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ON A MAXIMUM PRINCIPLE IN POTENTIAL THEORY

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A very interesting theorem in potential theory due to Ugaheri asserts:

For every R^m there exists a positive constant M such that for every nonnegative nonincreasing function L defined on the interval $\langle 0, \infty\rangle$ and for every Radon measure p in R^m the following estimate holds:

$$\sup_{x\in R^m} \int_{R^m} L(|x - y|) \, \mathrm{d}p(y) \leqq M \sup_{x\in \operatorname{spt} p} \int_{R^m} L(|x - y|) \, \mathrm{d}p(y) .$$

In this paper we extend this result to more general kernels $K(x, y) \geqq 0$ on $R^m \times R^m$ satisfying the following conditions: $K(x, .)$ is Borel measurable for every $x \in R^m$ and there is a seminorm q on R^m and a constant $c > 0$ such that

$$q(x - y) \geqq q(z - y) \Rightarrow K(x, y) \leqq c\, K(z, y) .$$

We prove that for such K there is a constant $M > 0$ (depending on m only) such that

$$\sup_{x\in R^m} \int_{R^m} K(x, y) \, \mathrm{d}p(y) \leqq cM \sup_{x\in \operatorname{spt} p} \int_{R^m} K(x, y) \, \mathrm{d}p(y)$$

for every Radon measure p; at the same time we present estimates (which are the best possible in certain cases) for the corresponding M.

We use the following notation in the whole text: $|\;|$ – Euclidean norm; α_m – volume of the unitary ball (with respect to the Euclidean norm) in R^m; σ_m – surface of the unitary spher (with respect to the Euclidean norm) in R^m; λ_m – Lebesgue measure on R^m; $U(x; r)$ – open ball with the centre x and the radius r with respect to a certain metric; spt p – support of the measure p.

Definition 1. *Let $K(x, y)$ be a nonnegative function on $R^m \times R^m$ and let $K(x, .)$ be Borel-measurable for each $x \in R^m$. Then for every Radon measure p on R^m we define*

$$K\,p(x) = \int_{R^m} K(x, y)\,\mathrm{d}p(y)$$

for each $x \in R^m$.

Definition 2. *Let q be a seminorm on R^m. A positive constant M is called admissible for q, if every nonnegative function $K(x, y)$ on $R^m \times R^m$, such that $K(x, .)$ is Borel-measurable and there is a positive constant c such that*

$$q(x - y) \geqq q(z - y) \Rightarrow K(x, y) \leqq c\,K(z, y)\,,$$

satisfies

$$\sup_{x \in R^m} K\,p(x) \leqq cM \sup_{x \in \operatorname{spt} p} K\,p(x)$$

for every nonzero Radon measure p on R^m.

Theorem 1. *The constant $5^m - 3^m$ is admissible for every seminorm on R^m.*

Proof. 1) First we prove that the smallest number of elements of a $\frac{1}{2}$-net on the unitary sphere is smaller than or equal to $5^m - 3^m$ for every norm $\| \ \|$ on R^m.

Let $x^1, \ldots, x^k$ be points on the unitary sphere such that the distance of every two different points is greater than $\frac{1}{2}$ and there is no point on the unitary sphere which has a distance greater than $\frac{1}{2}$ from each of these points. Such points exist. The points $x^1, \ldots, x^k$ form a $\frac{1}{2}$-net on the unitary sphere. It suffices to prove that $k \leqq 5^m - 3^m$.

The balls with centres $x^1, \ldots, x^k$ and radius $\frac{1}{4}$ are disjoint subsets of the set $U(0; \frac{5}{4}) - U(0; \frac{3}{4})$. The sum of their Euclidean volumes is smaller than or equal to the volume of the set $U(0; \frac{5}{4}) - U(0; \frac{3}{4})$:

$$k(\tfrac{1}{4})^m\,V \leqq (\tfrac{5}{4})^m\,V - (\tfrac{3}{4})^m\,V\,,$$

where V is the volume of the unitary ball. Therefore

$$k \leqq 5^m - 3^m\,.$$

2) Let q be a seminorm on R^m and let c be a positive number. Let $K(x, y)$ be a nonnegative function on $R^m \times R^m$ such that $K(x, .)$ is Borel-measurable for each $x \in R^m$ and

$$q(x - y) \geqq q(z - y) \Rightarrow K(x, y) \leqq c\,K(z, y)\,.$$

Further, let p be a nonzero Radon measure with a compact support in R^m.

If $q = 0$ then arbitrary $x, y \in R^m$, $z \in \operatorname{spt} p$ fulfil $K(x, y) \leqq c\,K(z, y)$ and therefore

$$K\,p(x) \leqq cK\,p(z) \leqq \sup_{y \in \operatorname{spt} p} cK\,p(y) \leqq c(5^m - 3^m) \sup_{y \in \operatorname{spt} p} K\,p(y)\,.$$

Let there be $y \in R^m$ such that $q(y) \neq 0$.

$Y = \{y \in R^m;\ q(y) = 0\}$ is a linear space. Let Z be the direct complement of Y in R^m. The dimension of Z is n; q is a norm on Z. According to 1) there are points $x^1, \ldots, x^k$, which form a $\frac{1}{2}$-net on the unitary sphere in Z, where

$$k \leqq 5^n - 3^n \leqq 5^m - 3^m . \tag{1}$$

Put $L = Y + \operatorname{spt} p$. Then L is a closed set (in the Euclidean metric). Let $x \in R^m$ be an arbitrary fixed point.

