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ISOMORPHISM OF PROJECTIVE PLANES AND ISOTOPISM OF PLANAR TERNARY RINGS

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In the last years isotopism of planar ternary rings was studied especially by G. E. MARTIN. The theory of planar ternary rings was developed together with the theory of projective planes so that there is the close connection between isotopism of planar ternary rings and isomorphism of projective planes. This article is supposed to be a contribution to that problem.

I. PROJECTIVE PLANES, PLANAR TERNARY RINGS

Under the term *projective plane* we understand an ordered triple $(\mathscr{P}, \mathscr{L}, \mathscr{I})$ where $\mathscr{P}, \mathscr{L}$ are disjoint sets and $\mathscr{I}$ is a binary relation from $\mathscr{P}$ to $\mathscr{L}$ (i.e. $\mathscr{I} \subseteq \subseteq \mathscr{P} \times \mathscr{L}$) and the following axioms are satisfied: $(\mathbf{P}_1)$ to every two distinct elements $p, q \in \mathscr{P}$ there exists just one element $L \in \mathscr{L}$ such that $p, q \mathscr{I} L$, $(\mathbf{P}_2)$ to every two distinct elements $L, M \in \mathscr{L}$ there exists just one element $p \in \mathscr{P}$ such that $p \mathscr{I} L, M$, and $(\mathbf{P}_3)$ there exist four elements from $\mathscr{P}$ such that no three of them are in the relation $\mathscr{I}$ with the same element of $\mathscr{L}$. The elements of $\mathscr{P}$ are called points, the elements of $\mathscr{L}$ lines, and the relation $\mathscr{I}$ is called incidence. The phrases "lies on" or "passes through" are also used for the relation $\mathscr{I}$. From the axioms of the projective plane it follows that the number of points on every line is equal to the number of lines passing through each point.

Let $\boldsymbol{\mathscr{P}}_1 = (\mathscr{P}_1, \mathscr{L}_1, \mathscr{I}_1)$, $\boldsymbol{\mathscr{P}}_2 = (\mathscr{P}_2, \mathscr{L}_2, \mathscr{I}_2)$ be projective planes. Then each ordered couple of bijections (π, λ), $\pi : \mathscr{P}_1 \to \mathscr{P}_2$, $\lambda : \mathscr{L}_1 \to \mathscr{L}_2$ such that $p \mathscr{I}_1 L$ holds if and only if $p^\pi \mathscr{I}_2 L^\lambda$ is called an isomorphism of projective plane $\boldsymbol{\mathscr{P}}_1$ onto the projective plane $\boldsymbol{\mathscr{P}}_2$. With respect to the symmetry of the relation "to be isomorphic onto" it is possible to speak about isomorphic projective planes.

A *planar ternary ring* PTR is an ordered couple $(\mathscr{S}, \mathrm{T})$ where $\mathscr{S}$ is a set, $|\mathscr{S}| \geqq 2$, and T is a ternary operation defined on $\mathscr{S}$, i.e. a mapping $\mathrm{T} : \mathscr{S}^3 \to \mathscr{S}$ satisfying the following axioms:

(i) $\forall a, b, c \in \mathscr{S}\ \exists!\ x \in \mathscr{S}\quad \mathrm{T}(a, b, x) = c$,

(ii) $\forall a, b, c, d \in \mathscr{S},\ a \neq c\ \exists!\ x \in \mathscr{S}\quad \mathrm{T}(x, a, b) = \mathrm{T}(x, c, d)$,

(iii) $\forall a, b, c, d \in \mathscr{S},\ a \neq c\ \exists!\ (x, y) \in \mathscr{S}^2\quad \mathrm{T}(a, x, y) = b,\ \mathrm{T}(c, x, y) = d$.

If it holds in a PTR that

(iv) $\exists O^l \in \mathscr{S}\ \forall b \in \mathscr{S}\ \exists b' \in \mathscr{S}\ \forall m \in \mathscr{S}\quad \mathrm{T}(O^l, m, b) = b'$,

(v) $\exists O^r \in \mathscr{S}\ \forall b \in \mathscr{S}\ \exists b'' \in \mathscr{S}\ \forall x \in \mathscr{S}\quad \mathrm{T}(x, O^r, b) = b''$,

then the PTR is called an intermediate ternary ring (ITR). The elements O^l and O^r are called respectively the left and the right quasizero of PTR. The left (right) quasizero is unique in the PTR $\mathbf{R} = (\mathscr{S}, \mathrm{T})$. If there exists an element $O \in \mathscr{S}$ such that $\mathrm{T}(O, m, b) = \mathrm{T}(x, O, b) = b$ for all $x, m, b \in \mathscr{S}$, then O is called the zero of the PTR.

