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INTERCHANGEABILITY OF UNBOUNDED LINEAR OPERATORS: GENERAL THEORY OF TRANSMUTATIVITY

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This paper deals with the notions of interchangeability of general linear, not necessarily everywhere defined or continuous operators from a Banach space into itself. These general linear operators are frequently called unbounded linear operators, even if a little inaccurately.

Since the simple interchangeability of superpositions of these general linear operators, called here and usually commutativity, does not provide an effective notion of interchangeability beyond the system of everywhere defined continuous operators, we introduce a new notion of interchangeability, called here transmutativiy, which has many needed properties and naturally coincides with commutativity for everywhere defined continuous operators. Moreover, it includes various notions of interchangeability introduced more or less accidentally by different technical tools for some special classes of operators.

There are the following sections: 1. Sets, spaces and operators, 2. Basic types of interchangeability: commutativity, submutativity, transmutativity, 3. Transmutativity of general operators, 4. Resolventive operators and their transmutativity, 5. Scalar operators and their transmutativity, 6. Cumulants of operators, 7. Reactive operators and their transmutativity.

Section 1 is of an auxiliary character. Sections 2 and 3 contain the basic definitions and results. Sections 4 and 5 are devoted to some special classes of operators and are independent of each other. Sections 6 and 7 synthetize and generalize some special results from Sections 2, 4 and 5.

1. SETS, SPACES AND OPERATORS

1.1. In the whole paper, E will usually denote an arbitrary Banach space over the complex number field C with the norm $\|\cdot\|$.

For this Banach space E, we denote:

$\mathsf{L}^+(E)$ – the set of all linear operators with the linear nonvacuous domain in E and with the range also in E,

$\mathsf{L}(E)$ – the set of all everywhere defined continuous operators from $\mathsf{L}^+(E)$.

For an operator $A \in \mathsf{L}^+(E)$, we denote by $\mathsf{D}(A)$ its domain and by $\mathsf{R}(A)$ its range. The everywhere defined identity operator in E is denoted by I.

1.2. For $A, B \in \mathsf{L}^+(E)$, we define:

the inclusion $A \subseteq B$: $\mathsf{D}(A) \subseteq \mathsf{D}(B)$ and $Ax = Bx$ for every $x \in \mathsf{D}(A)$;

the sum $A + B$: $\mathsf{D}(A + B) = \mathsf{D}(A) \cap \mathsf{D}(B)$ and $(A + B)x = Ax + Bx$ for every $x \in \mathsf{D}(A) \cap \mathsf{D}(B)$;

the product AB: $\mathsf{D}(AB) = \{x : x \in \mathsf{D}(B) \text{ and } Bx \in \mathsf{D}(A)\}$ and $(AB)x = A(Bx)$ for every $x \in \mathsf{D}(B)$ such that $Bx \in \mathsf{D}(A)$.

For a one-to-one operator $A \in \mathsf{L}^+(E)$ we define:

the inverse operator A^{-1}: $\mathsf{D}(A^{-1}) = \mathsf{R}(A)$ and $AA^{-1}x = x$ for every $x \in \mathsf{R}(A)$.

1.3. For $A \in \mathsf{L}^+(E)$, we define $A^\sim$ as the set of all operators $S \in \mathsf{L}(E)$ such that

(I) $SA \subseteq AS$,

(II) $SG = GS$ for every $G \in \mathsf{L}(E)$ with $GA \subseteq AG$.

1.4. Proposition. *Let $A \in \mathsf{L}^+(E)$. Then the set $A^\sim$ always contains all multiples of the everywhere defined identity operator.*

Proof. Immediately from 1.3.

1.5. Proposition. *Let $A \in \mathsf{L}^+(E)$. If the operator A is everywhere defined continuous, then $A \in A^\sim$.*

Proof. Immediately from 1.3.

1.6. Proposition. *There exist a Banach space E and an operator $A \in \mathsf{L}^+(E)$ such that*

(a) *the operator A is densely defined closed,*

(b) *every operator $G \in \mathsf{L}(E)$ such that $GA \subseteq AG$ is a multiple of the identity operator.*

Proof. Such space and operator have been constructed by Fuglede ([4], Theorem II):

(i) $E = \mathsf{L}_2(-\infty, \infty)$,

(ii) the operator A is defined as follows: $x \in \mathsf{D}(A)$ and $Ax = y$ if and only if $x \in \in L_2(-\infty, \infty)$, $y \in L_2(-\infty, \infty)$, x' exists almost everywhere and $y(\xi) = = \xi\, x(\xi) + x'(\xi)$ for almost every ξ, $-\infty < \xi < \infty$.

2. BASIC TYPES OF INTERCHANGEABILITY: COMMUTATIVITY, SUBMUTATIVITY, TRANSMUTATIVITY

2.1. Let $A, B \in \mathsf{L}^+(E)$. The operators A, B are called *commutative* if $AB = BA$.

2.2. Let $A, B \in \mathsf{L}^+(E)$. The operators A, B are called *submutative* if $AB \subseteq BA$.

2.3. Theorem. *Let $A, B \in \mathsf{L}^+(E)$. Then the following statements* (A) *and* (B) *are equivalent:*

(A) *the operators A, B are commutative;*

(B) *the operators A, B and B, A are submutative.*

2.4. Theorem. *Let $A, B \in \mathsf{L}^+(E)$. If the operators A, B are everywhere defined, then the following statements* (A), (B) *and* (C) *are mutually equivalent:*

(A) *the operators A, B are commutative;*

(B) *the operators A, B are submutative;*

(C) *the operators B, A are submutative.*

2.5. Let $A, B \in \mathsf{L}^+(E)$. The operators A, B are called *transmutative* if $ST = TS$ for every $S, T \in \mathsf{L}(E)$ such that

(I) $SA \subseteq AS$,

(II) $SG = GS$ for every $G \in \mathsf{L}(E)$ with $GA \subseteq AG$,

(III) $TB \subseteq BT$,

(IV) $TH = HT$ for every $H \in \mathsf{L}(E)$ with $HB \subseteq BH$.

2.6. Proposition. *Let $A, B \in \mathsf{L}^+(E)$. Then the following statements* (A) *and* (B) *are equivalent:*

(A) *$ST = TS$ for every $S \in A^{\sim}$ and $T \in B^{\sim}$;*

(B) *the operators A, B are transmutative.*

Proof. An immediate consequence of 1.3 and 2.5.

2.7. Theorem. *Let $A, B \in \mathsf{L}^+(E)$. If*

(α) *the operator A is everywhere defined continuous,*

(β) *the operators A, B are submutative,*

then the operators A, B are transmutative.

Proof. By 1.3 (II) and 2.2 we easily obtain from (α) and (β) that

(1) $AT = TA$ for every $T \in B^{\sim}$.

Using again 1.3 (II), we see from (1) that

(2) $ST = TS$ for every $S \in A^{\sim}$ and $T \in B^{\sim}$.