If $x \in L$, there are $z \in \operatorname{spt} p$ and $y \in Y$ such that $x = z + y$. Then each $u \in \operatorname{spt} p$ fulfils $q(x - u) = q(z - u + y) = q(z - u)$. Therefore $K(x, u) \leqq c\,K(z, u)$ for each $u \in \operatorname{spt} p$. Therefore

$$K\,p(x) \leqq cK\,p(z) \leqq c \sup_{u \in \operatorname{spt} p} K\,p(u) \leqq c(5^m - 3^m) \sup_{u \in \operatorname{spt} p} K\,p(u) .$$

Assume now that $x \notin L$. Put

$$M_i = \left\{y \in Z + x - \{x\};\ q\left(\frac{y - x}{q(y - x)} - x^i\right) \leqq \frac{1}{2}\right\},\ L_i = M_i + Y \text{ for } i = 1, \ldots, k .$$

Since $x^1, \ldots, x^k$ is a $\frac{1}{2}$-net on the unitary sphere in Z, we have

$$\bigcup_{i=1}^{k} M_i = Z + x - \{x\} .$$

Hence

$$\bigcup_{i=1}^{k} L_i \supset L .$$

Clearly,

$$K\,p(x) = \int_L K(x, y)\,\mathrm{d}p(y) \leqq \sum_{i=1}^{k} \int_{L_i \cap L} K(x, y)\,\mathrm{d}p(y) .$$

With respect to (1) it suffices to prove that

$$\int_{L_i \cap L} K(x, y)\,\mathrm{d}p(y) \leqq c \sup_{y \in \operatorname{spt} p} K\,p(y) \quad \text{for} \quad i = 1, \ldots, k .$$

If $L_i \cap L = \emptyset$ then the inequality is evidently valid. Let $L_i \cap L \neq \emptyset$.

$M = L \cap (Z + x)$ is a closed subset of $Z + x$. Further, $L = M + Y$. Hence $L \cap L_i = (M \cap M_i) + Y$, where $M \cap M_i$ is a closed subset of $Z + x$. There is $u \in M \cap M_i$ such that $q(u - x) \leqq q(z - x)$ for every $z \in M \cap M_i$. Since $u \in M$ and $M + Y = L = Y + \operatorname{spt} p$, there is $y^1 \in Y$ such that $u + y^1 \in \operatorname{spt} p$. Let us denote $v = u + y^1$. For every $z \in L \cap L_i$ we have

$$q(v - x) \leqq q(z - x) . \tag{2}$$

If $z \in L \cap L_i$ then there are $z^1 \in M \cap M_i$ and $y \in Y$ such that $z^1 + y = z$. Then $q(v - x) = q(u - x) \leqq q(z^1 - x) = q(z - x)$. Since

$$q\left(\frac{u-x}{q(u-x)}-x^i\right)\leqq\frac{1}{2},\quad q\left(\frac{z^1-x}{q(z^1-x)}-x^i\right)\leqq\frac{1}{2},$$

we have

$$q\left(\frac{u-x}{q(u-x)}-\frac{z^1-x}{q(z^1-x)}\right)\leqq 1\,. \tag{3}$$

Further,

$$q(z-v)=q(z^1-u)=q(z^1-x-(u-x))\leqq \tag{4}$$

$$\leqq q\left((z^1-x)-\frac{q(u-x)}{q(z^1-x)}(z^1-x)\right)+q\left(\frac{q(u-x)}{q(z^1-x)}\right)(z^1-x-(u-x))=$$

$$=\left(1-\frac{q(u-x)}{q(z^1-x)}\right)q(z^1-x)+q(u-x)\,q\left(\frac{u-x}{q(u-x)}-\frac{z^1-x}{q(z^1-x)}\right)\leqq$$

$$\leqq q(z^1-x)-q(u-x)+q(u-x)=q(z^1-x)=q(z-x)$$

according to (2), (3). Thus

$$K(x,z)\leqq c\,K(v,z)\,.$$

Hence

$$\int_{L\cap L_i}K(x,y)\,\mathrm{d}p(y)\leqq\int_{L\cap L_i}c\,K(v,y)\,\mathrm{d}p(y)\leqq cK\,p(v)\leqq c\sup_{y\in\operatorname{spt}p}K\,p(y)\,,$$

because $v\in\operatorname{spt}p$.

3) Let q be a seminorm on R^m. Let K be a function on $R^m\times R^m$ with the same properties as in 2). Let p be a Radon measure on R^m which has not a compact support.

For every integer n we define a measure p_n on R^m by $p_n(A)=p(A\cap U(0;n))$ for each Borel set A. Then p_n has a compact support and

$$K\,p_n(x)\leqq c(5^m-3^m)\sup_{y\in\operatorname{spt}p_n}K\,p_n(y)\leqq c(5^m-3^m)\sup_{y\in\operatorname{spt}p_n}K\,p(y)\leqq$$

$$\leqq c(5^m-3^m)\sup_{y\in\operatorname{spt}p}K\,p(y)$$

for each $x\in R^m$.

$$K\,p(x)=\lim_{n\to\infty}K\,p_n(x)\leqq c(5^m-3^m)\sup_{y\in\operatorname{spt}p}K\,p(y)\,.$$

Theorem 2. *The smallest constant admissible for the maximum norm on R^m is 2^m.*

Proof. 1) First we prove that 2^m is an admissible constant for the maximum norm $\|\ \|$ on R^m.

