces of degree d'_2 containing v; therefore the regularity degree of such a graph must be even. First consider graphs without multiple edges. In this case the regularity degree of a planar graph cannot be greater than five. We have only two possibilities: two or four. But a graph of the regularity degree two is a circuit and has only two faces, both with the same degree. Thus the regularity degree of the required graph must be four. If n is the number of its vertices, then the number of its edges is $2n$. As each edge belongs exactly to one face of degree d'_1, the number of faces of degree d'_1 is $2n/d'_1$. Analogously the number of

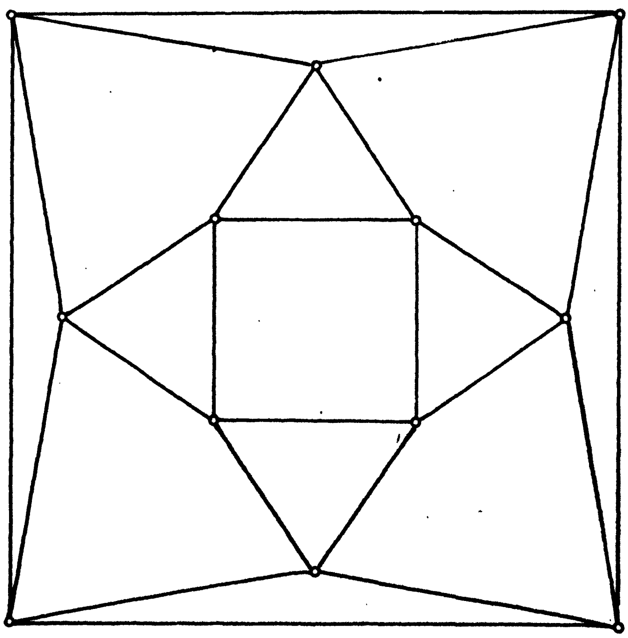

Fig. 1.

faces of degree d'_2 is $2n/d'_2$ and the total number of faces is $2n/d'_1 + 2n/d'_2$. Substituting into Euler's formula for planar graphs we obtain

$$n + 2n/d'_1 + 2n/d'_2 - 2n = 2 .$$

This implies

$$n = 2d'_1d'_2/(2d'_1 + 2d'_2 - d'_1d'_2) .$$

In a finite planar graph of the regularity degree four at least one face must have a degree less than six. Without loss of generality let $d'_1 < d'_2$; then $d'_1 < 6$. Obviously $d'_1 \geqq 3$, because $F(G)$ contains no loops and no multiple edges. For $d'_1 = 3$ we obtain $n = 6d'_2(6 - d'_2)$. The possible values of d'_2 are 4 and 5; for greater values od d'_2 the number n would be undefined or negative. If $d'_2 = 4$, then $n = 12$. The graph satisfying these conditions is in Fig. 1; we can easily see that it is the uniqe graph satisfying them. If $d'_2 = 5$, then $n = 30$ and we have also a unique possible graph satisfying this in Fig. 2. For $d'_1 = 4$ we obtain $n = 4d'_2/(4 - d'_2)$; for $d'_1 = 5$ we have $n = 10d'_2/(10 - 3d'_2)$. For each $d'_2 > d'_1$ these expressions are negative; the required graphs do not exist.

Now consider graphs with multiple edges. Then $d'_1 = 2$ and such a graph is obtained by doubling all edges of a regular planar graph in which all faces have equal degrees different from two; these graphs were listed above.

We see that $F(G)$ must be isomorphic to some of the above described graphs. To any of these graphs H we construct $F^{-1}(H)$ by the operation which is inverse to F; we shall not describe it in words; it is shown in Fig. 3. If we consider H a graph of a polyhedron, we can imagine F^{-1} as "cutting off all angles".

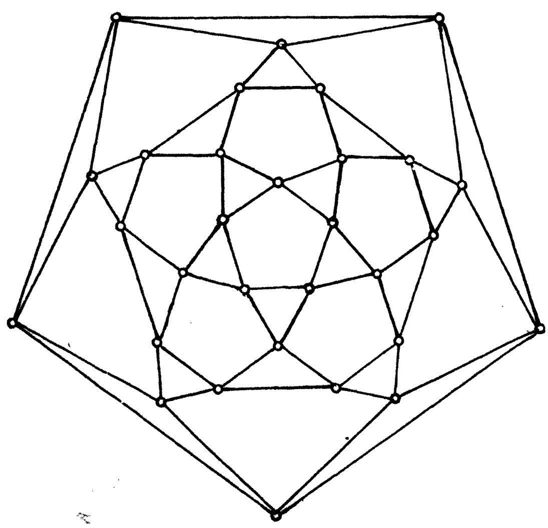

Fig. 2.