Then (2) implies the conclusion by virtue of 2.6(A) $\Rightarrow$ (B).

2.8. Theorem. *Let* $A, B \in \mathsf{L}^+(E)$. *If*

(α) *the operator* A *is everywhere defined continuous,*

(β) *the operator* B *is everywhere defined continuous,*

then the following statements (A) *and* (B) *are equivalent:*

(A) *the operators* A, B *are commutative;*

(B) *the operators* A, B *are transmutative.*

Proof. (A) $\Rightarrow$ (B): By 2.3, we see from (A) that the operators A, B are submutative. This together with (α) enables us to apply 2.7 and the implication (A) $\Rightarrow$ (B) follows.

(B) $\Rightarrow$ (A): By 1.5, we see from (α) and (β) that $A \in A^{\sim}$ and $B \in B^{\sim}$. Using these relations, we obtain from (B) by 2.6(B) $\Rightarrow$ (A) that $AB = BA$ which proves the implication (B) $\Rightarrow$ (A) in view of 2.1.

2.9. Note. In the sequel, we frequently and constantly use the notions of commutativity and submutativity, but the terms commutative and submutative, introduced in 2.1 and 2.2, explicitly occur only in some cases. In technical conditions, we usually prefer to write down the corresponding relations in concise form directly by the defining formulas.

2.10. Synopsis of notions of interchangeability. The notion of commutativity of operators from $\mathsf{L}^+(E)$, as defined above in 2.1, is one of the simplest forms possible of intechangeability of general operators from $\mathsf{L}^+(E)$.

It is a formal extension of the notion of interchangeability frequently used for the everywhere defined continuous operators from $\mathsf{L}^+(E)$.

Regrettably, it is necessary to say that the notion of commutativity within the framework of the whole system $\mathsf{L}^+(E)$ is practically without importance since it has almost no reasonable and useful properties beyond the scope of the everywhere defined continuous operators.

The same is true for various weakenings of the notion of commutativity for operators $A, B \in \mathsf{L}^+(E)$ as e.g. $ABx = BAx$ for every $x \in \mathsf{D}(AB) \cap \mathsf{D}(BA)$ or $ABx = BAx$ for every x from a dense subset of $\mathsf{D}(AB) \cap \mathsf{D}(BA)$, or similarly.

The notion of submutativity of operators from $\mathsf{L}^+(E)$, as defined above in 2.2, is sometimes also introduced as an autonomous unit of interchangeability of general operators from $\mathsf{L}^+(E)$ even if it has practically the same disadvantages within the framework of the whole system $\mathsf{L}^+(E)$ as the notion of commutativity and, moreover, it is unsymmetric. Nevertheless, it appears useful in a certain degree provided the first operator of the couple is everywhere defined continuous.

Since both the previous notions of interchangeability – commutativity and submutativity – appear disadvantageous, we introduced in 2.5 a new notion of interchangeability, called here transmutativity. The basic idea of its definition is more intuitively described in 2.6 by means of the commutativity of certain sets of everywhere defined continuous operators from $\mathsf{L}^+(E)$, associated with arbitrary operator from $\mathsf{L}^+(E)$ by 1.3.

In view of 2.1 and 2.2, the above mentioned set defined by 1.3 can be called a ***bisubmutant*** in analogy with the already used term ***bicommutant*** denoting the set which is obtained if the operator inclusions in 1.3 (submutativity in the sense of 2.2) are replaced by the identities (commutativity in the sense of 2.1). The name bisubmutant is more easily to understand if we take the set of all everywhere defined continuous operators submutative with the given operator for the submutant.

Using the notion of bisubmutant, we can say that, by 2.6, the transmutativity of two operators from $\mathsf{L}^+(E)$ means the commutativity of their bisubmutants. In this way, the notion of bisubmutant becomes of basic importance in all our subsequent considerations.

On the contrary, it should be said that the above mentioned notion of bicommutant is of little importance within the framework of the whole system $\mathsf{L}^+(E)$. It is mostly introduced only for everywhere defined continuous operators from $\mathsf{L}^+(E)$ where it coincides with the bisubmutant by 2.4 even if the general definition described above is formally quite possible.

Naturally, the notion of transmutativity must in the first place and unconditionally coincide with the notion of commutativity for everywhere defined continuous operators from $\mathsf{L}^+(E)$, which actually happens as shown above in 2.8.

But unlike the commutativity and submutativity, the transmutativity has many further reasonable and useful properties within the framework of the whole system $\mathsf{L}^+(E)$. The most important of them will be dealt with in the sequel.

Elementary properties of transmutativity are summarized in Section 3.

It is urgent that the notion of transmutativity should coincide with various forms of interchangeability, occasionally introduced by sundry technical tools for some special classes of operators. This coincidence with commutativity was established in the present Section 2 for everywhere defined continuous operators. In Sections 4 and 5, we deal with this problem for further important classes of operators: resolventive and scalar. Moreover, we shall give some other new criteria of transmutativity for these operators in a subsequent paper (Interchangeability of unbounded operators: special criteria of transmutativity, to appear in this Journal).

In Sections 2, 4 and 5 it is also shown that the transmutativity of some special classes of operators, as everywhere defined continuous, resolventive or scalar, is completely determined by the commutativity of certain simpler associated systems of everywhere defined continuous operators than those of bisubmutants. Analogous systems are generally singled out in Section 6 under the name of cumulants of operators and their principal role for transmutativity of a large class of the so-called reactive operators is shown in Section 7.

A *cumulant* of a given operator from $\mathsf{L}^+(E)$ is defined in Section 6 as any set of everywhere defined continuous operators from $\mathsf{L}^+(E)$ which is equipotent with the bisubmutant of this operator with respect to the commutativity. This means that an everywhere defined continuous operator is commutative with every operator from a cumulant of the given operator if and only if it is commutative with every operator from its bisubmutant. It is clear that the bisubmutant of an arbitrary operator from $\mathsf{L}^+(E)$ is always one of its cumulants. Moreover, special operators as everywhere defined continuous, resolventive or scalar ones have simple cumulants which will be indicated.

The *reactivity* of a given operator from $\mathsf{L}^+(E)$, as defined in Section 7, means that any everywhere defined continuous operator commutative with every operator from the bisubmutant of the given operator is submutative with the given operator itself. Two main properties of reactive operators explain our special interest in them. First, the class of reactive operators includes the special classes of everywhere defined continuous, resolventive and scalar operators, even if on the other hand, not every "respectable" (e.g. densely defined closed) operator is reactive. Secondly, the transmutativity of reactive operators is completely determined by the commutativity and submutativity of their arbitrary cumulants.

3. TRANSMUTATIVITY OF GENERAL OPERATORS

3.1. Theorem. *Let $A, B \in \mathsf{L}^+(E)$. Then the following statements* (A) *and* (B) *are equivalent*:

(A) *the operators A, B are transmutative*;

(B) *the operators B, A are transmutative.*

Proof. Obvious from 2.6.

