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ON AN INTEGRAL FORMULA

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One of the main tools used in the global differential geometry is the integral formula (1.14.1) of [1] for the so-called Codazzi-tensors. In the following paper, I present a more (possibly the most) general integral formula; the above mentioned formula appears then as its simple corollary.

Given a Riemannian manifold (M, g) , $\dim M = n$. Let ∇ be its associated Euclidean connection. In each coordinate neighborhood U of M , we may choose orthonormal sections $\{v_1 \dots v_n\}$ of $T(U)$ such that ∇ is given by

$$(1) \quad \nabla m = \omega^i v_i, \quad \nabla v_i = \omega_i^j v_j; \quad \omega_i^i + \omega_j^j = 0$$

and we have

$$(2) \quad d\omega^i = \omega^j \wedge \omega_j^i, \quad d\omega_i^j = \omega_i^k \wedge \omega_k^j - \frac{1}{2} R_{ikl}^j \omega^k \wedge \omega^l, \quad R_{ikl}^j + R_{ilk}^j = 0.$$

The curvature tensor (at $m \in U$)

$$(3) \quad R : T_m(M) \times T_m(M) \times T_m(M) \rightarrow T_m(M),$$

$$R(y^i v_i, z^i v_i)(x^i v_i) = R_{ikl}^j x^i y^k z^l v_j$$

satisfies (2₃) and

$$(4) \quad R_{ikl}^j + R_{jkl}^i = 0, \quad R_{ikl}^j = R_{kij}^l, \quad R_{ikl}^j + R_{ilj}^k + R_{ijl}^k = 0,$$

i.e.,

$$(5) \quad R(y, z)x + R(z, y)x = 0, \quad g(x, R(z, t)y) + g(y, R(z, t)x) = 0,$$

$$g(y, R(z, t)x) = g(t, R(x, y)z),$$

$$g(y, R(z, t)x) + g(z, R(t, y)x) + g(t, R(y, z)x) = 0$$

for $x, y, z, t \in T_m(M)$.

Let (E, g^*) , $\dim E = n + N$ be a Euclidean vector bundle over M ; on E , let a metric connection ∇^* satisfying

$$(6) \quad \nabla_x^* g^*(\xi, \eta) = g^*(\nabla_x^* \xi, \eta) + g^*(\xi, \nabla_x^* \eta)$$

for each $x \in T_m(M)$, $m \in M$ and local sections $\xi, \eta : M \rightarrow E$ defined in a neighborhood of m be given. Suppose that E is trivial over U , and let $w_\alpha : U \rightarrow E$ ($\alpha, \beta, \dots = 1, \dots, N$) be orthonormal sections; ∇^* is then given by

$$(7) \quad \nabla^* w_\alpha = \tau_\alpha^\beta w_\beta, \quad \tau_\alpha^\beta + \tau_\beta^\alpha = 0.$$

We have

$$(8) \quad d\tau_\alpha^\beta = \tau_\alpha^\gamma \wedge \tau_\gamma^\beta - \frac{1}{2} S_{\alpha ij}^\beta \omega^i \wedge \omega^j, \quad S_{\alpha ij}^\beta + S_{\alpha ji}^\beta = 0,$$

the curvature tensor of ∇^* being

$$(9) \quad S : T_m(M) \times T_m(M) \times E_m \rightarrow E_m, \\ S(x^i v_i, y^j v_j) (\xi^\alpha w_\alpha) = S_{\alpha ij}^\beta \xi^\alpha x^i y^j w_\beta.$$

Evidently,

$$(10) \quad S_{\alpha ij}^\beta + S_{\beta ij}^\alpha = 0,$$

i.e., S satisfies

$$(11) \quad S(x, y) \xi + S(y, x) \xi = 0, \quad g^*(\xi, S(x, y) \eta) + g^*(\eta, S(x, y) \xi) = 0$$

for each $\xi, \eta \in E_m$; $x, y \in T_m(M)$.

