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ON QUASIPERIODIC MOTIONS IN A ONE-DIMENSIONAL TWO-PHASE STEFAN PROBLEM

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1. INTRODUCTION

A function $\varphi : R \to R$ is called quasiperiodic with basic frequencies $\nu_1, \ldots, \nu_n$ if it can be represented in the form

$$\varphi(t) = \Phi(\nu_1 t, \ldots, \nu_n t),$$

where $\Phi(\theta_1, \ldots, \theta_n)$ is a continuous function of period 2π in $\theta_1, \ldots, \theta_n$.

In recent years, the existence of quasiperiodic solutions to partial differential equations has bee investigated in a number of papers. For some references in this respect we refer to [1].

In this paper, we extend the result of [2], where the existence of ω-periodic solutions to a Stefan problem was studied. We shall deal with the system

(1.1) $$U_{1t} - a_1^2 U_{1xx} = 0, \quad 0 < x < s(t), \quad t \in R,$$

(1.2) $$U_1(t, 0) = T_1 + \chi_1(t), \quad t \in R,$$

(1.3) $$U_1(t, s(t)) = 0, \quad t \in R,$$

(1.4) $$U_{2t} - a_2^2 U_{2xx} = 0, \quad s(t) < x < b, \quad t \in R,$$

(1.5) $$U_2(t, s(t)) = 0, \quad t \in R,$$

(1.6) $$U_2(t, b) = -T_2 + \chi_2(t), \quad t \in R,$$

(1.7) $$s'(t) = m_2 U_{2x}(t, s(t)) - m_1 U_{1x}(t, s(t)), \quad t \in R,$$

(1.8) $$0 < s(t) < b, \quad t \in R.$$

We shall suppose that
a_j, m_j, T_j and b are positive constants and that χ_j are quasiperiodic functions with basic frequencies $\nu_1, \ldots, \nu_n$, i.e., $\chi_j(t) = h_j(\nu_1 t, \ldots, \nu_n t)$, where the functions $h_j(\theta_1, \ldots, \theta_n)$ are smooth and 2π-periodic in $\theta_1, \ldots, \theta_n$.

Let us denote

(1.9) $$s_0 = T_1 m_1 b/(T_1 m_1 + T_2 m_2)\,,$$

(1.10) $$u_1^{st}(x) = \{T_1 m_1(b - x) - T_2 m_2 x\}/bm_1 \quad \text{for} \quad 0 \leqq x \leqq s_0\,,$$

(1.11) $$u_2^{st}(x) = \{T_1 m_1(b - x) - T_2 m_2 x\}/bm_2 \quad \text{for} \quad s_0 \leqq x \leqq b\,.$$

It is obvious that

(1.12) if $\chi_1 = \chi_2 = 0$, then $s = s_0$, $U_j(t, x) = u_j^{st}(x)$, $j = 1, 2$, is a solution to (1.1)–(1.8).

The functions s, U_1 and U_2 which are to satisfy (1.1)–(1.8) will be represented in the form

(1.13) $$s(t) = \sigma(\nu_1 t, \ldots, \nu_n t)\,,$$

(1.14) $$U_1(t, x) = u_1(\nu_1 t, \ldots, \nu_n t, s_0 x/s(t))\,,$$

(1.15) $$U_2(t, x) = u_2(\nu_1 t, \ldots, \nu_n t, b - (b - s_0)(b - x)/(b - s(t)))\,,$$

where the functions $\sigma(\theta_1, \ldots, \theta_n)$, $u_1(\theta_1, \ldots, \theta_n, \xi)$, $u_2(\theta_1, \ldots, \theta_n, \xi)$ are defined respectively on R^n, $R^n \times [0, s_0]$, $R^n \times [s_0, b]$ and periodic in $\theta_1, \ldots, \theta_n$ with period 2π.