Let c be a positive constant. Let $K(x,y)$ be a nonnegative function on $R^m\times R^m$, let $K(x,.)$ be Borel-measurable for each $x\in R^m$ and suppose that $\|x-y\|\leqq$

$\leqq \|z - y\| \Rightarrow K(z, y) \leqq c\, K(x, y)$. Let p be a non-zero Radon measure on R^m. Let $x \in R^m$ be an arbitrary fixed point. If $x \in \operatorname{spt} p$ then

$$K\, p(x) \leqq \sup_{y \in \operatorname{spt} p} K\, p(y)\,,$$

and since either $K = 0$ or $c \geqq 1$, we obtain

$$K\, p(x) \leqq \sup_{y \in \operatorname{spt} p} K\, p(y) \leqq c \sup_{y \in \operatorname{spt} p} K\, p(y) \leqq c\, 2^m \sup_{y \in \operatorname{spt} p} K\, p(y)\,.$$

Now let $x \notin \operatorname{spt} p$.

Put $M = \{-1; 1\}^m$. $M = \{x^i; i = 1, \ldots, 2^m\}$, where x^i, $i = 1, \ldots, 2^m$, are mutually different points on the unitary sphere determined by the norm $\| \;\|$. Put

$$M_i = \left\{ y \in R^m - \{x\};\; \left\| \frac{y - x}{\|y - x\|} - x^i \right\| \leqq 1 \right\} \quad \text{for} \quad i = 1, \ldots, 2^m\,.$$

Evidently

$$\bigcup_{i=1}^{2^m} M_i = R^m - \{x\}\,.$$

We obtain

$$K\, p(x) = \int_{\operatorname{spt} p} K(x, y)\, \mathrm{d}p(y) \leqq \sum_{i=1}^{2^m} \int_{M_i \cap \operatorname{spt} p} K(x, y)\, \mathrm{d}p(y)\,.$$

Thus it suffices to show that

$$(5) \qquad \int_{M_i \cap \operatorname{spt} p} K(x, y)\, \mathrm{d}p(y) \leqq c \sup_{z \in \operatorname{spt} p} K\, p(z) \quad \text{for} \quad i = 1, \ldots, 2^m\,.$$

If $M_i \cap \operatorname{spt} p = \emptyset$ then (5) is evidently valid. Let $M_i \cap \operatorname{spt} p \neq \emptyset$. Then there is $z \in M_i \cap \operatorname{spt} p$ such that for every $y \in M_i \cap \operatorname{spt} p$,

$$(6) \qquad \|x - z\| \leqq \|x - y\|\,.$$

For $u, v \in R^m$, $\|u\| = \|v\| = 1$, $\|u - x^i\| \leqq 1$, $\|v - x^i\| \leqq 1$ we prove

$$(7) \qquad \|u - v\| \leqq 1\,.$$

Then in the same way as we have proved the inequality (4) from (2), (3) in the proof of Theorem 1, we may prove from (6), (7) that $\|x - y\| \geqq \|z - y\|$ for every $y \in M_i \cap \operatorname{spt} p$. According to the assumption, $K(x, y) \leqq c\, K(z, y)$. Since $z \in \operatorname{spt} p$, we obtain

$$\int_{M_i \cap \operatorname{spt} p} K(x, y)\, \mathrm{d}p(y) \leqq c \int_{M_i \cap \operatorname{spt} p} K(z, y)\, \mathrm{d}p(y) \leqq cK\, p(z) \leqq c \sup_{y \in \operatorname{spt} p} K\, p(y)$$

and thus the inequality (5) is true.

We prove the inequality (7).

Let $u, v \in R^m$, $\|u\| = \|v\| = 1$, $\|u - x^i\| \leqq 1$, $\|v - x^i\| \leqq 1$. We assume for the simplicity that $x^i = \{1\}^m$. Since $\|u\| = \|v\| = 1$, we have $u_j \leqq 1$, $v_j \leqq 1$ for $j = 1, \ldots, m$. Since $\|u - x^i\| \leqq 1$, $\|v - x^i\| \leqq 1$, we have $0 \leqq u_j$, $0 \leqq v_j$ for $j = 1, \ldots, m$. Thus $\|u - v\| = \max_{j=1,\ldots,m} |u_j - v_j| \leqq 1$. The inequality (7) is valid.

2) Now we prove that every admissible constant for the maximum norm is greater than or equal to 2^m.

Let us define

$$\begin{aligned} L(t) &= 1 \quad &&\text{for} \quad 0 \leqq t \leqq 1\,, \\ & 2 - t \quad &&\text{for} \quad 1 < t < 2\,, \\ & 0 \quad &&\text{for} \quad t \geqq 2 \end{aligned}$$

on the interval $\langle 0; \infty)$.