For each $m \in M$, let a p -linear mapping

$$(12) \quad F : \times^p T_m(M) \rightarrow E_m$$

be given. The mappings

$$(13) \quad F^{(1)} : \times^{p+1} T_m(M) \rightarrow E_m, \quad F^{(2)} : \times^{p+2} T_m(M) \rightarrow E_m$$

let be introduced by

$$(14) \quad F^{(1)}(x_{(1)}, \dots, x_{(p)}, x) = \nabla_x^* F(x'_{(1)}, \dots, x'_{(p)}) - \\ - \sum_{r=1}^p F(x_{(1)}, \dots, x_{(r-1)}, \nabla_x x'_{(r)}, x_{(r+1)}, \dots, x_{(p)}), \\ F^{(2)} = (F^{(1)})^{(1)},$$

where $x_{(1)}, \dots, x_{(p)}, x \in T_m(M)$ and $x'_{(1)}, \dots, x'_{(p)}$ are local sections of $T(M)$ such that $x'_{(1)}(m) = x_{(1)}, \dots, x'_{(p)}(m) = x_{(p)}$.

Lemma. The mappings $F^{(1)}$ and $F^{(2)}$ are well defined, i.e., they do not depend on the choice of the sections $x'_{(1)}, \dots, x'_{(p)}$. Further,

$$(15) \quad F^{(2)}(x_{(1)}, \dots, x_{(p)}, y, x) - F^{(2)}(x_{(1)}, \dots, x_{(p)}, x, y) = \\ = \sum_{r=1}^d F(x_{(1)}, \dots, x_{(r-1)}, R(x, y) x_{(r)}, x_{(r+1)}, \dots, x_{(p)}) - S(x, y) F(x_{(1)}, \dots, x_{(p)}).$$

Proof. Let us write, in U ,

$$(16) \quad F(x_{(1)}^i v_i, \dots, x_{(p)}^i v_i) = F_{i_1 \dots i_p}^\alpha x_{(1)}^{i_1} \dots x_{(p)}^{i_p} \omega_\alpha.$$

The components $F_{i_1 \dots i_p}^\alpha$ let be introduced by

$$(17) \quad dF_{i_1 \dots i_p}^\alpha - \sum_{r=1}^p F_{i_1 \dots i_{r-1} i_{r+1} \dots i_p}^\alpha \omega_{i_r}^i + F_{i_1 \dots i_p}^\beta \tau_\beta^\alpha = F_{i_1 \dots i_p}^\alpha \omega^i.$$

Then it is easy to see that

$$(18) \quad F^{(1)}(x_{(1)}^i v_i, \dots, x_{(p)}^i v_i, x^i v_i) = F_{i_1 \dots i_p}^\alpha x_{(1)}^{i_1} \dots x_{(p)}^{i_p} x^i \omega_\alpha.$$

The exterior differentiation of (17) yields

$$(19) \quad (dF_{i_1 \dots i_p}^\alpha - F_{i_1 \dots i_p}^\alpha \omega_i^j - \sum_{r=1}^p F_{i_1 \dots i_{r-1} j i_{r+1} \dots i_p}^\alpha \omega_{i_r}^j + F_{i_1 \dots i_p}^\beta \tau_\beta^\alpha) \wedge \omega^i = \\ = \frac{1}{2} \left(\sum_{r=1}^p F_{i_1 \dots i_{r-1} k i_{r+1} \dots i_p}^\alpha R_{i_r j i}^k - F_{i_1 \dots i_p}^\beta S_{\beta j i}^\alpha \right) \omega^j \wedge \omega^i.$$

The components $F_{i_1 \dots i_p i j}^\alpha$ let be defined by

$$(20) \quad dF_{i_1 \dots i_p i}^\alpha - F_{i_1 \dots i_p j}^\alpha \omega_i^j - \sum_{r=1}^p F_{i_1 \dots i_{r-1} j i_{r+1} \dots i_p}^\alpha \omega_{i_r}^j + F_{i_1 \dots i_p}^\beta \tau_\beta^\alpha = F_{i_1 \dots i_p i j}^\alpha \omega^j.$$