We shall prove that for every h_1, h_2 which are 2π-periodic in $\theta_1, \ldots, \theta_n$, sufficiently smooth and close to zero, there exist smooth functions s, U_1 and U_2, 2π-periodic in $\theta_1, \ldots, \theta_n$, close to s_0, u_1^{st} and u_2^{st}, respectively, and such that s, U_1 and U_2 given by (1.13)–(1.15) satisfy (1.1)–(1.8). This shows that the function s, describing the position of the phase interface, is quasiperiodic with basic frequencies $\nu_1, \ldots, \nu_n$. In other words, it will be proved that small quasiperiodic perturbations of constant boundary temperatures give rise to quasiperiodic changes of the position of the phase interface.

The main result of the paper is formulated and proved in Section 5 as a consequence of the Implicit Function Theorem applied in the spaces introduced in Section 3. In section 2, the transformation of the spatial variable indicated by (1.13)–(1.15) is accomplished and the system (1.1)–(1.8) is converted to a system of equations for functions on fixed spatial intervals. Some auxiliary results are derived in Section 4.

2. TRANSFORMATIONS OF VARIABLES

Throughout the paper we will denote

$$D_\theta = \sum_{j=1}^{n} \nu_j \frac{\partial}{\partial \theta_j}\,.$$

Inserting the expresions (1.13)–(1.15) for s, U_1 and U_2 into (1.1)–(1.8), we obtain the following system of equations:

$$D_\theta u_1 - \{a_1 s_0/\sigma\}^2 u_{1\xi\xi} - (D_\theta\sigma)\,\xi u_{1\xi}/\sigma = 0 \quad \text{for} \quad 0 < \xi < s_0\,, \tag{2.1}$$

$$u_1(\cdot, 0) = T_1 + h_1\,, \tag{2.2}$$

$$u_1(\cdot, s_0) = 0\,, \tag{2.3}$$

$$D_\theta u_2 - \{a_2(b - s_0)/(b - \sigma)\}^2 u_{2\xi\xi} - (D_\theta\sigma)(b - \xi)\,u_{2\xi}/(b - \sigma) = 0 \quad \text{for} \quad s_0 < \xi < b\,, \tag{2.4}$$

$$u_2(\cdot, s_0) = 0\,, \tag{2.5}$$

$$u_2(\cdot, b) = -T_2 + h_2\,, \tag{2.6}$$

$$D_\theta\sigma + m_1 s_0 u_{1\xi}(\cdot, s_0)/\sigma - m_2(b - s_0)\,u_{2\xi}(\cdot, s_0)/(b - \sigma) = 0\,, \tag{2.7}$$

$$0 < \sigma < b\,. \tag{2.8}$$

Conversely, if the functions u_1, u_2 and σ are solutions to (2.1)–(2.8), then it is easy to verify that the functions U_1, U_2 and s given by (1.13)–(1.15) satisfy (1.1)–(1.8).

In this perturbation study, it is convenient to look for the functions u_1, u_2 and σ in the form

$$u_1(\theta, \xi) = v_1(\theta, \xi) + u_1^{st}(\xi) + h_1(\theta)(s_0 - \xi)/s_0\,, \tag{2.9}$$

$$u_2(\theta, \xi) = v_2(\theta, \xi) + u_2^{st}(\xi) + h_2(\theta)(\xi - s_0)/(b - s_0)\,, \tag{2.10}$$

$$\sigma(\theta) = s_0 + r(\theta)\,. \tag{2.11}$$

Inserting these relations into (2.1)–(2.8), we obtain a number of equations which v_1, v_2 and r are to satisfy. Four of them say that the boundary values of the functions v_1 and v_2 are zero and together with the conditions of 2π-periodicity in $\theta_1, \ldots, \theta_n$ they will be used when introducing the spaces involved.