All the finite planar connected vertex-transitive graphs of the regularity degree three in the cases (β) and (γ) must be among these graphs $F^{-1}(H)$ and, as we easily see, any of these graphs $F^{-1}(H)$ has the required properties. Therefore in the cases

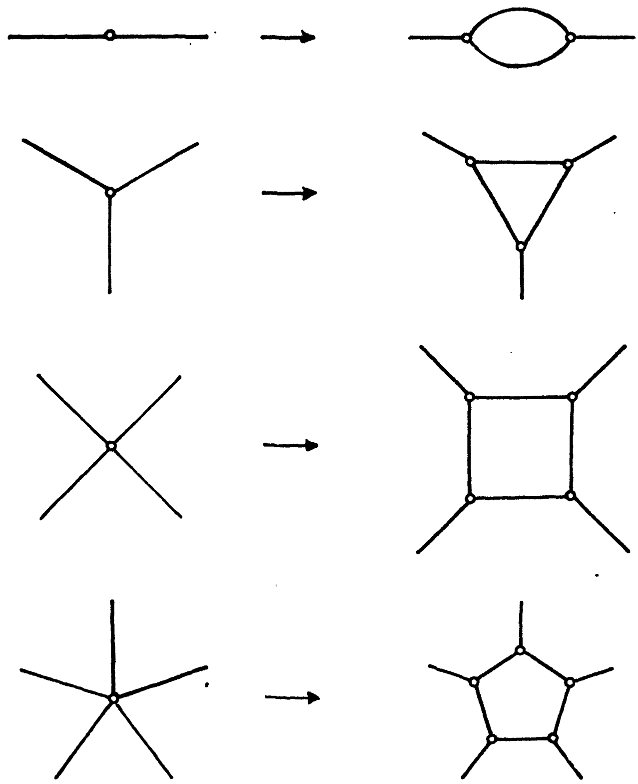

Fig. 3.

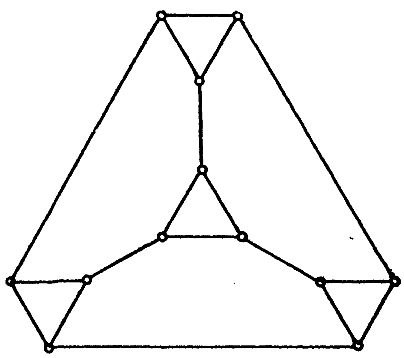

Fig. 4.

(β) and (γ) the finite planar connected vertex-transitive graphs of the regularity degree three are the graphs of n-side prismata for all positive integers $n \geqq 2$ and the graphs in Figs. 4–11. We have proved a theorem.

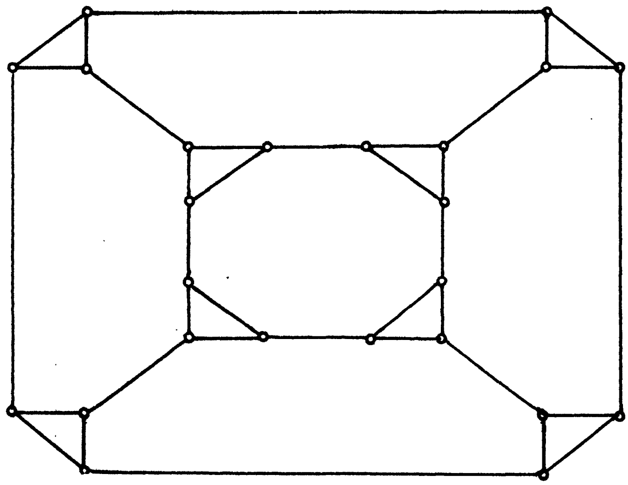

Fig. 5.