Put $K(x, y) = L(\|x - y\|)$. Then $K(x, y)$ is a nonnegative continuous function on $R^m \times R^m$. Further,

$$\|x - y\| \geqq \|z - y\| \Rightarrow K(x, y) \leqq K(z, y)\,.$$

Put $A = \{-1; 1\}^m$. We define $p(B) =$ number of elements of $A \cap B$ for every set B. Then p is a Radon measure on R^m.

Let M be an admissible constant for the maximum norm on R^m. Then

$$K\,p(x) \leqq M \sup_{y \in \operatorname{spt} p} K\,p(y) \tag{8}$$

for each $x \in R^m$; clearly $\operatorname{spt} p = A$. For every $y \in A$, $K\,p(y) = 1$. Further, $K\,p(0) = 2^m$. If we substitute into (8) then we obtain $2^m \leqq M$.

Theorem 3. *$\frac{1}{2}(2 + \sqrt{(2 + \sqrt{3})})\,(m - 1)\,(2\sqrt{(2 + \sqrt{3})})^{m-1}$ is admissible for every Hilbert norm on R^m, where $m \geqq 3$. Every constant which is admissible for a Hilbert norm on R^m, $m \geqq 3$, is greater than or equal to $2(2/\sqrt{3})^{m-1}$. For every norm on R^1, the smallest admissible constant is equal to 2. The constant 6 is admissible for each Hilbert norm on R^2. Every constant which is admissible for a Hilbert norm on R^2 is greater than or equal to 5.*

Proof. 1) First of all we pass through some auxiliary calculations. Let $m \geqq 3$. The volume of the part of the ball with the radius R limited by the $(m - 1)$-dimensional plane with a distance h from the centre of the ball is

$$V(h, R) = \int_{\{[x_1,\ldots,x_m];\, x_1^2 + \ldots + x_m^2 \leqq R^2,\, x_1 \geqq h\}} 1 \,\mathrm{d}\lambda_m =$$

$$= \int_h^R \int_{\{[x_2,\ldots,x_m];\, x_2^2 + \ldots + x_m^2 \leqq R^2 - x_1^2\}} 1 \,\mathrm{d}\lambda_{m-1}\,\mathrm{d}x_1 = \alpha_{m-1} \int_h^R (R^2 - x^2)^{(m-1)/2}\,\mathrm{d}x\,.$$

The surface of the $(m-1)$-dimensional spherical cap corresponding to this body is the derivative of the function V with respect to R:

$$S(h, R) = \alpha_{m-1}(m-1) R \int_h^R (R^2 - x^2)^{(m-3)/2} \, dx .$$

For $R = 1$ we obtain

$$S(h) = \alpha_{m-1}(m-1) \int_h^1 (1 - x^2)^{(m-3)/2} \, dx . \tag{9}$$

For $h = 0$ we obtain

$$\sigma_m = 2\alpha_{m-1}(m-1) \int_0^1 (1 - x^2)^{(m-3)/2} \, dx . \tag{10}$$

2) Now we prove that the smallest number of elements of a $\sqrt{(2 - \sqrt{3})}$-net on the unitary sphere in R^m, where $m \geqq 3$, with respect to the Euclidean norm is smaller than or equal to $\frac{1}{2}(2 + \sqrt{(2 + \sqrt{3})}) (m-1) (2\sqrt{(2 + \sqrt{3})})^{m-1}$.

Let $x^1, \ldots, x^k$ be points on the unitary sphere such that the distance of every two different points is greater than $\sqrt{(2 - \sqrt{3})}$ (i.e., their radiusvectors enclose an angle greater than $\pi/6$) and there is no point on the unitary sphere which has the distance greater than $\sqrt{(2 - \sqrt{3})}$ from each of these points. Such points exist. The points $x^1, \ldots, x^k$ form a $\sqrt{(2 - \sqrt{3})}$-net on the unitary sphere. It suffices to prove that

$$k \leqq \frac{2 + \sqrt{(2 + \sqrt{3})}}{2} (m-1) (2\sqrt{(2 + \sqrt{3})})^{m-1} .$$

Denote by A_i, $i = 1, \ldots, k$ the set of the points on the unitary sphere such that their radiusvectors enclose with the radiusvector of the point x^i angles smaller than or equal to $\pi/12$. Since A_i, $i = 1, \ldots, k$, are disjoint subsets of the unitary sphere, the sum of the surfaces of A_i is smaller than or equal to the surface of the unitary sphere. A_i is a spherical cap such that the plane limiting the part of ball corresponding to this spherical cap has a distance from the point 0 equal to $h = \cos \pi/12 = \frac{1}{2}\sqrt{(2 + \sqrt{3})}$. Thus $k\, S(h) \leqq \sigma_m$. According to (9),

$$k \leqq \frac{\sigma_m}{\alpha_{m-1}(m-1) \int_h^1 (1 - x^2)^{(m-3)/2} \, dx} .$$

According to (10),

$$\sigma_m = 2\alpha_{m-1}(m-1) \int_0^1 (1 - x^2)^{(m-3)/2} \, dx \leqq 2\alpha_{m-1}(m-1) .$$