Denoting

$$T = T_1 m_1 + T_2 m_2\,, \quad A_j = T/bm_j\,, \quad j = 1, 2\,,$$

we find that the functions v_1, v_2 and r, all 2π-periodic in $\theta_1, \ldots, \theta_n$, are to satisfy the system

$$G_1(v_1, r, h_1) \equiv D_\theta v_1 - \{a_1 s_0/(s_0 + r)\}^2 v_{1\xi\xi} - (D_\theta r)\,\xi(v_{1\xi} - A_1)/(s_0 + r) + (D_\theta h_1)(s_0 - \xi)/s_0 + (D_\theta r)\,h_1\xi/\{(s_0 + r)\,s_0\} = 0\,, \quad 0 < \xi < s_0\,, \tag{2.12}$$

$$v_1(\cdot, 0) = v_1(\cdot, s_0) = 0\,, \tag{2.13}$$

(2.14) $$G_2(v_2, r, h_2) \equiv D_\theta v_2 - \{a_2(b - s_0)/(b - s_0 - r)\}^2 v_{2\xi\xi} - (D_\theta r)(b - \xi)(v_{2\xi} - A_2)/(b - s_0 - r) + (D_\theta h_2)(\xi - s_0)/(b - s_0) - (D_\theta r) h_2(b - \xi)/\{(b - s_0 - r)(b - s_0)\} = 0\,, \quad s_0 < \xi < b\,,$$

(2.15) $$v_2(\cdot, s_0) = v_2(\cdot, b) = 0\,,$$

(2.16) $$G_3(v_1, v_2, r, h_1, h_2) \equiv D_\theta r + Tr/\{(s_0 + r)(b - s_0 - r)\} + m_1 s_0 v_{1\xi}(\cdot, s_0)/(s_0 + r) - m_2(b - s_0) v_{2\xi}(\cdot, s_0)/(b - s_0 - r) - m_1 h_1/(s_0 + r) - m_2 h_2/(b - s_0 - r) = 0\,,$$

(2.17) $$|r| < \min(s_0, b - s_0)\,.$$

We shall return to this system in Section 5 when the function spaces introduced in the next section and some auxiliary results of Section 4 will be available. This section is concluded with one obvious remark.

(2.18) If $h_1 = h_2 = 0$, then $v_1 = 0$, $v_2 = 0$ and $r = 0$ satisfy (2.12)–(2.17).

3. FUNCTION SPACES

We shall denote by Z^+ the set of nonnegative integers and by N the set of positive integers. For $n \in N$, we put

$$T_n = [0, 2\pi]^n\,.$$

In what follows the following standard notation will be used: $\alpha = (\alpha_1, \ldots, \alpha_n) \in (Z^+)^n$, $|\alpha| = \sum_{j=1}^{n} \alpha_j$, and $D_\theta^\alpha = D_{\theta_1}^{\alpha_1} \ldots D_{\theta_n}^{\alpha_n}$. Given a $k \in Z^+$, we denote by P_k the space of all real-valued functions $\sigma(\theta_1, \ldots, \theta_n)$ of period 2π in $\theta_1, \ldots, \theta_n$, such that their generalized derivatives $D_\theta^\alpha \sigma$ are locally square-integrable on R^n for all α, $|\alpha| \leqq k$. The space P_k is a Hilbert space if equipped with the inner product

$$\langle \sigma, \eta \rangle_{P_k} = \sum_{|\alpha| \leqq k} \langle D_\theta^\alpha \sigma, D_\theta^\alpha \eta \rangle_{P_0}\,,$$

where

$$\langle \sigma, \eta \rangle_{P_0} = \int_{T_n} \sigma(\theta)\, \eta(\theta)\, d\theta_1 \ldots d\theta_n\,.$$

The corresponding norm in P_k is given by

$$\|\sigma\|_{P_k} = \langle \sigma, \sigma \rangle_{P_k}^{1/2}\,.$$

Further, we denote by $\mathscr{P}_k$ the space consisting of all functions $\sigma \in P_k$ such that $D_\theta \sigma \in P_k$. $\mathscr{P}_k$ will be equipped with the inner product

$$\langle \sigma, \eta \rangle_{\mathscr{P}_k} = \langle \sigma, \eta \rangle_{P_k} + \langle D_\theta \sigma, D_\theta \eta \rangle_{P_k}$$

and the norm

$$\|\sigma\|_{\mathscr{P}_k} = \langle\sigma, \sigma\rangle_{\mathscr{P}_k}^{1/2} .$$