Clearly,

$$(21) \quad F^{(2)}(x_{(1)}^i v_i, \dots, x_{(p)}^i v_i, x^i v_i, y^i v_i) = F_{i_1 \dots i_p i j}^\alpha x_{(1)}^{i_1} \dots x_{(p)}^{i_p} x^i y^j \omega_\alpha,$$

and (19) implies

$$(22) \quad F_{i_1 \dots i_p j i}^\alpha - F_{i_1 \dots i_p i j}^\alpha = \sum_{r=1}^p F_{i_1 \dots i_{r-1} k i_{r+1} \dots i_p}^\alpha R_{i_r i j}^k - F_{i_1 \dots i_p}^\beta S_{\beta i j}^\alpha,$$

i.e., (15). QED.

Let us introduce the following notation. For $H : \times^q T_m(M) \rightarrow \mathbb{R}$ a q -linear mapping, $q \geq 2$, $1 \leq q_1 < q_2 \leq q$, write

$$(23) \quad H(x_{(1)}, \dots, x_{(q_1-1)}, A, x_{(q_1+1)}, \dots, x_{(q_2-1)}, A, x_{(q_2+1)}, \dots, x_{(q)}) = \\ = \sum_{i=1}^n H(x_{(1)}, \dots, x_{(q_1-1)}, v_i, x_{(q_1+1)}, \dots, x_{(q_2-1)}, v_i, x_{(q_2+1)}, \dots, x_{(q)}),$$

where $\{v_1, \dots, v_n\}$ is an arbitrary orthonormal basis of $T_m(M)$; (23) is well defined.

Theorem. Let (M, g) , (E, g^*, ∇^*) be as above. At each $m \in M$, let p -linear mappings $F, G : \times^p T_m(M) \rightarrow E_m$ be given. On M , consider the 1-form

$$(24) \quad \varphi(x) = g^*(F(A_1, \dots, A_{p-1}, x), G^{(1)}(A_1, \dots, A_{p-1}, B, B)) - \\ - g^*(F(A_1, \dots, A_{p-1}, B), G^{(1)}(A_1, \dots, A_{p-1}, x, B)).$$

Let ∂M be the boundary of M and $*$ the Hodge operator. Then

$$(25) \quad \int_{\partial M} * \varphi = \int_M \{g^*(F^{(1)}(A_1, \dots, A_{p-1}, B, B), G^{(1)}(A_1, \dots, A_{p-1}, C, C)) - \\ - g^*(F^{(1)}(A_1, \dots, A_{p-1}, B, C), G^{(1)}(A_1, \dots, A_{p-1}, C, B)) + \\ + g^*(F(A_1, \dots, A_{p-1}, B), \sum_{r=1}^{p-1} G(A_1, \dots, A_{r-1}, R(B, C) A_r, A_{r+1}, \dots, A_{p-1}, C) + \\ + G(A_1, \dots, A_{p-1}, R(B, C) C) - S(B, C) G(A_1, \dots, A_{p-1}, C))\} do,$$

do being the volume element on M .

Proof. On U we get

$$(26) \quad \varphi = \delta_{\alpha\beta} \delta^{i_1 j_1} \dots \delta^{i_{p-1} j_{p-1}} \delta^{kl} (F_{i_1 \dots i_{p-1} i}^\alpha G_{j_1 \dots j_{p-1} kl}^\beta - F_{i_1 \dots i_{p-1} k}^\alpha G_{j_1 \dots j_{p-1} il}^\beta) \omega^i.$$