We obtain

$$k \leqq \frac{2}{\int_h^1 (1 - x^2)^{(m-3)/2}\,dx} \leqq \frac{2}{(1 + h)^{(m-3)/2} \int_h^1 (1 - x)^{(m-3)/2}\,dx} =$$

$$= \frac{(m - 1)}{2} \frac{(1 + h)\,2}{(\sqrt{(1 - h^2)})^{m-1}}.$$

If we substitute h then we obtain

$$k \leqq \frac{2 + \sqrt{(2 + \sqrt{3})}}{2} (m - 1)\,(2\sqrt{(2 + \sqrt{3})})^{m-1}.$$

3) In this part of the proof we prove that the greatest number of points on the unitary sphere in R^m, $m \geqq 3$, with respect to the Euclidean norm, such that the distance of every two different points is greater than 1, is greater than or equal to $2(2/\sqrt{3})^{m-1}$. Let $x^1, \ldots, x^k$ be points on the unitary sphere such that the distance of every two different points is greater than 1 and there is no point on the unitary sphere which has a distance greater than 1 from each of these points. Such points exist. It suffices to prove that $k \leqq 2(2/\sqrt{3})^{m-1}$.

The points $x^1, x^2, \ldots, x^k$ form a 1-net on the unitary sphere. The sum of the surfaces of the intersections of the unitary sphere with the $U(x^i; 1)$, $i = 1, \ldots, k$, is greater than or equal to the surface of the unitary sphere. Since the plane limiting the body corresponding to this spherical cap (the intersection of the unitary sphere with $U(x^1; 1)$) has the distance from the point 0 equal to $\frac{1}{2}$, we obtain $k\,S(\frac{1}{2}) \geqq \sigma_m$. According to (9),

$$k \geqq \frac{\sigma_m}{\alpha_{m-1}(m - 1) \int_{1/2}^1 (1 - x^2)^{(m-3)/2}\,dx}.$$

Accrording to (10),

$$\sigma_m = 2\alpha_{m-1}(m - 1) \int_0^1 (1 - x^2)^{(m-3)/2}\,dx \geqq$$

$$\geqq 2\alpha_{m-1}(m - 1) \int_0^1 (1 - x)^{(m-3)/2}\,dx = 4\alpha_{m-1}.$$

Thus

$$k \geqq \frac{4}{(m - 1) \int_{1/2}^1 (1 - x^2)^{(m-3)/2}\,dx} = \frac{4}{(m - 1) \int_0^{\pi/3} (\sin^2 x)^{(m-3)/2} \sin x\,dx} \geqq$$

$$\geqq \frac{2}{(m - 1) \int_0^{\pi/3} \sin^{m-2} x \cos x\,dx} = \frac{2}{(m - 1) \int_0^{\sqrt{3}/2} y^{m-2}\,dy} = 2\left(\frac{2}{\sqrt{3}}\right)^{m-1}.$$

4) Now we prove the first part of the theorem. We may identify every norm on R^1 with the maximum norm. Therefore 2 is the smallest admissible constant for every norm on R^1. Since we may identify every Hilbert norm with the Euclidean norm we shall consider the Euclidean norm only. According to the second part of the proof there are points $x^1, \ldots, x^k$ which form a $\sqrt{(2 - \sqrt{3})}$-net on the unitary sphere in R^m, where $m \geqq 3$, such that $k \leqq \frac{1}{2}(2 + \sqrt{(2 + \sqrt{3})})(m - 1)(2\sqrt{(2 + \sqrt{3})})^{m-1}$. Further, there are points $x^1, \ldots, x^k$ which form a $\sqrt{(2 - \sqrt{3})}$-net on the unitary sphere in R^2 such that $k = 6$.

It suffices to prove that k is an admissible constant for the Euclidean norm. Let c be a positive number. Let $K(x, y)$ be a nonnegative function on $R^m \times R^m$ such that $K(x, .)$ is Borel-measurable for each $x \in R^m$ and

$$|x - y| \geqq |z - y| \Rightarrow K(x, y) \leqq c\, K(z, y)\,.$$

Further, let p be a nonzero Radon measure in R^m. Let $x \in R^m$. If $x \in \operatorname{spt} p$ then $K\,p(x) \leqq ck \sup_{y \in \operatorname{spt} p} K\,p(y)$ evidently. Let $x \notin \operatorname{spt} p$.

Put

$$L_i = \left\{ z \in R^m - \{x\};\ \left| \frac{z - x}{|z - x|} - x^i \right| \leqq \sqrt{(2 - \sqrt{3})} \right\} \quad \text{for} \quad i = 1, \ldots, k\,.$$

Then

$$K\,p(x) = \int_{R^m} K(x, y)\, \mathrm{d}p(y) \leqq \sum_{i=1}^{k} \int_{L_i \cap \operatorname{spt} p} K(x, y)\, \mathrm{d}p(y)\,.$$

It suffices to prove that

$$\int_{L_i \cap \operatorname{spt} p} K(x, y)\, \mathrm{d}p(y) \leqq c \sup_{y \in \operatorname{spt} p} K\,p(y)\,.$$

If $L_i \cap \operatorname{spt} p = \emptyset$ then the inequality evidently holds.