Given s_0 and b, $0 < s_0 < b$, we put

$$I_1 = [0, s_0] \quad \text{and} \quad I_2 = [s_0, b] .$$

Thus, $\partial I_1 = \{0, s_0\}$ and $\partial I_2 = \{s_0, b\}$. To simplify the notation we denote by ${}_jH^0$ the space of all real-valued functions on $R^n \times I_j$ which have period 2π in first n variables and are square-integrable on $T_n \times I_j$. For $u, v \in {}_jH^0$, we put

$$\langle u, v\rangle_{{}_jH^0} = \int_{T_n}\int_{I_j} u(\theta, \xi)\, v(\theta, \xi)\, \mathrm{d}\xi\, \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n$$

and

$$\|u\|_{{}_jH^0} = \langle u, u\rangle_{{}_jH^0}^{1/2} .$$

The space consisting of all functions $u \in {}_jH^0$ for which

$$\|u\|_{{}_jH^k} \equiv \{\sum_{|\alpha|+\beta\leq k} \|D_\theta^\alpha D_\xi^\beta u\|_{{}_jH^0}^2\}^{1/2}$$

is finite will be denoted by ${}_jH^k$. We denote by $\mathscr{C}_j$ the space of all real-valued and smooth functions $u(\theta, \xi)$ on $R^n \times I_j$ which have period 2π in $\theta_1, \ldots, \theta_n$ and vanish for $\xi \in \partial I_j$. Let B_j be the completion of $\mathscr{C}_j$ with respect to the norm

$$|||u||| \equiv \|u\|_{{}_jH^0} + \|D_\xi^1 u\|_{{}_jH^0} .$$

Finally, we denote by ${}_j\mathscr{H}^k$ the space consisting of all functions u satisfying $u \in {}_jH^k \cap B_j$, $D_\theta u \in {}_jH^k$, $D_\xi^2 u \in {}_jH^k$. The space ${}_j\mathscr{H}^k$ will be equipped with the norm

$$\|u\|_{{}_j\mathscr{H}^k} = \{\|u\|_{{}_jH^k}^2 + \|D_\theta u\|_{{}_jH^k}^2 + \|D_\xi^2 u\|_{{}_jH^k}^2\}^{1/2} .$$

4. AUXILIARY ASSERTIONS

Lemma 4.1. *Let $p \in Z^+$ and $j = 1$ or 2. For every $g \in {}_jH^p$, there exists a unique $w \in {}_j\mathscr{H}^p$ such that*

$$(D_\theta - a_j^2 D_\xi^2)\, w = g .$$

Moreover,

$$\|w\|_{{}_j\mathscr{H}^p} \leq c_{p,j}\|g\|_{{}_jH^p} \tag{4.1}$$

with $c_{p,j}$ independent of g.

Proof. We can restrict ourselves to the case $j = 1$ and $s_0 = \pi$, i.e., $I_1 = (0, \pi)$. To begin with, let us suppose $p = 0$. If the functions g and w are written in the form

$$g(\theta, \xi) = \sum_{l\in Z^n}\sum_{k\in N} g_{lk} \mathrm{e}^{il\theta} \sin k\xi , \quad w(\theta, \xi) = \sum_{l\in Z^n}\sum_{k\in N} w_{lk} \mathrm{e}^{il\theta} \sin k\xi ,$$

we immediately obtain

$$w_{lk} = g_{lk}/\{ivl + k^2\},$$

where $l = (l_1, \ldots, l_n)$, $vl = v_1 l_1 + \ldots + v_n l_n$ and Z stands for the set of integers. The last relation implies $w \in {}_1\mathscr{H}^0$ and the lemma is proved for $p = 0$. Further, we shall proceed by induction with respect to p. Let (4.1) be satisfied for $p - 1$, $p \in N$. Given any $g \in {}_1H^p$, we have $D^1_{\theta_j} g \in {}_1H^{p-1}$ for $j = 1, \ldots, n$. We now apply the difference operators in the variables $\theta_1, \ldots, \theta_n$, in which all the functions are 2π-periodic. Let δ^j_h be the difference operator in the variable θ_j, i.e.,