Lemma 1. *If O^l and O^r are respectively the left and the right quasizero of* PTR $\mathbf{R} = (\mathscr{S}, \mathrm{T})$, *then* $\mathrm{T}(O^l, m, b) = \mathrm{T}(x, O^r, b)$ *for any* $b, x, m \in \mathscr{S}$.

Proof. For each $b \in \mathscr{S}$, $\mathrm{T}(O^l, O^r, b) = b'$ and $\mathrm{T}(O^l, O^r, b) = b''$ according to the axioms (iv) or (v). Since T is a ternary operation, we have $b' = b''$.

Lemma 2. *If* $\mathbf{R} = (\mathscr{S}, \mathrm{T})$ *is an* ITR, *then the left (right) quasizero induces in* $\mathscr{S}$ *a permutation* φ *for which it holds: for each* $b \in \mathscr{S}$ $\mathrm{T}(O^l, m, b) = b^\varphi$ *for all* $m \in \mathscr{S}$ $(\mathrm{T}(x, O^r, b) = b^\varphi$ *for all* $x \in \mathscr{S})$.

Proof. The fact that φ is injective in $\mathscr{S}$ follows from the definition of the PTR. If $c \in \mathscr{S}$, then there is $b \in \mathscr{S}$ such that $\mathrm{T}(O^l, O^r, b) = c$ according to the axiom (i). Hence $b^\varphi = c$ and φ is surjective, too.

We consider all the PTR's of the same cardinality to be always constructed over the same set $\mathscr{S}$.

If $(\mathscr{S}, \mathrm{T}_1)$ is a PTR and $\alpha, \beta, \gamma, \delta$ arbitrary permutations in $\mathscr{S}$, then the system $(\mathscr{S}, \mathrm{T}_2)$ defined by the relation

$$(*)\qquad [\mathrm{T}_2(x, m, b)]^\alpha = \mathrm{T}_1(x^\beta, m^\gamma, b^\delta)$$

holding for all $x, m, b \in \mathscr{S}$ is called isotopic to the PTR $(\mathscr{S}, \mathrm{T}_1)$. By verifying the axioms (i)–(iii), $(\mathscr{S}, \mathrm{T}_2)$ may be easily shown to be a PTR, too. With respect to the symmetry of the relation "to be isotopic to" it is possible to speak about isotopic PTR's. An ordered quadruple of permutations $(\alpha, \beta, \gamma, \delta)$, where $\alpha, \beta, \gamma, \delta$ are permutations in $\mathscr{S}$ satisfying the relation $(*)$, is called an isotopism of the PTR $(\mathscr{S}, \mathrm{T}_1)$ onto the PTR $(\mathscr{S}, \mathrm{T}_2)$.

2. ISOMORPHISM OF PROJECTIVE PLANES AND ISOTOPISM OF PLANAR TERNARY RINGS

Theorem 1. *Let* $\mathbf{R} = (\mathcal{S}, \mathrm{T})$ *be a* PTR, *then the ordered triple* $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ *defined in the following way is a projective plane*:

$$\mathcal{P} = \{(x, y) \mid (x, y) \in \mathcal{S}^2\} \cup \{(z) \mid z \in \mathcal{S}\} \cup \{(\infty)\}, \quad \infty \notin \mathcal{S},$$
$$\mathcal{L} = \{[m, b] \mid (m, b) \in \mathcal{S}^2\} \cup \{[n] \mid n \in \mathcal{S}\} \cup \{[\infty]\}, \quad \infty \notin \mathcal{S},$$

$\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$ *and the following holds*:

(a) $(x, y)\,\mathcal{I}[m, b] \Leftrightarrow \mathrm{T}(x, m, b) = y$, (d) $(m)\,\mathcal{I}[\infty]$ *for all* $m \in \mathcal{S}$,
(b) $(x, y)\,\mathcal{I}[x]$ *for all* $x, y \in \mathcal{S}$, (e) $(\infty)\,\mathcal{I}[x]$ *for all* $x \in \mathcal{S}$,
(c) $(m)\,\mathcal{I}[m, b]$ *for all* $m, b \in \mathcal{S}$, (f) $(\infty)\,\mathcal{I}[\infty]$.