Let $L_i \cap \operatorname{spt} p \neq \emptyset$. Then there is $z \in L_i \cap \operatorname{spt} p$ such that $|z - x| \leqq |y - x|$ for each $y \in L_i \cap \operatorname{spt} p$. For every $y \in L_i \cap \operatorname{spt} p$ we have $|(y - x)/|y - x| - x^i| \leqq \leqq \sqrt{(2 - \sqrt{3})}$, i.e., the radiusvector of $(y - x)/|y - x|$ encloses with the radiusvector of the point x^i an angle smaller than or equal to $\pi/6$. Therefore the radiusvector of the point $(y - x)/|y - x|$ encloses with the radiusvector of the point $(z - x) : : |z - x|$ an angle smaller than or equal to $\pi/3$, i.e., $|(z - x)/|z - x| - (y - x) : : |y - x|| \leqq 1$. In the same way as we have proved the inequality (4) from (2), (3) in the proof of Theorem 1, we may prove the analogous inequality. The proof proceeds as that of Theorem 1.

5) Now we prove the second part of the theorem. Since we may identify the Hilbert norm with the Euclidean norm, we shall consider the Euclidean norm only.

According to the third part of the proof there are points $x^1, \ldots, x^k$ on the unitary sphere in R^m, where $m \geqq 3$, such that the distance of every two different points is greater than 1 and $k \geqq 2(2/\sqrt{3})^{m-1}$. Further, there are points $x^1, \ldots, x^k$ on the unitary sphere in R^2 such that the distance of every two different points is greater than 1 and $k = 5$. It suffices to prove that every admissible constant for the Euclidean norm is greater than or equal to k.

For every set A in R^m we define

$$p(A) = \sum_{x^i \in A} 1 .$$

Then p is a Radon measure in R^m, spt $p = \{x^1, \ldots, x^k\}$. Further $c_i = \min_{j \neq i} |x^i - x^j| > 1$, $c = \min_{i=1,\ldots,k} c_i > 1$. Put

$$L(t) = \begin{cases} 1 & \text{for } t \in \langle 0; 1 \rangle , \\ \dfrac{t}{1-c} + \dfrac{c}{c-1} & \text{for } t \in (1; c) , \\ 0 & \text{for } t \in \langle c; \infty) . \end{cases}$$

L is a nonnegative continuous nonincreasing function defined on the interval $\langle 0; \infty)$. Put $K(x, y) = L(|x - y|)$ for each $x, y \in R^m$. Then $K\,p(x^i) = 1$, $K\,p(0) = k$. If M is an admissible constant for the Euclidean norm, then $M \geqq k$.

Remark 1. *We can find a smaller constant than in Theorem* 1 *which is admissible for the Euclidean norm. For example,* 26 *is admissible for the Euclidean norm on* R^3. *We can find a better lower estimate of the admissible constant for the Euclidean norm. For example, every admissible constant for the Euclidean norm in* R^m, *where* $m \geqq 3$, *is greater than or equal to* $4m$, *which is a better lower estimate of the admissible constant for* $m = 3, \ldots, 29$.

Proof. 1) Now we prove the first part of the remark. According to the fourth part of the proof of Theorem 3 it suffices to prove that the smallest number of elements of a $\sqrt{(2 - \sqrt{3})}$-net on the unitary sphere in R^3 is smaller than or equal to 26. We denote by S the unitary sphere. Put

$$A = \{1, -1, 0, 1/\sqrt{2}, -1/\sqrt{2}, 1/\sqrt{3}, -1/\sqrt{3}\}^3 \cap S .$$

Number of elements of A is $2^3 + 2^2 . 3 + 2 . 3 = 26$. We prove that A is a $\sqrt{(2 - \sqrt{3})}$-net on S. Suppose that A is not a $\sqrt{(2 - \sqrt{3})}$-net on S. Then there is $x \in S$ such that $|x - y| > \sqrt{(2 - \sqrt{3})}$ for every $y \in A$. We assume for simplicity that

$$0 \leqq x_1 \leqq x_2 \leqq x_3 . \tag{11}$$

$|x - [0, 0, 1]| > \sqrt{(2 - \sqrt{3})}$. Therefore

$$x_3 < \frac{\sqrt{3}}{2} . \tag{12}$$

$|x - [0, 1/\sqrt{2}, 1/\sqrt{2}]| > \sqrt{(2 - \sqrt{3})}$. Therefore

(13) $$x_2 + x_3 < \sqrt{(\tfrac{3}{2})}\,.$$

$|x - [1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}]| > \sqrt{(2 - \sqrt{3})}$. Therefore

(14) $$x_1 + x_2 + x_3 < \tfrac{3}{2}\,.$$

We have $x_2 = \sqrt{(1 - x_1^2 - x_3^2)}$. If we substitute into (13), we obtain $x_3 + \sqrt{(1 - x_1^2 - x_3^2)} < \sqrt{(\tfrac{3}{2})}$.

Therefore

$$2x_3^2 - \sqrt{6}x_3 + x_1^2 + \tfrac{1}{2} > 0\,.$$

Thus, just one of the following three possibilities occurs:

a) $2 - 8x_1^2 < 0$,

b) $2 - 8x_1^2 \geqq 0$, $x_3 < \dfrac{\sqrt{6} - \sqrt{(2 - 8x_1^2)}}{4}$,

c) $2 - 8x_1^2 \geqq 0$, $x_3 > \dfrac{\sqrt{6} + \sqrt{(2 - 8x_1^2)}}{4}$.

ad a) $x_1 > \tfrac{1}{2}$. According to (11), $x_2 > \tfrac{1}{2}$, $x_3 > \tfrac{1}{2}$. Thus $x_1 + x_2 + x_3 > \tfrac{3}{2}$, which contradicts (14).

ad b) Since $0 \leqq x_1 \leqq x_2 \leqq x_3$ and $x_1^2 + x_2^2 + x_3^2 = 1$, we have $\sqrt{(1 - x_1^2)}/\sqrt{2} \leqq x_3$.