3.2. Lemma. *Let $A \in \mathsf{L}^+(E)$. Then $ST = TS$ for every $S, T \in A^{\sim}$.*

Proof. Evident from 1.3.

3.3. Theorem. *Let $A \in \mathsf{L}^+(E)$. Then the operators A, A are transmutative.*

Proof. An immediate consequence of 2.6 and 3.2.

3.4. Lemma. *Let $A \in \mathsf{L}^+(E)$. Then $(\alpha A)^{\sim} = \alpha A^{\sim}$ for every $\alpha \in \mathrm{C}$, $\alpha \neq 0$.*

Proof. Immediately from 1.3.

3.5. Theorem. *Let $A, B \in \mathsf{L}^+(E)$ and $\alpha, \beta \in \mathsf{C}$.* If $\alpha \neq 0$, $\beta \neq 0$, *then the following statements* (A) *and* (B) *are equivalent*:

(A) *the operators A, B are transmutative*;

(B) *the operators αA, βB are transmutative.*

Proof. An immediate consequence of 2.6 and 3.4.

3.6. Lemma. *Let $A \in \mathsf{L}^+(E)$. Then $(\alpha I + A)^{\sim} = A^{\sim}$ for every $\alpha \in \mathsf{C}$.*

Proof. Immediately from 1.3.

3.7. Theorem. *Let $A, B \in \mathsf{L}^+(E)$ and $\alpha, \beta \in \mathsf{C}$. Then the following statements* (A) *and* (B) *are equivalent*:

(A) *the operators A, B are transmutative*;

(B) *the operators $\alpha I + A$, $\beta I + B$ are transmutative.*

Proof. An immediate consequence of 2.6 and 3.6.

3.8. Sublemma. *Let $A \in \mathsf{L}^+(E)$. If the operator A is one-to-one, then for every $G \in \mathsf{L}^+(E)$, the following statements* (A) *and* (B) *are equivalent*:

(A) $GA \subseteq AG$;

(B) $GA^{-1} \subseteq A^{-1}G$.

Proof. Let us first fix a $G \in \mathsf{L}^+(E)$.

(A) $\Rightarrow$ (B): Then (A) can be written in the form

(1) $y \in \mathsf{D}(G)$, $Gy \in \mathsf{D}(A)$ and $GAy = AGy$ for every $y \in \mathsf{D}(A)$.

Taking $y = A^{-1}x$ for $x \in \mathsf{D}(A^{-1})$, we get from (1) that

(2) $GA^{-1}x \in \mathsf{D}(A)$, $x \in \mathsf{D}(G)$ and $Gx = GAA^{-1}x = AGA^{-1}x$ for every $x \in \mathsf{D}(A^{-1})$.

Now we see from (2) that $Gx \in \mathsf{R}(A) = \mathsf{D}(A^{-1})$ and $A^{-1}Gx = A^{-1}AGA^{-1}x = GA^{-1}x$ for every $x \in \mathsf{D}(A^{-1})$, which is (B).

(B) $\Rightarrow$ (A). In an analogous way as (A) $\Rightarrow$ (B).

3.9. Lemma. *Let $A \in \mathsf{L}^+(E)$. If the operator A is one-to-one, then $A^{\sim} = (A^{-1})^{\sim}$.*

Proof. Immediately from 1.3 and 3.8.

3.10. Theorem. *Let $A, B \in \mathsf{L}^+(E)$. If the operator B is one-to-one, then the following statements* (A) *and* (B) *are equivalent*:

(A) *the operators A, B are transmutative*;

(B) *the operators A, B^{-1} are transmutative.*

Proof. An immediate consequence of 2.6 and 3.9.

3.11. Theorem. *Let $A, B \in \mathsf{L}^+(E)$. If the operators A, B are one-to-one, then the following statements* (A) *and* (B) *are equivalent*:

(A) *the operators A, B are transmutative;*

(B) *the operators A^{-1}, B^{-1} are transmutative.*

Proof. An immediate consequence of 2.6 and 3.9.

3.12. Lemma. *Let $A \in \mathsf{L}^+(E)$. If $GA \subseteq AG$ for every $G \in \mathsf{L}(E)$, then either $\mathsf{D}(A) = \{0\}$ or the operator A is a multiple of the everywhere defined identity operator.*

Proof. We first prove that

(1) for every $x \in \mathsf{D}(A)$, there exists an $\alpha \in \mathrm{C}$ such that $Ax = \alpha x$.

Indeed, since (1) is true if $\mathsf{D}(A) = \{0\}$, we can suppose $\mathsf{D}(A) \neq \{0\}$ and proceed indirectly. Thus, let us admit that there exists an $x_0 \in \mathsf{D}(A)$ such that $x_0 \neq 0$ and $Ax_0 \neq \alpha x_0$ for every $\alpha \in \mathrm{C}$. This implies that $Ax_0 \neq 0$ and that the vectors x_0, Ax_0 are independent. Using the Hahn-Banach Theorem, we fix an $l \in E^*$ such that $l(x_0) = 1$ and $l(Ax_0) = 0$. Now let us define $Gx = l(x)\, x_0$ for $x \in E$. Then clearly $G \in \mathsf{L}(E)$. On the other hand, $x_0 \in \mathsf{D}(AG)$, $AGx_0 = l(x_0)\, Ax_0 = Ax_0 \neq 0$ and $x_0 \in \mathsf{D}(GA)$, $GAx_0 = l(Ax_0)\, x_0 = 0$. But these conclusions contradict the supposition of our lemma and thus, (1) is valid.

Now, we prove that

(2) there exists an $\alpha \in \mathrm{C}$ such that $Ax = \alpha x$ for every $x \in \mathsf{D}(A)$.

Indeed, (2) is clear if $\mathsf{D}(A) = \{0\}$. Further, (2) follows from (1) if $\mathsf{D}(A)$ is a one-dimensional subspace. Hence let us suppose that $\mathsf{D}(A)$ is at least a two-dimensional subspace and that (2) does not hold. Now let us first fix an $x_1 \in \mathsf{D}(A)$ so that $x_1 \neq 0$. By (1), we can find an $\alpha_1 \in \mathrm{C}$ such that $Ax_1 = \alpha_1 x_1$. Since $x_1 \in \mathsf{D}(A)$, $x_1 \neq 0$, $Ax_1 = \alpha_1 x_1$, $\mathsf{D}(A)$ is at least two-dimensional and (2) is not tıue by supposition, we can find an $x_2 \in \mathsf{D}(A)$ so that the vectors x_1, x_2 are independent and $Ax_2 \neq \alpha_1 x_2$. By (1), we can find an $\alpha_2 \in \mathrm{C}$ so that $Ax_2 = \alpha_2 x_2$ and a $\gamma \in \mathrm{C}$ so that $A(x_1 + x_2) = \gamma(x_1 + x_2)$. Consequently, we get $\alpha_1 x_1 + \alpha_2 x_2 = \gamma(x_1 + x_2)$. Since the vectors x_1, x_2 are independent, it follows from this identity that $\alpha_1 = \alpha_2 = \gamma$. Hence $Ax_2 = \alpha_2 x_2 = \alpha_1 x_2$ which contradicts $Ax_2 \neq \alpha_1 x_2$. The statement (2) is proved.