The proof may be done by verifying the validity of the axioms $(\mathbf{P}_1)-(\mathbf{P}_3)$ in $(\mathcal{P}, \mathcal{L}, \mathcal{I})$.

The projective plane from the above theorem will be called the projective plane over the PTR $\mathbf{R}$ and denoted by $\mathcal{P}_{\mathbf{R}}$.

Now let us have an ITR $\mathbf{R} = (\mathcal{S}, \mathrm{T})$ and a projective plane $\mathcal{P}_{\mathbf{R}}$ over it. According to Theorem 1 all the points incident with the line $[\infty]$ are of the (m) shape for all $m \in \mathcal{S}$, and all the lines passing through the point (∞) are of the $[x]$ shape for all $x \in \mathcal{S}$. Now let us choose a significant point (O^r) and a significant line $[O^l]$ where $O^l, O^r \in \mathcal{S}$, O^l is the left quasizero and O^r is the right quasizero of the PTR $\mathbf{R}$. According to Theorem 1, each point incident with the line $[O^l]$ is of the (O^l, a) shape. However, to each $a \in \mathcal{S}$ there exist just one $b \in \mathcal{S}$ such that $b^\varphi = a$ where φ is the permutation in $\mathcal{S}$ induced by the quasizeros. Each point (O^l, a) may be recorded in the (O^l, b^φ) shape. Each line passing through the point (O^r) is of the $[O^r, a]$ shape. Since $\mathrm{T}(O^l, O^r, a) = a^\varphi$, it holds $(O^l, a^\varphi)\,\mathcal{I}[O^r, a]$. The line passing through the points (O^l, c^φ) and (m) is denoted by $[m, c]$ because $\mathrm{T}(O^l, m, c) = c^\varphi$. The point lying on the lines $[O^r, d]$ and $[x]$ is denoted by (x, d^φ) because $\mathrm{T}(x, O^r, d) = d^\varphi$.

The following two theorems showing the reciprocal connection between isomorphisms of projective planes and isotopisms of PTR's are given without any proof.

Theorem 2. *If the* PTR's $\mathbf{R}_1$ *and* $\mathbf{R}_2$ *are isotopic, then the projective planes* $\mathcal{P}_{\mathbf{R}_1}, \mathcal{P}_{\mathbf{R}_2}$ *are isomorphic.*

Proof. See [4], page 288, Theorem 9.3.3.

However, the inversion of this theorem is impossible. An example of isomorphic projective planes over nonisotopic PTR's is mentioned in [4], Chapter 11.4. The following theorem answers the question which isomorphisms of projective planes over the PTR's with zero make the corresponding PTR's isotopic.

Theorem 3. *If there exists an isomorphism (π, λ) of a projective plane $\mathcal{P}_{\mathbf{R}_1}$ onto a projective plane $\mathcal{P}_{\mathbf{R}_2}$ such that $(\infty)^\pi = (\infty)$, $(O_1)^\pi = (O_2)$, $(O_1, O_1)^\pi = (O_2, O_2)$ where O_1 is the zero of the* PTR $\mathbf{R}_1$ *and O_2 is the zero of the* PTR $\mathbf{R}_2$, *then* $\mathbf{R}_1$ *and* $\mathbf{R}_2$ *are isotopic.*

Proof. See [2].

A suitable criterion can be formulated also for ITR's.

Theorem 4. *If there exists an isomorphism (π, λ) of a projective plane $\mathcal{P}_{\mathbf{R}_1} = (\mathcal{P}_1, \mathcal{L}_1, \mathcal{I}_1)$ onto a projective plane $\mathcal{P}_{\mathbf{R}_2} = (\mathcal{P}_2, \mathcal{L}_2, \mathcal{I}_2)$ such that $(\infty_1)^\pi = (\infty_2)$, $(O_1^r)^\pi = (O_2^r)$, $[O_1^l]^\lambda = [O_2^l]$ where O_1^l and O_1^r is the left and the right quasizero of the* PTR $\mathbf{R}_1 = (\mathcal{S}, \mathrm{T}_1)$, *while O_2^l and O_2^r is the left and the right quasizero of the* PTR $\mathbf{R}_2 = (\mathcal{S}, \mathrm{T}_2)$, *then* $\mathbf{R}_1$ *and* $\mathbf{R}_2$ *are isotopic.*

Proof. The assumptions of this theorem imply that:

(1) there exists a bijection π' that is the restriction of π on $\{(x) \mid x \in \mathcal{S}\}$ such that $(x)^{\pi'} = (y)$. It means that π' induces on $\mathcal{S}$ a permutation γ for which $x^\gamma = y$.

