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REMARKS ON HETEROGENEOUS ALGEBRAS

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1. INTRODUCTION

The aim of this paper is to pick out some new applications of heterogeneous algebras. Particularly, the system of all blocks of a compatible relation on a complete algebra may be considered a heterogeneous algebra of the so called full type. We prove that to any heterogeneous algebra $\mathfrak{A}$ a heterogeneous algebra $\mathfrak{B}$ of full type can be constructed on a complete algebra in such a way that the subalgebras of $\mathfrak{A}$ and $\mathfrak{B}$ are in a close connection. A special case of this construction is the well-known construction of a deterministic acceptor to a given acceptor in such a way that both accept the same language.

2. HETEROGENEOUS ALGEBRAS

In what follows, we recall some well-known definitions; only new definitions and results are numbered.

Let $I \neq \emptyset$ be a set. Suppose that a set A_i is assigned to any i in I. Then we say that the function A is an *indexed family of sets of type* I; it is denoted by $(A_i)_{i\in I}$. If i, j in I and $i \neq j$ imply $A_i \neq A_j$, we say that $(A_i)_{i\in I}$ is an *indexed family of mutually different sets* (of type I). An indexed family $(B_i)_{i\in I}$ of type I is said to be a *subfamily* of $(A_i)_{i\in I}$ if $B_i \subseteq A_i$ for any i in I. The set of all subfamilies of a family is a complete lattice if the relation "is less than or equal to" is understood as "is a subfamily of".

In what follows, we often omit the expression "indexed" if it is clear from the context.

Let $A^{(k)} = (A_i^{(k)})_{i\in I}$ be an indexed subfamily of a fixed family of type I for any $k \in K$ and let $(A^{(k)})_{k\in K}$ be a family of these subfamilies. Then $A^{(0)} = (A_i^{(0)})_{i\in I}$ is the greatest lower bound of $(A^{(k)})_{k\in K}$ in the above mentioned complete lattice if and only if $A_i^{(0)} = \bigcap_{k\in K} A_i^{(k)}$ for every $i \in I$.

We identify ordered n-tuples with words of length n so that relations are considered as sets of words.

1. Definition. Let $I \neq \emptyset$, $T \neq \emptyset$ be sets, a a function of T into the set of nonnegative integers, and ω a function of T into the set of all relations on I such that, for any $t \in T$, $\omega(t)$ is a relation on I of arity $a(t) + 1$. Then the ordered quadruple (I, T, a, ω) is said to be a *heterogeneous algebra type.*

2. Definition. Let (I, T, a, ω) be a heterogeneous algebra type, $(A_i)_{i \in I}$ an indexed family of sets of type I. Then the ordered triple $\mathfrak{A} = ((A_i)_{i \in I}, (f_t^{\mathfrak{A}})_{t \in T}, \omega)$ is said to be a *heterogeneous algebra of type* (I, T, a, ω) whenever the following condition is satisfied. For any $t \in T$, $f_t^{\mathfrak{A}}$ is a partial operation on the set $\bigcup_{i \in I} A_i$ of arity $a(t)$ such that for any $i(0), i(1), \ldots, i(a(t))$ in I with the property $i(0)\, i(1) \ldots i(a(t)) \in \omega(t)$, $f_t^{\mathfrak{A}}$ maps the set $A_{i(1)} \times A_{i(2)} \times \ldots \times A_{i(a(t))}$ into $A_{i(0)}$.

The sets A_i $(i \in I)$ are called *components* (or *phyla*) of $\mathfrak{A}$.

Remarks. (1) If $a(t) = 0$, then $f_t^{\mathfrak{A}}$ is a constant such that $f_t^{\mathfrak{A}} \in A_i$ for any $i \in \omega(t)$.

(2) If $a(t) > 0$ and $i(1), \ldots, i(a(t))$ in I are such that $i(0)\, i(1) \ldots i(a(t)) \notin \omega(t)$ for every $i(0)$ in I, then the operation $f_t^{\mathfrak{A}}$ may be defined in some points of $A_{i(1)} \times \times A_{i(2)} \times \ldots \times A_{i(a(t))}$, and need not be defined in others. The defined values are not subjected to any condition; for instance, they may be in different components.

(3) In [1], $\omega(t)$ is supposed to contain exactly one element for every $t \in T$; in [8] only mutually disjoint sets A_i $(i \in I)$ are admitted. We dispense with these restrictions.