Finally, we prove that

(3) $\mathsf{D}(A) = \{0\}$ or $\mathsf{D}(A) = E$.

Indeed, let $\{0\} \neq \mathsf{D}(A) \neq E$. Let us fix an arbitrary $x_0 \in \mathsf{D}(A)$ such that $x_0 \neq 0$. By the Hahn-Banach Theorem we choose an $l \in E^*$ such that $l(x_0) = 1$. Then we put $Gx = l(x)\,(x_0)$ for $x \in E$. Cleaıly $G \in \mathsf{L}(E)$. Further, $\mathsf{D}(AG) = \{0\}$ and $\mathsf{D}(GA) = \mathsf{D}(A) \neq \{0\}$. But these conclusions contradict the supposition of our lemma and thus, (3) is valid.

The statement of our lemma now follows at once from (2) and (3).

3.13. Theorem. *Let $A \in \mathsf{L}^+(E)$. Then the following statements* (A), (B) *and* (C) *are mutually equivalent*:

(A) (I) *the operator A is everywhere defined continuous,*

(II) *for every $B \in \mathsf{L}(E)$, the operators A, B are commutative*;

(B) (I) *the operator A is everywhere defined continuous,*

(II) *for every $B \in \mathsf{L}^+(E)$, the operators A, B are transmutative*;

(C) *the operator A is a multiple of the everywhere defined identity operator.*

Proof Use 1.3, 2.6 and 3.12.

3.14. Theorem. *Let $A \in \mathsf{L}^+(E)$. Then the following statements* (A), (B) *and* (C) *are mutually equivalent*:

(A) *for every $B \in \mathsf{L}(E)$, the operators A, B are transmutative*;

(B) *for every $B \in \mathsf{L}^+(E)$, the operators A, B are transmutative*;

(C) *every operator $S \in \mathsf{L}(E)$ such that $SA \subseteq AS$ and $SG = GS$ for every $G \in \mathsf{L}(E)$ with $GA \subseteq AG$ is a multiple of the everywhere defined identity operator.*

Proof. Use 1.3, 2.6 and 3.12.

3.15. Proposition. *Let $A \in \mathsf{L}^+(E)$. Then the following statements* (A) *and* (B) *are equivalent*:

(A) *the operator A is a multiple of the everywhere defined identity operator*;

(B) (I) *the operator A is everywhere defined continuous,*

(II) *every operator $S \in \mathsf{L}(E)$ such that $SA \subseteq AS$ and $SG = GS$ for every $G \in \mathsf{L}(E)$ with $GA \subseteq AG$ is a multiple of the everywhere defined identity operator.*

Proof. The implication (A) $\Rightarrow$ (B) follows from 3.12, the implication (B) $\Rightarrow$ (A) is obvious from 1.3 and 1.5.

3.16. Proposition. *There exist a Banach space E and an operator $A \in \mathsf{L}^+(E)$ such that*

(a) *the operator A is densely defined closed,*

(b) $\mathsf{D}(A) \neq E$,

(c) *every operator $S \in \mathsf{L}(E)$ such that $SA \subseteq AS$ and $SG = GS$ for every $G \in \mathsf{L}(E)$ with $GA \subseteq AG$ is a multiple of the everywhere defined identity operator.*

Proof. Immediately from 1.6.

3.17. Theorem. *Let $A, B \in \mathsf{L}^+(E)$. If there exists a set $U \subseteq \mathsf{L}(E)$ such that*

(α) *$ZA \subseteq AZ$ for every $Z \in \mathsf{L}(E)$ such that $ZU = UZ$ for every $U \in U$,*

(β) *$UB \subseteq BU$ for every $U \in \boldsymbol{U}$,*

then the operators A, B are transmutative.

Proof. Let us fix a set $\boldsymbol{U} \subseteq \mathsf{L}(E)$ such that (α) and (β) hold.

By 1.3(II), we see from (β) that

(1) $UT = TU$ for every $U \in \boldsymbol{U}$ and $T \in B^{\sim}$.

Then it follows from (α) and (1) that

(2) $TA \subseteq AT$ for every $T \in B^{\sim}$.

Using again 1.3(II) we get from (2) that

(3) $ST = TS$ for every $S \in A^{\sim}$ and $T \in B^{\sim}$.

But (3) enables us to apply 2.6 and the conclusion follows.

3.18. Theorem. *Let $A, B \in \mathsf{L}^+(E)$. If there exist two sets $\boldsymbol{U}, \boldsymbol{V} \subseteq \mathsf{L}(E)$ such that*

(α) *$ZA \subseteq AZ$ for every $Z \in \mathsf{L}(E)$ such that $ZU = UZ$ for every $U \in \boldsymbol{U}$,*

(β) *$ZB \subseteq BZ$ for every $Z \in \mathsf{L}(E)$ such that $ZV = VZ$ for every $V \in \boldsymbol{V}$,*

(γ) *$UV = VU$ for every $U \in \boldsymbol{U}$ and $V \in \boldsymbol{V}$,*

then the operators A, B are transmutative.

Proof. Let us fix sets $\boldsymbol{U}, \boldsymbol{V} \subseteq \mathsf{L}(E)$ such that (α) – (γ) hold.

It follows from (β) and (γ) that $UB \subseteq BU$ for every $U \in \boldsymbol{U}$.

This property together with (α) enables us to apply 3.17 and the conclusion follows.

3.19. Theorem. *Let $\boldsymbol{U} \subseteq \mathsf{L}(E)$ and $B \in \mathsf{L}^+(E)$. If there exists an operator $A \in \mathsf{L}^+(E)$ such that*

(α) *the operators A, B are transmutative,*

(β_1) *$UA \subseteq AU$ for every $U \in \boldsymbol{U}$,*

(β_2) *$UG = GU$ for every $U \in \boldsymbol{U}$ and $G \in \mathsf{L}(E)$ with $GA \subseteq AG$,*

then $UT = TU$ for every $U \in \boldsymbol{U}$ and $T \in \mathsf{L}(E)$ such that $TB \subseteq BT$ and $TH = HT$ for every $H \in \mathsf{L}(E)$ with $HB \subseteq BH$.

Proof. Let us fix an operator $A \in \mathsf{L}^+(E)$ so that (α), (β_1) and (β_2) hold.

By 1.3, we see from (β_1) and (β_2) that

(1) $\boldsymbol{U} \subseteq A^{\sim}$.

By 2.6, it follows from (α) and (1) that

(2) $UT = TU$ for every $U \in \boldsymbol{U}$ and $T \in B^{\sim}$,

The conclusion follows from (2) by 1.3.