(2) there exists a bijection π'' that is the restriction of π on $\{(O_1^l, x) \mid x \in \mathcal{S}\}$ such that $(O_1^l, x)^{\pi''} = (O_2^l, y)$. It means that π'' induces on $\mathcal{S}$ a permutation α for which $x^\alpha = y$.

(3) there exists a bijection λ' that is the restriction of λ on $\{[x] \mid x \in \mathcal{S}\}$ such that $[x]^{\lambda'} = y$. It means that λ' induces on $\mathcal{S}$ a permutation β for which $x^\beta = y$.

Thus $(m)^\pi = (m^\gamma)$, $(O_1^l, b^{\varphi_1})^\pi = (O_2^l, b^{\varphi_1\alpha})$, $[x]^\lambda = [x^\beta]$ for all $x, m, b \in \mathcal{S}$.*)

It is necessary to determine the images of the remaining points and lines with respect to the permutations α, β, γ. Since $(O_1^r)^\pi = (O_2^r)$ and $(\infty_1)^\pi = (\infty_2)$, it is $[\infty_1]^\lambda = [\infty_2]$. Since $(m)^\pi = (m^\gamma)$ and $(O_1^l, b^{\varphi_1})^\pi = (O_2^l, b^{\varphi_1\alpha})$, it is $[m, b]^\lambda = [m^\gamma, c]$. Furthermore, it holds: $\mathrm{T}_2(O_2^l, m^\gamma, c) = b^{\varphi_1\alpha}$ and $\mathrm{T}_2(O_2^l, m^\gamma, c) = c^{\varphi_2}$ which implies $c = b^{\varphi_1\alpha\varphi_2^{-1}}$ and thus $[m, b]^\lambda = [m^\gamma, b^{\varphi_1\alpha\varphi_2^{-1}}]$. Since $[x]^\lambda = [x^\beta]$ and $[O_1^r, y]^\lambda = [O_2^r, y^{\varphi_1\alpha\varphi_2^{-1}}]$, it is $(x, y^{\varphi_1})^\pi = (x^\beta, z)$. It holds that $\mathrm{T}_2(x^\beta, O_2^r, y^{\varphi_1\alpha\varphi_2^{-1}}) = z$ and $\mathrm{T}_2(x^\beta, O_2^r, z^{\varphi_2^{-1}}) = z$. This implies that $z = y^{\varphi_1\alpha}$ and thus $(x, y^{\varphi_1})^\pi = (x^\beta, y^{\varphi_1\alpha})$.

Since (π, λ) is an isomorphism of $\mathcal{P}_{\mathbf{R}_1}$ onto $\mathcal{P}_{\mathbf{R}_2}$, it is true that $(x, y^{\varphi_1})\, \mathcal{I}_1 [m, b] \Leftrightarrow (x, y^{\varphi_1})^\pi\, \mathcal{I}_2 [m, b]^\lambda \Rightarrow (x^\beta, y^{\varphi_1\alpha})\, \mathcal{I}_2 [m^\gamma, b^{\varphi_1\alpha\varphi_2^{-1}}]$. This means that for all $x, y^{\varphi_1}, m, b \in \mathcal{S}$ for which the above mentioned relation holds, $\mathrm{T}_1(x, m, b) = y^{\varphi_1}$ and $\mathrm{T}_2(x^\beta, m^\gamma, b^{\varphi_1\alpha\varphi_2^{-1}}) = y^{\varphi_1\alpha}$. Let $\varphi_1\alpha\varphi_2^{-1} = \delta$, then we can write $[\mathrm{T}_1(x, m, b)]^\alpha = \mathrm{T}_2(x^\beta, m^\gamma, b^\delta)$ for all $x, m, b \in \mathcal{S}$.