Hence

$$(27) \quad d * \varphi = \delta_{\alpha\beta} \delta^{i_1 j_1} \dots \delta^{i_{p-1} j_{p-1}} \delta^{kl} \delta^{ij} (F_{i_1 \dots i_{p-1} ij}^\alpha G_{j_1 \dots j_{p-1} kl}^\beta - F_{i_1 \dots i_{p-1} k}^\alpha G_{j_1 \dots j_{p-1} il}^\beta + \\ + F_{i_1 \dots i_{p-1} i}^\alpha G_{j_1 \dots j_{p-1} kl}^\beta - F_{i_1 \dots i_{p-1} k}^\alpha G_{j_1 \dots j_{p-1} il}^\beta) do = \\ = \delta_{\alpha\beta} \delta^{i_1 j_1} \dots \delta^{i_{p-1} j_{p-1}} \delta^{kl} \delta^{ij} \{F_{i_1 \dots i_{p-1} ij}^\alpha G_{j_1 \dots j_{p-1} kl}^\beta - F_{i_1 \dots i_{p-1} k}^\alpha G_{j_1 \dots j_{p-1} il}^\beta + \\ + F_{i_1 \dots i_{p-1} i}^\alpha (G_{j_1 \dots j_{p-1} kl}^\beta - G_{j_1 \dots j_{p-1} kjl}^\beta)\} do;$$

using (22) for G , we get (25). QED.

Corollary. Let (M, g) be a Riemannian manifold. Let $T(x, y)$ be a symmetric tensor on M , and let there be a function $\tau : \partial M \rightarrow \mathbb{R}$ such that $T(x, y) = \tau g(x, y)$ on ∂M . Further, let $T(x, y)$ have, at $m \in M$, orthonormal eigen-vectors $\{v_1, \dots, v_n\}$ with the corresponding eigen-values k_1, \dots, k_n , and let K_{ij} be the sectional curvature corresponding to the 2-direction $\{v_i, v_j\}$. Then

$$(28) \quad \int_M \{T^{(1)}(A, B, B) T^{(1)}(A, C, C) - T^{(1)}(A, B, C) T^{(1)}(A, C, B) - \\ - \sum_{1 \leq i < j \leq n} K_{ij} (k_i - k_j)^2\} do = 0.$$

Proof. Set $E = M \times \mathbb{R}$, $F(x, y) = G(x, y) = T(x, y)$. Then

$$\begin{aligned}\varphi(x) &= T(A, x) T^{(1)}(A, B, B) - T(A, B) T^{(1)}(A, x, B) = \\ &= \delta^{ik} \delta^{lm} T_{ij} T_{klm} x^j - \delta^{ik} \delta^{jm} T_{ij} T_{klm} x^l.\end{aligned}$$

At $m \in \partial M$ we have $T_{ij} = \tau \delta_{ij}$ and

$$\varphi(x) = \tau \delta^{lm} (T_{ilm} - T_{ilm}) x^j = 0.$$

Further, at $m \in M$,

$$\begin{aligned}T(A, B) \{T(R(B, C) A, C) + T(A, R(B, C) C)\} &= \\ &= \delta^{im} \delta^{jp} \delta^{lq} T_{ij} T_{kl} R_{mpq}^k + \delta^{ik} \delta^{jp} \delta^{qm} T_{ij} T_{kl} R_{mpq}^l = \\ &= \sum_{i,k} \delta^{im} \delta^{ip} \delta^{kq} k_i k_k R_{mpq}^k + \sum_{i,k} p^{ik} \delta^{ip} \delta^{qm} k_i k_k R_{mpq}^k = \\ &= \sum_{i,k} k_i k_k R_{iik}^k + \sum_i \delta^{qm} k_i^2 R_{miq}^i = \sum_{i,k} k_i k_k K_{ik} - \sum_{i,j} k_i^2 K_{ji} = - \sum_{1 \leq i < j \leq n} (k_i - k_j)^2 K_{ij}.\end{aligned}$$

Inserting these into (25), we get (28). QED.

Bibliography

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