3.20. Theorem. *Let $\mathbf{U}, \mathbf{V} \subseteq \mathsf{L}(E)$. If there exist two operators $A, B \in \mathsf{L}^+(E)$ such that*

(α) *the operators A, B are transmutative,*

(β_1) *$UA \subseteq AU$ for every $U \in \mathbf{U}$,*

(β_2) *$UG = GU$ for every $U \in \mathbf{U}$ and $G \in \mathsf{L}(E)$ with $GA \subseteq AG$,*

(γ_1) *$VB \subseteq BV$ for every $V \in \mathbf{V}$,*

(γ_2) *$VH = HV$ for every $V \in \mathbf{V}$ and $H \in \mathsf{L}(E)$ with $HB \subseteq BH$,*

then $UV = VU$ for every $U \in \mathbf{U}$ and $V \in \mathbf{V}$.

Proof. Let us fix operators $A, B \in \mathsf{L}^+(E)$ so that (α), (β_1), (β_2), (γ_1) and (γ_2) hold. By 1.3, we see from (β_1) (β_2) and (γ_1) (γ_2) that $\mathbf{U} \subseteq A^\sim$ and $\mathbf{V} \subseteq B^\sim$. This property together with (α) implies the conclusion by virtue of 2.6.

3.21. Remark. Theorems 3.17 – 3.20 may appear a little artificial, but they will play an important role in the subsequent sections when studying the transmutativity of some special classes of operators.

4. RESOLVENTIVE OPERATORS AND THEIR TRANSMUTATIVITY

4.1. Let $A \in \mathsf{L}^+(E)$ and $z \in \mathbf{C}$. The number z is called a ***resolvent point of the operator** A* if the operator $zI + A$ is one-to-one and $(zI + A)^{-1} \in \mathsf{L}(E)$.

4.2. Let $A \in \mathsf{L}^+(E)$. The operator A is called ***resolventive*** if there exists a number $z \in \mathbf{C}$ which is a resolvent point of the operator A.

4.3. Lemma. *Let $A \in \mathsf{L}^+(E)$ and $z \in \mathbf{C}$. If the number z is a resolvent point of the operator A, then*

(a) $(zI + A)^{-1} A \subseteq A(zI + A)^{-1}$,

(b) $G(zI + A)^{-1} = (zI + A)^{-1} G$ *for every $G \in \mathsf{L}(E)$ such that $GA \subseteq AG$,*

(c) $ZA \subseteq AZ$ *for every $Z \in \mathsf{L}(E)$ such that $Z(zI + A)^{-1} = (zI + A)^{-1} Z$.*

Proof. First we have

$$(zI + A)^{-1} A = (zI + A)^{-1} (zI + A - zI) \subseteq I - z(zI + A)^{-1} = A(zI + A)^{-1},$$

which proves (a).

Further, let $G \in \mathsf{L}(E)$ and $GA \subseteq AG$. Then $(zI + A) G = zG + AG \supseteq zG + GA = G(zI + A)$. Multiplying this inclusion by $(zI + A)^{-1}$ from the left and right we obtain $G(zI + A)^{-1} \supseteq (zI + A)^{-1} G$, which implies (b).

Finally, let $Z \in \mathsf{L}(E)$ and $Z(zI + A)^{-1} = (zI + A)^{-1} Z$. Multiplying this identity by $zI + A$ from the left and right we get easily that $(zI + A) Z \supseteq Z(zI + A)$, which implies $ZA \subseteq AZ$ as was necessary in order to prove (c).

4.4. Theorem. *Let $A, B \in \mathsf{L}^+(E)$ and $\alpha, \beta \in \mathrm{C}$. If*

(α) *the number α is a resolvent point of the operator A,*

(β) *the number β is a resolvent point of the operator B,*

then the following statements (A) *nad* (B) *are equivalent:*

(A) *the operators $(\alpha I + A)^{-1}$, $(\beta I + B)^{-1}$ are commutative;*

(B) *the operators A, B are transmutative.*

Proof. Let us put $U = \{(\alpha I + A)^{-1}\}$ and $V = \{(\beta I + B)^{-1}\}$.

Then the implication (A) $\Rightarrow$ (B) is an immediate consequence of 2.1, 3.18 and 4.3(c) and the implication (B) $\Rightarrow$ (A) of 2.1, 3.20 and 4.3(a), (b).

4.5. Remark. Theorem 4.4 permits to prove that two semigroups of operators (say strongly continuous, but the same is true for arbitrary regular distribution semigroups) are commutative (in the sense that every operator from the range of the one is commutative with every operator from the range of the other) if and only if their generators are transmutative. It suffices to prove that the above described commutativity of semigroups is equivalent to the commutativity of resolvents of their generators, which again follows from the known relations between a semigroup and the resolvent of its generator. An analogous statement is valid also for cosine operator functions.

5. SCALAR OPERATORS AND THEIR TRANSMUTATIVITY

5.1. By $\mathrm{B}(\mathrm{C})$ we denote the set of all Borel subsets of C. Further, the set of all mappings of $\mathrm{B}(\mathrm{C})$ into $\mathsf{L}(E)$ will be denoted by $\mathrm{B}(\mathrm{C}) \to \mathsf{L}(E)$.

5.2. Let $A \in \mathsf{L}^+(E)$ and $\mathscr{E} \in \mathrm{B}(\mathrm{C}) \to \mathsf{L}(E)$. The function $\mathscr{E}$ is called a *(spectral) resolution of the operator A* if

(I) the function $\mathscr{E}(\cdot)\, x$ is a σ-additive vector-valued measure on $\mathrm{B}(\mathrm{C})$ for every $x \in E$,

(II) $\mathscr{E}(X \cap Y) = \mathscr{E}(X)\,\mathscr{E}(Y)$ for every $X, Y \in \mathrm{B}(\mathrm{C})$,

(III) $\mathscr{E}(\mathrm{C}) = I$,

(IV) $x \in \mathsf{D}(A)$ if and only if the integral $\int_{\mathrm{C}} \sigma \mathscr{E}(\mathrm{d}\sigma)\, x$ exists,

(V) $Ax = \int_{\mathrm{C}} \sigma \mathscr{E}(\mathrm{d}\sigma)\, x$ for every $x \in \mathsf{D}(A)$.

5.3. Let $A \in \mathsf{L}^+(E)$. The operator is called *scalar* if there exists a function $\mathscr{E} \in \mathrm{B}(\mathrm{C}) \to \mathsf{L}(E)$ which is a resolution of the operator A.