$$(\delta^j_h v)(\theta, x) = \{v(\theta + he^j, x) - v(\theta, x)\}/h,$$

where $e^j = (\delta_{j1}, \ldots, \delta_{jn})$ and δ_{jk} stands for the Kronecker symbol. Applying the difference operator to the equation in question, we get

$$(\mathsf{D}_\theta - a_1^2 D_\xi^2)\, \delta^j_h w = \delta^j_h g$$

and therefore, by the induction hypothesis,

$$\|\delta^j_h w\|_{{}_1\mathscr{H}^{p-1}} \leqq c_{p-1,1} \|\delta^j_h g\|_{{}_1H^{p-1}}.$$

As

$$\|\delta^j_h g\|_{{}_1H^{p-1}} \leqq \|D^1_{\theta_j} g\|_{{}_1H^{p-1}},$$

we immediately get $D^1_{\theta_j} w \in {}_1\mathscr{H}^{p-1}$ and

$$\|D^1_{\theta_j} w\|_{{}_1\mathscr{H}^{p-1}} \leqq c_{p-1,1} \|D^1_{\theta_j} g\|_{{}_1H^{p-1}}.$$

A a consequence, the right-hand side of the equation

$$D_\xi^{p+2} w = \mathsf{D}_\theta D_\xi^p w - D_\xi^p g$$

is an element of ${}_1H^0$. Hence, $D_\xi^{p+2} w \in {}_1H^0$. This relation and the inequality show that the inductive step is completed and the lemma is proved.

The next lemma immediately follows from the preceding one.

Lemma 4.2. *The operator $\mathsf{D}_\theta - a_j^2 D_\xi^2$ is a linear homeomorphism of ${}_j\mathscr{H}^k$ onto ${}_jH^k$.*

The following lemma is standard.

Lemma 4.3. *The mapping $w \to w_\xi(\cdot, s_0)$ is a linear continuous mapping of ${}_j\mathscr{H}^k$ into P_k for both $j = 1$ and 2.*

For the proof see Lemma 3.3 of [2].

Let $d > 0$ be fixed. We denote

$$Lp = \mathsf{D}_\theta p + dp + m_1 w_{1\xi}(\cdot, s_0) - m_2 w_{2\xi}(\cdot, s_0)$$

for $p \in \mathscr{P}_k$ and $w_j \in {}_j\mathscr{H}^k$ satisfying

$$(4.2) \qquad (\mathsf{D}_\theta - a_1^2 D_\xi^2)\, w_1 = -A_1 \xi (\mathsf{D}_\theta p)/s_0,$$

$$(4.3) \qquad (\mathsf{D}_\theta - a_2^2 D_\xi^2)\, w_2 = -A_2 (b - \xi)(\mathsf{D}_\theta p)/(b - s_0).$$

Lemma 4.4. *For every* $p \in \mathscr{P}_0$, *we have*

$$\langle Lp, D_\theta p + p\rangle_{P_0} \geqq \|D_\theta p\|_{P_0}^2 + \|p\|_{P_0}^2 . \tag{4.4}$$

Proof. We begin by showing that

$$(-1)^{j+1}\langle w_{j\xi}(\cdot, s_0), D_\theta p\rangle_{P_0} \geqq 0 , \tag{4.5}$$

$$(-1)^{j+1}\langle w_{j\xi}(\cdot, s_0), p\rangle_{P_0} \geqq 0 \tag{4.6}$$

for both $j = 1$ and 2. Multiplying (4.2) by $-w_{1\xi\xi}$ and integrating over $T_n \times I_1$, we get

$$-\langle w_{1\xi\xi}, D_\theta w_1\rangle_{1H^0} + a_1^2\|w_{1\xi\xi}\|_{1H^0}^2 = As_0^{-1}\langle w_{1\xi\xi}, \xi D_\theta p\rangle_{1H^0} . \tag{4.7}$$