Let us state some corollaries of this theorem. Let be given two ITR's: $\mathbf{R}_1 = (\mathcal{S}, \mathrm{T}_1)$ with the left quasizero O_1^l and the right quasizero O_1^r, and $\mathbf{R}_2 = (\mathcal{S}, \mathrm{T}_2)$ with the left quasizero O_2^l and the right quasizero O_2^r. Let $\mathcal{P}_{\mathbf{R}_1} = (\mathcal{P}_1, \mathcal{L}_1, \mathcal{I}_1)$, $\mathcal{P}_{\mathbf{R}_2} = (\mathcal{P}_2, \mathcal{L}_2, \mathcal{I}_2)$ be the projective planes over the given PTR's.

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*) φ_1, φ_2 are the permutations in $\mathcal{S}$ induced by the quasizeros from $\mathbf{R}_1, \mathbf{R}_2$.

Corollary 1. *If there exists an isomorphism* (π, λ) *of the projective plane* $\mathscr{P}_{\mathbf{R}_1}$ *onto* $\mathscr{P}_{\mathbf{R}_2}$ *such that* $(\infty_1)^\pi = (\infty_2)$, $(O_1^l, b)^\pi = (O_2^l, c)$, $(O_1^r)^\pi = (O_2^r)$, *then* $\mathbf{R}_1$ *and* $\mathbf{R}_2$ *are isotopic.*

Corollary 2. *If there exists an isomorphism* (π, λ) *of the projective plane* $\mathscr{P}_{\mathbf{R}_1}$ *onto* $\mathscr{P}_{\mathbf{R}_2}$ *such that* $[\infty_1]^\lambda = [\infty_2]$, $[O_1^l]^\lambda = [O_2^l]$, $(O_1^r)^\pi = (O_2^r)$, *then* $\mathbf{R}_1$ *and* $\mathbf{R}_2$ *are isotopic.*

Corollary 3. *If there exists an isomorphism* (π, λ) *of the projective plane* $\mathscr{P}_{\mathbf{R}_1}$ *onto* $\mathscr{P}_{\mathbf{R}_2}$ *such that* $[\infty_1]^\lambda = [\infty_2]$, $[O_1^l]^\lambda = [O_1^l]$, $[O_1^r, b]^\lambda = [O_2^r, c]$, *then* $\mathbf{R}_1$ *and* $\mathbf{R}_2$ *are isotopic.*

Now let us apply the results established in Theorem 4 to two suitable PTR's and the corresponding projective planes over them. Let us introduce several concepts needed in the example below. In a PTR $\mathbf{R} = (\mathscr{S}, \mathrm{T})$ in which elements $O, 1 \in \mathscr{S}$, $O \neq 1$ exist where O is the zero in $\mathbf{R}$ and $\mathrm{T}(x, 1, O) = \mathrm{T}(1, x, O) = x$ for all $x = \mathscr{S}$, two binary operations $+, \circ$ may be defined by the relations $\mathrm{T}(x, m, O) = x \circ m$, $\mathrm{T}(1, m, b) = m + b$ for all $x, m, b \in \mathscr{S}$. If $\mathrm{T}(x, m, b) = x \circ m + b$ for all $x, m, b \in \mathscr{S}$ where the operation $\circ$ has the priority, then $\mathbf{R} = (\mathscr{S}, \mathrm{T})$ is called a linear PTR and we can write $\mathbf{R} = (\mathscr{S}, +, \circ)$.

Example. Let us have linear PTR's $\mathbf{R}_1 = (\mathscr{S}, +_1, \circ_1)$ and $\mathbf{R}_2 = (\mathscr{S}, +_2, \circ_2)$ where $\mathscr{S} = \{0, 1, 2, a, b, c, d, e, f\}$, in which the binary operations are given by means of the following tables:

Table 1

$+_1, +_2$	0	1	2	a	b	c	d	e	f
0	0	1	2	a	b	c	d	e	f
1	1	2	0	b	c	a	e	f	d
2	2	0	1	c	a	b	f	d	e
a	a	b	c	d	e	f	0	1	2
b	b	c	a	e	f	d	1	2	0
c	c	a	b	f	d	e	2	0	1
d	d	e	f	0	1	2	a	b	c
e	e	f	d	1	2	0	b	c	a
f	f	d	e	2	0	1	c	a	b

Table 2

$\circ_1$	0	1	2	a	b	c	d	e	f
0	0	0	0	0	0	0	0	0	0
1	0	1	2	a	b	c	d	e	f
2	0	2	1	d	f	e	a	c	b
a	0	a	d	b	1	f	c	2	e
b	0	b	f	e	c	1	2	d	a
c	0	c	e	1	d	a	f	b	2
d	0	d	a	f	2	b	e	1	c
e	0	e	c	2	a	d	b	f	1
f	0	f	b	c	e	2	1	a	d