5.4. Lemma. *Let $A \in \mathsf{L}^+(E)$ and $\mathscr{E} \in \mathrm{B}(\mathrm{C}) \to \mathsf{L}(E)$. If the function $\mathscr{E}$ is a resolution of the operator A, then*

(a) $\mathscr{E}(X)\, Y \subseteq A\, \mathscr{E}(X)$ *for every* $X \in \mathsf{B}(\mathsf{C})$,

(b) $G\, \mathscr{E}(X) = \mathscr{E}(X)\, G$ *for every* $X \in \mathsf{B}(\mathsf{C})$ *and* $G \in \mathsf{L}(E)$ *such that that* $GA \subseteq AG$,

(c) $ZA \subseteq AZ$ *for every* $Z \in \mathsf{L}(E)$ *such that* $Z\, \mathscr{E}(X) = \mathscr{E}(X)\, Z$ *for every* $X \in \mathsf{B}(\mathsf{C})$.

Proof. The statements (a) and (c) follow easily from the properties 5.2(I)–(V).

The statement (b) is essentially due to Fuglede [3], whose proof for the special case of normal operators can be easily adapted to this general situation. Another proof, based on analytic function theory, is to be found in [2], Part III.

5.5. Theorem. *Let* $A, B \in \mathsf{L}^+(E)$ *and* $\mathscr{E}, \mathscr{F} \in \mathsf{B}(\mathsf{C}) \to \mathsf{L}(E)$. *If*

(α) *the function* $\mathscr{E}$ *is a resolution of the operator* A,

(β) *the function* $\mathscr{F}$ *is a resolution of the operator* B,

then the following statements (A) *and* (B) *are equivalent:*

(A) *the operators* $\mathscr{E}(X)$, $\mathscr{F}(Y)$ *are commutative for every* $X, Y \in \mathsf{B}(\mathsf{C})$;

(B) *the operators* A, B *are transmutative.*

Proof. Let us put $U = \{\mathscr{E}(X) : X \in \mathsf{B}(\mathsf{C})\}$ and $V = \{\mathscr{F}(Y) : Y \in \mathsf{B}(\mathsf{C})\}$.

Then the implication (A) $\Rightarrow$ (B) is an immediate consequence of 2.1, 3.18 and 5.4(c) and the implication (B) $\Rightarrow$ (A) of 2.1, 3.20 and 5.4(a), (b).

5.6. Remark. In connection with our previous consideration, the question can be raised whether it is possible to say something more about the "inner" structure of bisubmutants of scalar operators. In general, this structure is not transparent, but in the case of scalar operators, we can attempt to exploit the very special character of these operators. Unfortunately, an exhausting answer is not known to us but some information can be given, especially for Hilbert spaces.

First we can ask for scalarity (normality) of elements of the bisubmutant of a given scalar (normal) operator. We have

Proposition. Let $A \in \mathsf{L}^+(E)$. *If*

(α) *the space E is a Hilbert space,*

(β) *the operator A is scalar* (*normal*),

then the elements of $A^{\sim}$ *are scalar* (*normal*) *operators.*

Proof. Thanks to the Wermer Theorem ([2], XV. 6.4) we can consider the normal operators only.

Thus, let A be a normal operator.

It is well-known that $G^*A \subseteq AG^*$ for every $G \in \mathsf{L}(E)$ for which $GA \subseteq AG$.

Let now $S \in A^{\sim}$. Using the previous property, we get from 1.3(II) that $SG^* = G^*S$ for every $G \in \mathsf{L}(E)$ with $GA \subseteq AG$. This implies $S^*G = GS^*$. Using again 1.3(II), we see that $S^* \in A^{\sim}$. Now by 1.3(I) and 1.3(II), $S^*S = SS^*$ which proves the normality of S.

Further, we can seek for an analytical description of elements of the bisubmutant of a given scalar operator. As a tool for such an analytical description we can use the notion of bounded Borel functions of a given scalar operator in the sense of the operational calculus developed e.g. in [2], XVIII. In this direction we have the following results.

PROPOSITION. *Let $A \in \mathsf{L}^+(E)$. If the operator A is scalar, then every bounded Borel function of A belongs to $A^{\sim}$.*

Proof. Easy from the above mentioned operational calculus.

PROPOSITION. *Let $A \in \mathsf{L}^+(E)$. If*

(α) *the space E is a separable Hilbert space,*

(β) *the operator A is scalar,*

then every element of $A^{\sim}$ is a bounded Borel function of the operator A.

Proof. Use the Wermer Theorem ([2], XV. 6.4) and the von Neumann Theorem ([1], p. 64).

PROPOSITION. *The supposition of separability in the previous proposition is indispensable.*

Proof. See Nakano's example ([1], p. 65).

PROPOSITION. *Let $A \in \mathsf{L}^+(E)$. If*

(α) *the space E is a Hilbert space,*

(β) *the operator A is scalar,*

then for every operator $T \in \mathsf{L}(E)$, the following statements (A) *and* (B) *are equivalent*:

(A) $T \in A^{\sim}$,

(B) *there exists a generalized sequence $T_\alpha \in \mathsf{L}(E)$ such that*

(I) $T_\alpha(x) \to T(x)$ *for every x from a dense subset of E,*

(II) *the sequence T_α is uniformly bounded,*

(III) *every operator T_α is a bounded Borel function of the operator A.*

Proof. Use the Wermer Theorem ([2], XV.6.4) and the von Neumann and Segal Theorems ([4], 12.1 and 12.2.3).

6. CUMULANTS OF OPERATORS

6.1. Let $A \in \mathsf{L}^+(E)$ and $U \subseteq \mathsf{L}(E)$. The set U is called a *cumulant* of the operator A if

(I) for every $Z \in \mathsf{L}(E)$ such that $ZU = UZ$ for every $U \in \boldsymbol{U}$, we have $ZS = SZ$ for every $S \in \mathsf{L}(E)$ satisfying $SA \subseteq AS$ and $SG = GS$ for every $G \in \mathsf{L}(E)$ with $GA \subseteq AG$,

(II) for every $Z \in \mathsf{L}(E)$ such that $ZS = SZ$ for every $S \in \mathsf{L}(E)$ satisfying $SA \subseteq AS$ and $SG = GS$ for every $G \in \mathsf{L}(E)$ with $GA \subseteq AG$, we have $ZU = UZ$ for every $U \in \boldsymbol{U}$.

6.2. Lemma. *Let $A \in \mathsf{L}^+(E)$ and $\boldsymbol{U} \subseteq \mathsf{L}(E)$. Then the following statements (A) and (B) are equivalent:*

(A) *the set $\boldsymbol{U}$ is a cumulant of the operator A;*

(B) (I) *for every $Z \in \mathsf{L}(E)$ such that $ZU = UZ$ for every $U \in \boldsymbol{U}$, we have $ZS = SZ$ for every $S \in A^\sim$,*

(II) *for every $Z \in \mathsf{L}(E)$ such that $ZS = SZ$ for every $S \in A^\sim$, we have $ZU = UZ$ for every $U \in \boldsymbol{U}$.*

Proof. Immediately from 1.3 and 6.1.