We now have

$$\langle w_{1\xi\xi}, D_\theta w_1\rangle_{1H^0} = -\sum_{j=1}^{n} v_j\langle D_{\theta_j}^1 w_{1\xi}, w_{1\xi}\rangle_{1H^0} = 0 \tag{4.8}$$

in virtue of the 2π-periodicity in θ_j. Further, since $w_j(\theta, \xi)$ vanishes for $(\theta, \xi) \in R^n \times \times \partial I_j$, we have

$$\langle w_{1\xi}, D_\theta p\rangle_{1H^0} = \left\langle \int_0^{s_0} w_{1\xi}(\cdot, \xi)\,\mathrm{d}\xi, D_\theta p\right\rangle_{P_0} = \langle 0, D_\theta p\rangle_{P_0} = 0$$

and therefore

$$\langle w_{1\xi\xi}, \xi D_\theta p\rangle_{1H^0} = \langle(\xi w_{1\xi})_\xi - w_{1\xi}, D_\theta p\rangle_{1H^0} = \tag{4.9}$$

$$= \left\langle \int_0^{s_0} (\xi w_{1\xi})_\xi\,\mathrm{d}\xi, D_\theta p\right\rangle_{P_0} = s_0\langle w_{1\xi}(\cdot, s_0), D_\theta p\rangle_{P_0} .$$

Using (4.8) and (4.9) in (4.7), we immediately get (4.5) for $j = 1$. To prove (4.6) with $j = 1$ we shall proceed similarly. Multiplying (4.2) by w_1 and integrating over $T_n \times T_1$, we get

$$\langle D_\theta w_1, w_1\rangle_{1H^0} - a_1^2\langle w_{1\xi\xi}, w_1\rangle_{1H^0} = -A_1 s_0^{-1}\langle w_1, \xi D_\theta p\rangle_{1H^0} ,$$

which after some arrangements similar to those used above, turns out to be

$$a_1^2\|w_{1\xi}\|_{1H^0}^2 = A_1 s_0^{-1}\langle \xi D_\theta w_1, p\rangle_{1H^0} .$$

By (4.2), $D_\theta w_1 = a_1^2 w_{1\xi\xi} - A_1 s_0^{-1}\xi D_\theta p$ which we substitute into the last relation and thus obtain

$$a_1^2\|w_{1\xi}\|_{1H^0}^2 = A_1 a_1^2 s_0^{-1}\langle \xi w_{1\xi\xi}, p\rangle_{1H^0} - A_1^2 s_0^{-2}\langle \xi^2 p, D_\theta p\rangle_{1H^0} =$$

$$= A_1 a_1^2 s_0^{-1}\langle(\xi w_{1\xi})_\xi - w_{1\xi}, p\rangle_{1H^0} = A_1 a_1^2\langle w_{1\xi}(\cdot, s_0), p\rangle_{P_0} ,$$

since $\langle \xi^2 p, D_\theta p\rangle_{1H^0} = 0$ in virtue of the periodicity in θ. The last equation proves (4.6) with $j = 1$. As far as $j = 2$ is concerned, (4.5) and (4.6) can be proved similarly when, instead of (4.2), equation (4.3) is taken into consideration. By (4.5) and (4.6), we easily get

$$\langle Lp, D_\theta p + p\rangle_{P_0} = \|D_\theta p\|^2_{P_0} + \|p\|^2_{P_0} +$$
$$+ \sum_{j=1}^{2} m_j(-1)^{j+1} \{\langle w_{j\xi}(\cdot, s_0), p\rangle_{P_0} + \langle w_{j\xi}(\cdot, s_0), D_\theta p\rangle_{P_0}\} \geqq$$
$$\geqq \|D_\theta p\|^2_{P_0} + \|p\|^2_{P_0}\,.$$

This completes the proof.