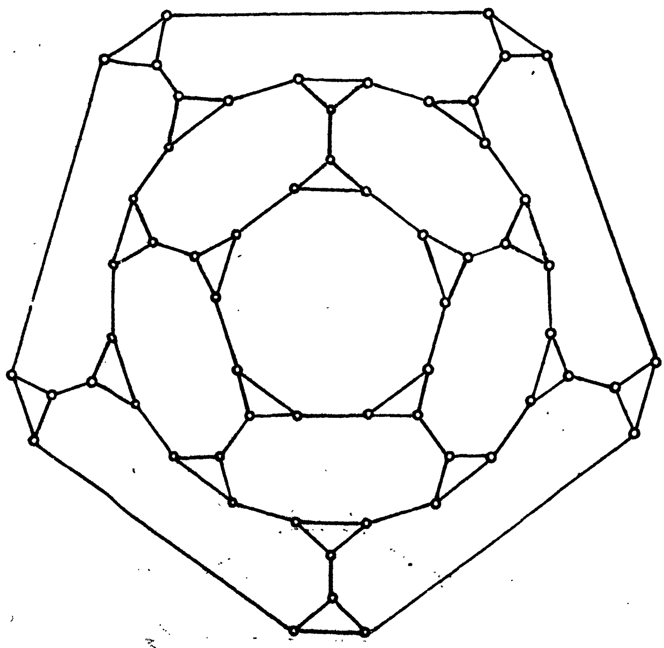

Fig. 6.

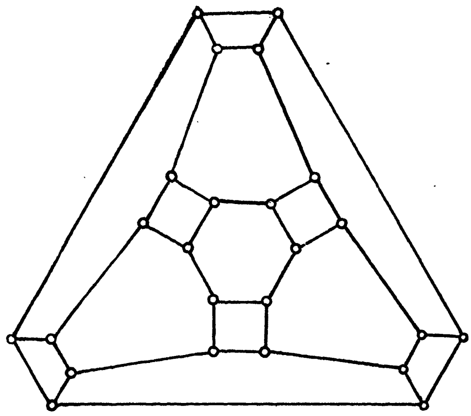

Fig. 7.

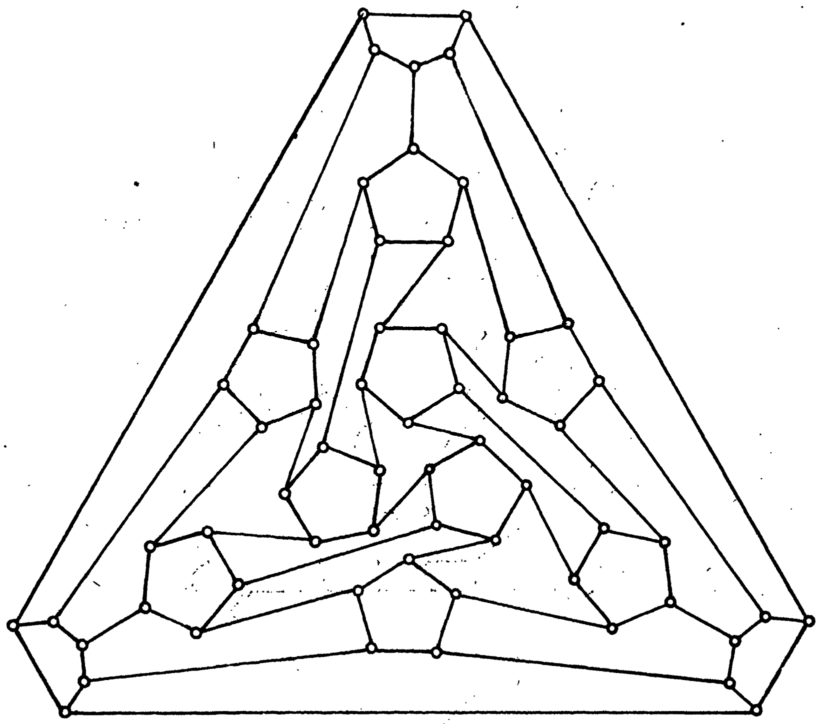

Fig. 8.