6.3. Theorem. *Let $A \in \mathsf{L}^+(E)$. Then there exists at least one cumulant of the operator A. Moreover, this cumulant can be chosen nonvacuous.*

Proof. The set $A^\sim$ is always a cumulant of the operator A as it is trivially seen from 1.3 (II) and 6.2 (B) $\Rightarrow$ (A).

The fact that the set $A^\sim$ is nonvacuous is shown in 1.4.

6.4. Proposition. *Let $A \in \mathsf{L}^+(E)$. Then the following statements (A) and (B) are equivalent:*

(A) *the vacuous set is a cumulant of the operator A;*

(B) *every operator $S \in \mathsf{L}(E)$ such that $SA \subseteq AS$ and $SG = GS$ for every $G \in \mathsf{L}(E)$ with $GA \subseteq AG$ is a multiple of the everywhere defined identity operator.*

Proof. (A) $\Rightarrow$ (B): Use 3.14.

(B) $\Rightarrow$ (A): Obvious.

6.5. Lemma. *Let $A \in \mathsf{L}^+(E)$ and $\boldsymbol{U} \subseteq \mathsf{L}(E)$. If*

(α) *$UA \subseteq AU$ for every $U \in \boldsymbol{U}$,*

(β) *$UG = GU$ for every $U \in \boldsymbol{U}$ and $G \in \mathsf{L}(E)$ with $GA \subseteq AG$,*

(γ) *$ZA \subseteq AZ$ for every $Z \in \mathsf{L}(E)$ such that $ZU = UZ$ for every $U \in \boldsymbol{U}$,*

then the set $\boldsymbol{U}$ is a cumulant of the operator A.

Proof. By 1.3 (II), it follows from (γ) that

(1) the condition 6.2(B) (I) holds.

By 1.3, it follows from (α) and (β) that

(2) $U \subseteq A^{\sim}$.

Then it is immediate from (2) that

(3) the condition 6.2(B) (II) holds.

Now the desired statement follows from (1) and (3) by 6.2(B) $\Rightarrow$ (A).

6.6. Theorem. *Let $A \in \mathsf{L}^+(E)$. If the operator A is everywhere defined continuous, then the set $\{A\}$ is a cumulant of the operator A.*

Proof. Immediately from 6.5.

6.7. Theorem. *Let $A \in \mathsf{L}^+(E)$. If the operator A is resolventive, then the set $\{(zI + A)^{-1}\}$ is a cumulant of the operator A for every $z \in \mathrm{C}$ which is a resolvent point of the operator A.*

Proof. Immediately from 4.3 and 6.5.

6.8. Theorem. *Let $A \in \mathsf{L}^+(E)$. If the operator A is scalar, then the set $\{\mathscr{E}(X) : X \in \in \mathrm{B}(\mathrm{C})\}$ is a cumulant of the operator A for every $\mathscr{E} \in \mathrm{B}(\mathrm{C}) \to \mathsf{L}(E)$ which is a resolution of the operator A.*

Proof. Immediately from 5.4 and 6.5.

7. REACTIVE OPERATORS AND THEIR TRANSMUTATIVITY

7.1. Let $A \in \mathsf{L}^+(E)$. The operator A is called **reactive** if $ZA \subseteq AZ$ for every $Z \in \mathsf{L}(E)$ such that $ZS = SZ$ for every $S \in \mathsf{L}(E)$ satisfying $SA \subseteq AS$ and $SG = GS$ for every $G \in \mathsf{L}(E)$ with $GA \subseteq AG$.

7.2. Lemma. *Let $A \in \mathsf{L}^+(E)$. Then the following statements* (A) *and* (B) *are equivalent*:

(A) *the operator A is reactive;*

(B) *$ZA \subseteq AZ$ for every $Z \in \mathsf{L}(E)$ such that $ZS = SZ$ for every $S \in A^{\sim}$.*

Proof. *Immediately from* 1.3 *and* 7.1.

7.3. Theorem. *Let $A \in \mathsf{L}^+(E)$. If the operator A is reactive, then*

($*$) *$ZA \subseteq AZ$ for every $Z \in \mathsf{L}(E)$ such that $ZG = GZ$ for every $G \in \mathsf{L}(E)$ with $GA \subseteq \subseteq AG$.*

Proof. By 7.2 (A) $\Rightarrow$ (B), we obtain from the assumption that

(1) $ZA \subseteq AZ$ for every $Z \in \mathsf{L}(E)$ such that $ZS = SZ$ for every $S \in A^{\sim}$.

It follows from 1.3(II) that

(2) for every $Z \in \mathsf{L}(E)$ such that $ZG = GZ$ foı every $G \in \mathsf{L}(E)$ with $GA \subseteq AG$, we have $ZS = SZ$ for every $S \in A^{\sim}$.

Now the statement of our theorem is an immediate consequence of (1) and (2).

7.4. Proposition. *There exist a Banach space E and an operator $A \in \mathsf{L}^{+}(E)$ such that*

(a) *the operator A is densely defined closed,*

(b) *the operator A is not reactive.*

Proof. Let us consider the space E and the operator $A \in \mathsf{L}^{+}(E)$ from 1.6.

The statement (a) is identical with 1.6(a).

Using 1.6(b) and 1.3(II), we easily obtain that

(1) $A^{\sim}$ is the set of all multiples of the everywhere defined identity operator.

On the other hand, it follows from the Hahn-Banach Theorem that there exists an operator Z such that

(2) $Z \in \mathsf{L}(E)$,

(3) Z is not a multiple of the everywhere defined identity operator.

It is obvious from (1) and (2) that

(4) $ZS = SZ$ for every $S \in A^{\sim}$.

To conclude the proof, suppose that the operator A is reactive. Then by 7.2(A) $\Rightarrow$ $\Rightarrow$ (B), we obtain from (2) and (4) that $ZA \subseteq AZ$. But this leads to a contradiction with (2) and (3) in view of 1.6(b). Hence the statement (b) holds.

7.5. Lemma. *Let $A \in \mathsf{L}^{+}(E)$. If there exists a set $\boldsymbol{U} \subseteq \mathsf{L}(E)$ such that*

(α) *$UA \subseteq AU$ for every $U \in \boldsymbol{U}$,*

(β) *$UG = GU$ for every $U \in \boldsymbol{U}$ and $G \in \mathsf{L}(E)$ with $GA \subseteq AG$,*

(γ) *$ZA \subseteq AZ$ for every $Z \in \mathsf{L}(E)$ such that $ZU = UZ$ for every $U \in \boldsymbol{U}$,*

then the operator A is reactive.

Proof. Let a set $\boldsymbol{U} \subseteq \mathsf{L}(E)$ with the properties (α), (β) and (γ) be fixed.

By 1.3, it follows from (α) and (β) that

(1) $\boldsymbol{U} \subseteq A^{\sim}$.

Then it follows from (1) and (γ) that

(2) the condition 7.2(B) holds.

Now the desired statement is obtained from (2) by use of 7.2.