Lemma 4.5. *The operator L is a linear homeomorphism of $\mathscr{P}_k$ onto P_k for every $k \in Z^+$.*

Proof. By Lemmas 4.2 and 4.3, the operator L is a linear continuous mapping of $\mathscr{P}_k$ into P_k. Lemma 4.4 implies that L is a one-to-one mapping. The lemma will be proved as soon as we show that $\mathbb{R}(L)$, the range of L, is equal to P_k. We shall proceed in two steps. Firstly we take up the case $k = 0$. By (4.4), we have

$$\|Lp\|_{P_0} \geqq \{\|D_\theta p\|^2_{P_0} + \|p\|_{P_0}\}^{1/2}/\sqrt{2}\,.$$

Thus, $\mathbb{R}(L)$ is a closed subspace of P_0. In fact, $\mathbb{R}(L) = P_0$ since otherwise we could take $q \in P_0$, $q \neq 0$, such that $\langle y, q\rangle_{P_0} = 0$ for every $y \in \mathbb{R}(L)$. Writing q as a Fourier series, $q(\theta) = \sum_{l \in Z^n} q_l e^{il\theta}$, we find that the function $p(\theta) = \sum_{l \in Z^n} q_l(ivl + 1)^{-1} e^{il\theta}$ satisfies $p \in \mathscr{P}_0$ and $D_\theta p + p = q$. By (4.4), we have

$$0 = \langle Lp, q\rangle_{P_0} = \langle Lp, D_\theta p + p\rangle_{P_0} \geqq \|D_\theta p\|^2_{P_0} + \|p\|^2_{P_0}$$

and therefore $p = 0$ which contradicts $q \neq 0$. Hence, $\mathbb{R}(L) = P_0$ which proves that L is a linear homeomorphism between $\mathscr{P}_0$ and P_0. Secondly, if $g \in P_k$ for some $k \in N$, we know, by the above reasoning, that there is a $p \in \mathscr{P}_0$ satisfying $Lp = g$. Using differences in the variables $\theta_1, \ldots, \theta_n$ as in the proof of Lemma 4.1, we easily find that p is actually an element of $\mathscr{P}_k$. This completes the proof.

5. THE MAIN RESULT

Let us put

$$k_0 = (n + 1)/2\,.$$

The following theorem is the main result of the paper.

Theorem 5.1. *Let the numbers $T_1, T_2, m_1, m_2, v_1, \ldots, v_n$ be positive. Let s_0 be defined by (1.9) and $k > k_0$, $k \in Z$. Then there exist two open sets $K \subset \mathscr{P}_k \times \mathscr{P}_k$ and $H \subset {}_1\mathscr{H}^k \times {}_2\mathscr{H}^k \times \mathscr{P}_k$ such that $(0, 0) \in K$, $(0, 0, 0) \in H$, and for every $(h_1, h_2) \in K$ there is a unique $(v_1, v_2, r) \in H$ which satisfies (2.12), (2.14) and (2.16). The correspondence $S : (h_1, h_2) \to (v_1, v_2, r)$ is a smooth mapping of K into H satisfying $S(0, 0) = (0, 0, 0)$.*

Proof. We put

$$B = \{r \in \mathscr{P}_k;\ \max_\theta |r(\theta)| < \min(s_0, b - s_0)\},$$

the topology on B being that induced from $\mathscr{P}_k$,

$$\mathscr{D} = {}_1\mathscr{H}^k \times {}_2\mathscr{H}^k \times B \times \mathscr{P}_k \times \mathscr{P}_k$$

and

$$G = (G_1, G_2, G_3).$$

It is not difficult to verify that $G : \mathscr{D} \to {}_1H^k \times {}_2H^k \times P_k$ is continuously Frèchet differentiable. Obviously, $G(0, 0, 0, 0, 0) = (0, 0, 0)$. For brevity we denote by $M = (M_1, M_2, M_3)$ the partial Frèchet derivative of G with respect to (v_1, v_2, r) at the point $(0, 0, 0, 0, 0)$. Under our assumptions