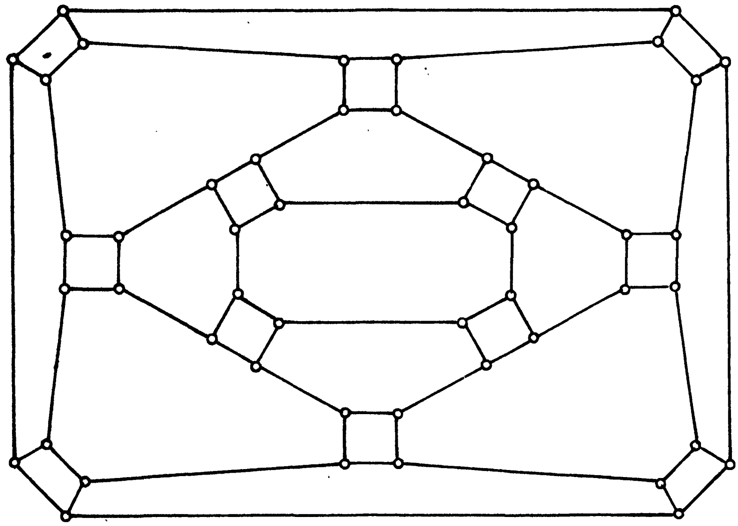

Fig. 9.

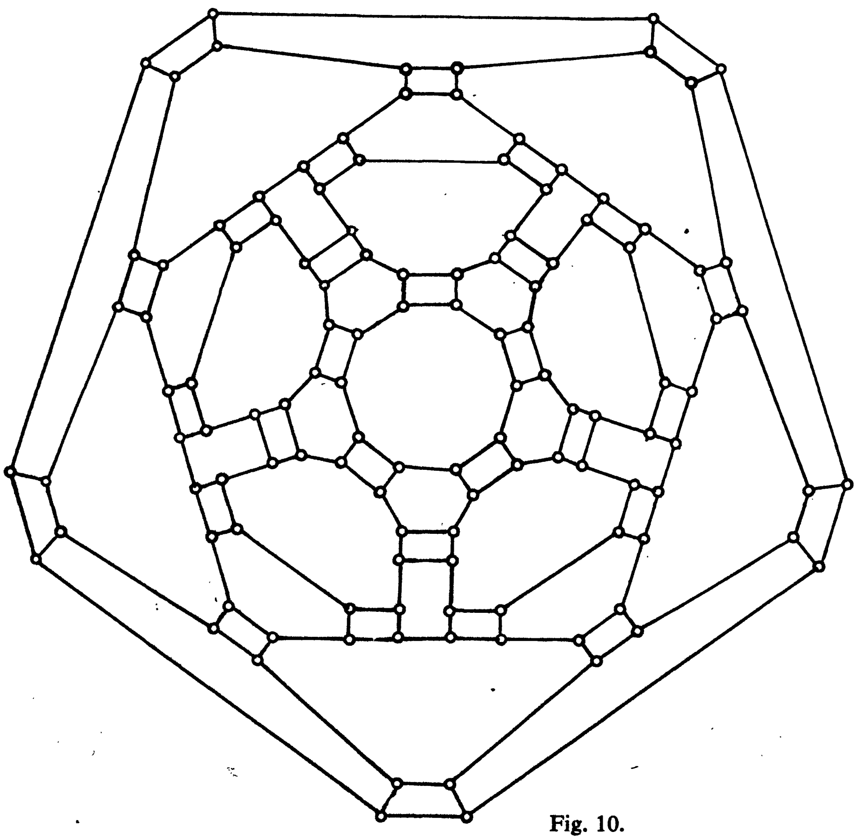

Fig. 10.

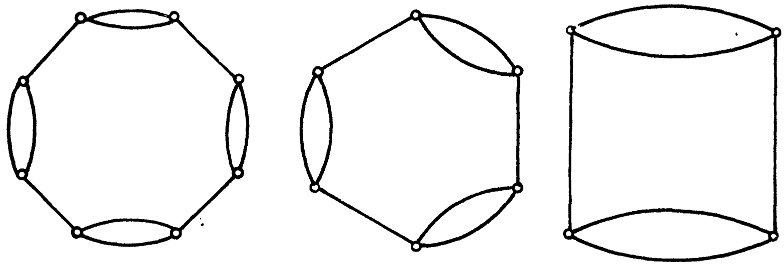

Fig. 11.

Theorem. *Let G be a finite planar connected vertex-transitive graph of the regularity degree three. Then G is isomorphic to one of the following graphs: the graph consisting of two vertices joined by three edges; the graph of the regular tetrahedron; the graph of the regular dodecahedron; the graph of an n-side prisma for* $n \geqq 2$; *some of the graphs in Figs.* 4–11.

Reference

[1] *В. Г. Визинг:* Некоторые нерешенные задачи в теории графов. Успехи мат. наук *23* (1968), 117–134.

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