Since

$$x_3 < \frac{\sqrt{2}(\sqrt{3} - \sqrt{(1 - 4x_1^2)})}{4},$$

we have

$$2\sqrt{(1 - x_1^2)} + \sqrt{(1 - 4x_1^2)} < \sqrt{3}\,.$$

We obtain

$$4\sqrt{((1 - x_1^2)(1 - 4x_1^2))} < 8x_1^2 - 2\,,$$

which contradicts the supposition b).

ad c) According to (10), $x_3 < \tfrac{1}{2}\sqrt{3}$. Therefore

$$\frac{\sqrt{6} + \sqrt{(2 - 8x_1^2)}}{4} < \frac{\sqrt{3}}{2}.$$

Thus

(15) $$\sqrt{\left(\frac{3\sqrt{2} - 4}{2}\right)} < x_1 \leqq \tfrac{1}{2}\,.$$

According to (11) and according to the supposition c) we obtain

$$x_1 + x_2 + x_3 > 2x_1 + \frac{\sqrt{6} + \sqrt{(2 - 8x_1^2)}}{4}. \tag{16}$$

We define the function $F(x) = 2x + \frac{1}{4}(\sqrt{6} + \sqrt{(2 - 8x^2)})$ on the interval $\langle\sqrt{(\frac{1}{2}(3\sqrt{2} - 4))}; \frac{1}{2}\rangle$. The function F is increasing on the interval $\sqrt{(\frac{1}{2}(3\sqrt{2} - 4))}; \sqrt{(\frac{2}{9})})$ and decreasing on the interval $(\sqrt{(\frac{2}{9})}; \frac{1}{2}\rangle$. Thus $F(x) \geqq \min\,(F(\sqrt{(\frac{1}{2}(3\sqrt{2}-4))}), F(\frac{1}{2})) > \frac{3}{2}$. According to (15) and according to (16) we obtain $x_1 + x_2 + x_3 > \frac{3}{2}$, which contradicts (14).

2) Now we prove the second part of the remark. According to the fifth part of the proof of Theorem 3 it suffices to prove that there are $4m$ points on the unitary sphere in R^m, $m \geqq 3$, such that the distance of every two different points is greater than 1. Such points are $[1/\sqrt{3}, \sqrt{(\frac{2}{3})}, 0]$, $[-1/\sqrt{3}, \sqrt{(\frac{2}{3})}, 0]$, $[1/\sqrt{3}, -\sqrt{(\frac{2}{3})}, 0]$, $[-1/\sqrt{3}, -\sqrt{(\frac{2}{3})}, 0]$, $[\sqrt{(\frac{2}{3})}, 0, 1/\sqrt{3}]$, $[-\sqrt{(\frac{2}{3})}, 0, 1/\sqrt{3}]$, $[\sqrt{(\frac{2}{3})}, 0, -1/\sqrt{3}]$, $[-\sqrt{(\frac{2}{3})}, 0, -1\sqrt{3}]$, $[0, 1/\sqrt{3}, \sqrt{(\frac{2}{3})}]$, $[0, -1/\sqrt{3}, \sqrt{(\frac{2}{3})}]$, $[0, 1/\sqrt{3}, -\sqrt{(\frac{2}{3})}]$, $[0, -1/\sqrt{3}, -\sqrt{(\frac{2}{3})}]$ in R^3.

Let $x^1, \ldots, x^{4m}$ be points on the unitary sphere in R^m such that the distance of every two different points is greater than 1. We may suppose that $|x_m^i| < 1$ for $i = 1, \ldots, 4m$. Then

$$a = \max_{i=1,\ldots,4m} |x_m^i| < 1.$$

Then there is b such that $\frac{1}{2} < b < 1/\sqrt{2}$ and $ab < \frac{1}{2}$. Put

$$\begin{aligned} y^i &= x^i \times \{0\} \quad \text{for} \quad i = 1, \ldots, 4m, \\ y^{4m+1} &= \{0\}^{m-1} \times [b, \sqrt{(1 - b^2)}], \\ y^{4m+2} &= \{0\}^{m-1} \times [b, -\sqrt{(1 - b^2)}], \\ y^{4m+3} &= \{0\}^{m-1} \times [-b, \sqrt{(1 - b^2)}], \\ y^{4m+4} &= \{0\}^{m-1} \times [-b, -\sqrt{(1 - b^2)}]. \end{aligned}$$

The points y^i, $i = 1, \ldots, 4m + 4$, are elements of unitary sphere in R^{m+1}. Now we prove that $|y^i - y^j| > 1$ for each $i \neq j$. If $i < j \leqq 4m$ then $|y^i - y^j| = |x^i - x^j| > 1$. If $4m < i < j$ then $|y^i - y^j| \geqq 2\min\,(b, \sqrt{(1 - b^2)}) > 1$. If $i \leqq 4m < j$ then $|y^i - y^j|^2 \geqq 2 - 2|bx_m^i| \geqq 2 - 2ab > 1$.