7.6. Theorem. *Let $A \in \mathsf{L}^+(E)$. If the operator A is everywhere defined continuous, then it is reactive.*

Proof. Immediately from 7.5.

7.7. Theorem. *Let $A \in \mathsf{L}^+(E)$. If the operator A is resolventive, then it is reactive.*

Proof. Immediately from 4.3 and 7.5.

7.8. Theorem. *Let $A \in \mathsf{L}^+(E)$. If the operator A is scalar, then it is reactive.*

Proof. Immediately from 5.4 and 7.5.

7.9. Theorem. *Let $A, B \in \mathsf{L}^+(E)$. If*

(α) *the operator A is everywhere defined continuous,*

(β) *the operator B is reactive,*

then the following statements (A) *and* (B) *are equivalent:*

(A) *the operators A, B are submutative;*

(B) *the operators A, B are transmutative.*

Proof. (A) $\Rightarrow$ (B): A special case of 2.7.

(B) $\Rightarrow$ (A): By 1.5 we obtain from (α) that

(1) $A \in A^{\sim}$.

By 2.6, it follows from (B) and (1) that

(2) $AT = TA$ for every $T \in B^{\sim}$.

Now by 7.2(B) $\Rightarrow$ (A), the conditions (β) and (2) imply (A).

7.10. Lemma. *Let $A \in \mathsf{L}^+(E)$ and $\mathbf{U} \subseteq \mathsf{L}(E)$. If*

(α) *the operator A is reactive,*

(β) *the set $\mathbf{U}$ is a cumulant of the operator A,*

then

(a) *$UA \subseteq AU$ for every $U \in \mathbf{U}$,*

(b) *$UG = GU$ for every $U \in \mathbf{U}$ and $G \in \mathsf{L}(E)$ with $GA \subseteq AG$,*

(c) *$ZA \subseteq AZ$ for every $Z \in \mathsf{L}(E)$ such that $ZU = UZ$ for every $U \in \mathbf{U}$.*

Proof. By 7.2(A) $\Rightarrow$ (B), we obtain from (α) that

(1) $ZA \subseteq AZ$ for every $Z \in \mathsf{L}(E)$ such that $ZS = SZ$ for every $S \in A^{\sim}$.

Further, by 6.2(A) $\Rightarrow$ (B) we obtain from (β) that

(2) for every $Z \in \mathsf{L}(E)$ such that $ZU = UZ$ for every $U \in \mathbf{U}$, we have $ZS = SZ$ for every $S \in A^{\sim}$,

(3) for every $Z \in \mathsf{L}(E)$ such that $ZS = SZ$ for every $S \in A^{\sim}$ we have $ZU = UZ$ for every $U \in \boldsymbol{U}$.

Moreover, it is immediate from 1.3 that

(4) $S_1S_2 = S_2S_1$ for every $S_1, S_2 \in A^{\sim}$.

Then (3) and (4) imply

(5) $US = SU$ for every $U \in \boldsymbol{U}$ and $S \in A^{\sim}$.

On the other hand, we have by 1.3(II) that

(6) $SG = GS$ for every $S \in A^{\sim}$ and $G \in \mathsf{L}(E)$ with $GA \subseteq AG$.

Now it is easy to see that (1) and (5) imply (a), (3) and (6) imply (b) and (1) and (2) imply (c). The proof is complete.

7.11. Theorem. *Let $A, B \in \mathsf{L}^+(E)$. If*

(α) *the operator A is reactive,*

(β) *there exists a set $\boldsymbol{U} \subseteq \mathsf{L}(E)$ such that*

  (i) *$\boldsymbol{U}$ is a cumulant of the operator A,*

  (ii) *the operators U, B are submutative for every $U \in \boldsymbol{U}$,*

then the operators A, B are transmutative.

Proof. An immediate consequence of 2.2, 3.17 and 7.10 (by virtue of the property 7.10(c)).

7.12. Theorem. *Let $A, B \in \mathsf{L}^+(E)$ and $\boldsymbol{U} \subseteq \mathsf{L}(E)$. If*

(α_1) *the operator A is reactive,*

(α_2) *the set $\boldsymbol{U}$ is a cumulant of the operator A,*

(β) *the operator B is reactive,*

then the following statements (A) *and* (B) *are equivalent:*

(A) *the operators U, B are submutative for every $U \in \boldsymbol{U}$;*

(B) *the operators A, B are transmutative.*

Proof. (A) ⇒ (B): An immediate consequence of 7.11.

(B) ⇒ (A): An immediate consequence of 2.2, 3.19 and 7.10 (by virtue of the properties 7.10(a), (b)).

7.13. Theorem. *Let $A, B \in \mathsf{L}^+(E)$ and $\boldsymbol{U}, \boldsymbol{V} \subseteq \mathsf{L}(E)$. If*

(α_1) *the operator A is reactive,*

(α_2) *the set $\boldsymbol{U}$ is a cumulant of the operator A,*

(β_1) *the operator B is reactive,*

(β_2) *the set V is a cumulant of the operator B,*

then the following statements (A) *and* (B) *are equivalent*:

(A) *the operators U, V are commutative for every $U \in \mathsf{U}$ and $V \in \mathsf{V}$;*

(B) *the operators A, B are transmutative.*

Proof. (A) $\Rightarrow$ (B): An immedaite consequence of 2.1, 3.18 and 7.10 (by virtue of the property 7.10(c)).

(B) $\Rightarrow$ (A): An immedaite consequence of 2.1, 3.20 and 7.10 (by virtue of the properties 7.10(a), (b)).

7.14. Note. Theorems 7.11, 7.12 and 7.13 can also serve as skeleton theorems for a series of transmutativity criteria if we specify one or both of the operators A, B to everywhere defined continuous, resolventive or scalar operators as is possible by 6.6–6.8 and 7.6–7.8.

Some of these special cases were independently proved earlier in 2.7, 2.8, 4.4, 5.5 and 7.9. Explicit formulations of further special cases are immediate and will be left to the reader. Naturally, the proofs of these further special cases can be based directly on 3.17–3.20 as well.

7.15. Proposition. *Let $A \in \mathsf{L}^+(E)$ and $\mathsf{U} \subseteq \mathsf{L}(E)$. Then the conditions* 7.10 (α) *and* (β) *are equivalent to the conditions* 7.10 (a), (b) *and* (c).

Proof. An immediate consequence of 6.5, 7.5 and 7.10.

7.16. Remark. By Theorem 7.3, we can say that for reactive operators, the condition 1.3(II) implies the condition 1.3(I).

7.17. Remark. The converse of Theorem 7.3 seems improbable, but we do not know any non-reactive operator with the property 7.3($*$).

7.18. Remark. It is easy to see from 1.6 that there exist a Banach space E and an operator $A \in \mathsf{L}^+(E)$ which is densely defined closed, but has not even the property 7.3 ($*$) (weaker than reactivity). Cf. 7.4.

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