Table 3

$\circ_2$	0	1	2	a	b	c	d	e	f
0	0	0	0	0	0	0	0	0	0
1	0	1	2	a	b	c	d	e	f
2	0	2	1	d	f	e	a	c	b
a	0	a	d	e	c	1	f	b	2
b	0	b	f	1	d	a	c	2	e
c	0	c	e	b	1	f	2	d	a
d	0	d	a	c	e	2	b	f	1
e	0	e	c	f	2	b	1	a	d
f	0	f	b	2	a	d	e	1	c

Let us remark that the element O has the property of both the left nd the right quasizero of the PTR's $\mathbf{R}_1$ and $\mathbf{R}_2$. According to Theorem 1 the projective planes $\mathscr{P}_{\mathbf{R}_1} = (\mathscr{P}_1, \mathscr{L}_1, \mathscr{I}_1)$ and $\mathscr{P}_{\mathbf{R}_2} = (\mathscr{P}_2, \mathscr{L}_2, \mathscr{I}_2)$ can be constructed in the above mentioned way. We show that if there exists an isomorphism of $\mathscr{P}_{\mathbf{R}_1}$ onto $\mathscr{P}_{\mathbf{R}_2}$ satisfying the assumptions of Theorem 4, then $\mathbf{R}_1$ and $\mathbf{R}_2$ are isotopic.

Let us define bijections $\pi : \mathcal{P}_1 \to \mathcal{P}_2$ and $\lambda : \mathcal{L}_1 \to \mathcal{L}_2$ such that $(x, y)^\pi = (x \circ_1 2, y)$, $[m, b]^\lambda = [m \circ_1 2, b]$, $(m)^\pi = (m \circ_1 2)$ and $[x]^\lambda = [x \circ_1 2]$. It is possible to verify that $x \circ_1 m = (x \circ_1 2) \circ_2 (m \circ_1 2)$ for all $x, m \in \mathcal{S}$. This statement together with the condition (a) of Theorem 1 imply the following results: if $(x, y) \mathcal{I}_1 [m, b]$, then $(x, y)^\pi \mathcal{I}_2 [m, b]^\lambda$. For the other couples of incident points and lines from $\mathcal{P}_{\mathbf{R}_1}$ the incidence in $\mathcal{P}_{\mathbf{R}_2}$ is obviously preserved. It means that (π, λ) is an isomorphism of the projective planes $\mathcal{P}_{\mathbf{R}_1}, \mathcal{P}_{\mathbf{R}_2}$. The following holds: $(\infty_1)^\pi = (\infty_2)$, $(O)^\pi = (O)$ and $[O]^\lambda = [O]$. The isomorphism (π, λ) satisfies the assumptions of Theorem 4.

If we construct an isotopism $(\alpha, \beta, \gamma, \delta)$ of PTR's $\mathbf{R}_1$ and $\mathbf{R}_2$ in the same way as in Theorem 4, then $x^\gamma = x \circ_1 2$, $x^\beta = x \circ_1 2$, $x^\alpha = x$ for all $x \in \mathcal{S}$. Since O is the zero in the PTR's $\mathbf{R}_1$ and $\mathbf{R}_2$, we have $\varphi_1 = \varphi_2 = \mathrm{id}_{\mathcal{S}}$. Then $\alpha = \delta = \mathrm{id}_{\mathcal{S}}$ and $\beta = \gamma$. Since (π, λ) is an isomorphism and $(x, y) \mathcal{I}_1 [m, b] \Leftrightarrow (x, y)^\pi \mathcal{I}_2 [m, b]^\lambda \Rightarrow (x^\beta, y) \mathcal{I}_2 [m^\beta, b]$, it means that for each $x, y, m, b \in \mathcal{S}$ for which the above mentioned relation holds, $x \circ_1 m +_1 b = y$ and $x^\beta \circ_2 m^\beta +_2 b = y$. We can write $x \circ_1 m +_1 b = x^\beta \circ_2 m^\beta +_2 b$ which holds for each $x, m, b \in \mathcal{S}$, as can be verified from Tables 1–3. Hence the planar ternary rings $\mathbf{R}_1$ and $\mathbf{R}_2$ are isotopic.

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