$$M(w_1, w_2, p) = \frac{\mathrm{d}}{\mathrm{d}t} G(0 + tw_1, 0 + tw_2, 0 + tp, 0, 0)\Big|_{t=0} =$$
$$= (M_1(w_1, p),\ M_2(w_2, p),\ M_3(w_1, w_2, p)),$$

where

$$M_1(w_1, p) = (\mathrm{D}_\theta - a_1^2 D_\xi^2)\, w_1 + A_1 \xi(\mathrm{D}_\theta p)/s_0,$$
$$M_2(w_2, p) = (\mathrm{D}_\theta - a_2^2 D_\xi^2)\, w_2 + A_2(b - \xi)(\mathrm{D}_\theta p)/(b - s_0),$$
$$M_3(w_1, w_2, p) = \mathrm{D}_\theta p + dp + m_1 w_{1\xi}(\cdot, s_0) - m_2 w_{2\xi}(\cdot, s_0)$$

and

$$d = T/\{(b - s_0)\, s_0\}.$$

We shall show that M is a linear homeomorphism between ${}_1\mathscr{H}^k \times {}_2\mathscr{H}^k \times \mathscr{P}_k$ and ${}_1H^k \times {}_2H^k \times P_k$. To this end we put

$$\Lambda_j = (\mathrm{D}_\theta - a_j^2 D_\xi^2)^{-1}$$

which, by Lemma 4.2, is a linear homeomorphism between ${}_j\mathscr{H}^k$ and ${}_jH^k$. Given any $(y_1, y_2, z) \in {}_1H^k \times {}_2H^k \times P_k$, we must find $(w_1, w_2, p) \in {}_1\mathscr{H}^k \times {}_2\mathscr{H}^k \times \mathscr{P}_k$ satisfying $M(w_1, w_2, p) = (y_1, y_2, z)$, i.e.,

$$(\mathrm{D}_\theta - a_1^2 D_\xi^2)\, w_1 = y_1 - A_1 \xi(\mathrm{D}_\theta p)/s_0,$$
$$(\mathrm{D}_\theta - a_2^2 D_\xi^2)\, w_2 = y_2 - A_2(b - \xi)(\mathrm{D}_\theta p)/(b - s_0),$$
$$\mathrm{D}_\theta p + dp + m_1 w_{1\xi}(\cdot, s_0) - m_2 w_{2\xi}(\cdot, s_0) = z.$$

The first two equations yield

(5.1) $$w_1 = \Lambda_1 y_1 - A_1 s_0^{-1} \Lambda_1(\xi \mathrm{D}_\theta p),$$

(5.2) $$w_2 = \Lambda_2 y_2 - A_2(b - s_0)^{-1} \Lambda_2((b - \xi)\, \mathrm{D}_\theta p),$$

which substituted into the third gives

$$(5.3) \qquad D_\theta p - m_1 A_1 s_0^{-1}\{A_1(\xi D_\theta p)\}_\xi(\cdot, s_0) + m_2 A_2(b - s_0)^{-1} \,.$$

$$. \{A_2((b - \xi) D_\theta p)\}_\xi(\cdot, s_0) = z + \sum_{j=1}^{2} (-1)^j m_j \{A_j y_j\}_\xi(\cdot, s_0) \,.$$

Obviously, the right-hand side of this equation is an element of P_k and the left-hand side is Lp, as defined in the preceding section. By Lemma 4.5, there is a unique $p \in \mathscr{P}_k$ satisfying (5.3). Hence, by (5.1), (5.2) and Lemma 4.2, we get a unique $(w_1, w_2, p) \in \in {}_1\mathscr{H}^k \times {}_2\mathscr{H}^k \times \mathscr{P}_k$ such that $M(w_1, w_2, p) = (y_1, y_2, z)$. Applying the Implicit Function Theorem, we complete the proof.

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