Remark 2. The smallest number of elements of a $\sqrt{(2 - \sqrt{3})}$-net on the unitary sphere with respect to the Euclidean norm is an admissible constant for the Euclidean norm. We can find estimates of the smallest number of elements of a $\sqrt{(2 - \sqrt{3})}$-net on the unitary sphere in [3], [4].

Remark 3. *Suppose that $K(x, y) \geqq 0$ on $R^m \times R^m$ and $K(x, .)$ is Borel measurable on R^m for each $x \in R^m$. The existence of a seminorm q in R^m and a $c > 0$ satisfying* *the implication*

$$q(x - y) \geqq q(z - y) \Rightarrow K(x, y) \leqq c\, K(z, y)$$

is not necessary for the existence of a constant $M > 0$ quaranteeing the validity of the estimate

$$\sup_{x \in R^m} K\, p(x) \leqq M \sup_{x \in \operatorname{spt} p} K\, p(x)$$

for all Radon measures p in R^m.

Proof. Let $x_0 \in R^m - \{0\}$. It suffices to put

$$K(x, y) = 0 \quad \text{for} \quad [x, y] \in R^m \times R^m - \{[0, 0], [x_0, 0]\}\,,$$
$$1 \quad \text{for} \quad [x, y] \in \{[0, 0], [x_0, 0]\}\,.$$

Example 1. On $R^1 \times R^1$ we define

$$K(x, y) = \frac{\sin(x - y) + 2}{|x - y|} \quad \text{for} \quad x \neq y\,,$$
$$+\infty \quad \text{for} \quad x = y\,.$$

K is a nonnegative Borel-measurable function. If $x = y$ then $K(x, y) \geqq K(z, y)$ for each $z \in R^1$. If $x \neq y$, $|x - y| \leqq |z - y|$, then $z \neq y$ as well and

$$K(z, y) = \frac{\sin(z - y) + 2}{|z - y|} \leqq \frac{3}{|z - y|} \leqq \frac{3}{|x - y|} \leqq 3\,\frac{\sin(x - y) + 2}{|x - y|} = 3\,K(x, y)\,.$$

Theorem 2 implies

$$K\, p(x) \leqq 6 \sup_{y \in \operatorname{spt} p} K\, p(y)$$

for each nonzero Radon measure p and for each $x \in R^1$.

Example 2. Let $r, s > 0$. We define on $R^m \times R^m$

$$K(x, y) = \left(\sum_{i=1}^{m} |x_i - y_i|^r\right)^{-s} \quad \text{for} \quad x \neq y\,,$$
$$+\infty \quad \text{for} \quad x = y\,.$$

K is a nonnegative Borel-measurable function. If $r \geqq 1$, we define the norm in R^m by

$$\|x\| = \left(\sum_{i=1}^{m} |x_i - y_i|^r\right)^{1/r}.$$

If $x = y$ then $K(x, y) \geqq K(z, y)$ for each $z \in R^m$. If $x \neq y$ and $\|x - y\| \leqq \|z - y\|$ then $\|x - y\|^{rs} \leqq \|z - y\|^{rs}$ and thus $K(x, y) \geqq K(z, y)$. Theorem 1 yields

$$\sup_{x \in R^m} K\, p(x) \leqq (5^m - 3^m) \sup_{x \in \operatorname{spt} p} K\, p(x)$$

for each nonzero Radon measure p on R^m.

If $0 < r < 1$, we define by $\|x\| = \max_{i=1,\dots,m} |x_i|$ a norm in R^m. If $x = y$ then $K(x, y) \geqq$ $\geqq K(z, y)$ for each z. If $x \neq y$ and $\|x - y\| \leqq \|z - y\|$ then $z \neq y$ and

$$K(x, y) = (\sum_{i=1}^{m} |x_i - y_i|^r)^{-s} \geqq (m\|x - y\|^r)^{-s} \geqq m^{-s}(\|z - y\|^r)^{-s} \geqq$$
$$\geqq m^{-s}(\sum_{i=1}^{m} |z_i - y_i|^r)^{-s} = m^{-s} K(z, y) .$$

Thus $\|x - y\| \leqq \|z - y\| \Rightarrow K(z, y) \leqq m^s K(x, y)$. Theorem 2 yields

$$\sup_{x \in R^m} K\, p(x) \leqq m^s\, 2^m \sup_{x \in \operatorname{spt} p} K\, p(x)$$

for each nonzero Radon measure p in R^m.

This inequality is true for $r \geqq 1$ as well.

References

[1] *S. Kametani, T. Ugaheri:* On a theorem of Evans in the theory of generalized potential. Sugaku *1* (1947), 30—31.

[2] *K. Kunugui:* Étude sur la théorie du potential généralisé. Osaka Math. J., vol. *2* (1950), No. 1, 63—103.

[3] *Г. А. Кабатянский, В. И. Левенштейн:* О границах для упаковок на сфере и в пространстве. Проблемы передачи информации *14* (1978), Но. 1, 3—25.

[4] *В. М. Сиделников:* Новые оценки плотнейшей упаковки шаров в n-мерном эвклидовом пространстве. Математический сборник (1974), *95* 148—158.

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