Let $\mathfrak{A} = ((A_i)_{i \in I}, (f_t^{\mathfrak{A}})_{t \in T}, \omega)$ be a heterogeneous algebra, $(B_i)_{i \in I}$ a subfamily of the family $(A_i)_{i \in I}$. The family $(B_i)_{i \in I}$ is said to be *closed* in $\mathfrak{A}$ if the following condition is satisfied. For any t in T, for arbitrary $i(0), i(1), \ldots, i(a(t))$ in I with the property $i(0)\, i(1) \ldots i(a(t)) \in \omega(t)$, and for arbitrary $x_1 \in B_{i(1)}, \ldots, x_{a(t)} \in B_{i(a(t))}$ the assertion $f_t^{\mathfrak{A}}(x_1, \ldots, x_{a(t)}) \in B_{i(0)}$ holds. For a t in T with $a(t) = 0$, this means that $f_t^{\mathfrak{A}} \in B_i$ for every i in $\omega(t)$.

Let $\mathfrak{A} = ((A_i)_{i \in I}, (f_t^{\mathfrak{A}})_{t \in T}, \omega)$ be a heterogeneous algebra, $(B_i)_{i \in I}$ a closed family in A. We put $B = \bigcup_{i \in I} B_i$ and $f_t^{\mathfrak{B}} = f_t^A \cap B^{a(t)+1}$ for any $t \in T$. Then, for any $i(0)$, $i(1), \ldots, i(a(t))$ in I with $i(0)\, i(1) \ldots i(a(t)) \in \omega(t)$ and any $x_1 \in B_{i(1)}, \ldots, x_{a(t)} \in \in B_{i(a(t))}$, we have $f_t^{\mathfrak{A}}(x_1, \ldots, x_{a(t)})\, x_1 \ldots x_{a(t)} \in f_t^{\mathfrak{A}} \cap (B_{i(0)} \times B_{i(1)} \times \ldots \times B_{i(a(t))}) \subseteq \subseteq f_t^{\mathfrak{A}} \cap B^{a(t)+1} = f_t^{\mathfrak{B}}$ which implies that $f_t^{\mathfrak{B}}(x_1, \ldots, x_{a(t)}) = f_t^{\mathfrak{A}}(x_1, \ldots, x_{a(t)})$. We put $\mathfrak{B} = ((B_i)_{i \in I}, (f_t^{\mathfrak{B}})_{t \in T}, \omega)$; then $\mathfrak{B}$ is a heterogeneous algebra which is called a *subalgebra* of $\mathfrak{A}$.

It is clear that the greatest lower bound of a nonempty family of closed subfamilies in a heterogeneous algebra $\mathfrak{A}$ is closed in $\mathfrak{A}$. It follows that for any heterogeneous algebra $\mathfrak{A} = ((A_i)_{i \in I}, (f_t^{\mathfrak{A}})_{t \in T}, \omega)$ and for any subfamily $(C_i)_{i \in I}$ of the family $(A_i)_{i \in I}$, there exists a least family $(B_i)_{i \in I}$ closed in $\mathfrak{A}$ such that $(C_i)_{i \in I}$ is a subfamily of $(B_i)_{i \in I}$. The subalgebra $\mathfrak{B} = ((B_i)_{i \in I}, (f_t^{\mathfrak{B}})_{t \in T}, \omega)$ is said to be *generated* by the family $(C_i)_{i \in I}$. Particularly, if $C_i = \emptyset$ for any $i \in I$, then $\mathfrak{B}$ is the *least subalgebra* and $(B_i)_{i \in I}$ the *least closed family* in $\mathfrak{A}$.

3. Definition. Let $T \neq \emptyset$ be a set and a a mapping of T into the set of nonnegative integers. Then the ordered pair (T, a) is said to be a *complete algebra type.*

4. Definition. If (T, a) is a complete algebra type and A a set, then the ordered pair $(A, (f_t)_{t\in T})$ is said to be a *complete algebra of type* (T, a) provided that f_t is a complete operation on A of arity $a(t)$ for any $t \in T$.

A complete algebra can be converted into a heterogeneous algebra in various ways. We describe some of them.

5. Definition. Let $\tau = (T, a)$ be a complete algebra type, $\sigma = (I, T, a', \omega)$ a heterogeneous algebra type. Then σ is said to be *admissible* to τ if $a' = a$.

6. Definition. Let $\mathfrak{H} = (A, (f_t^{\mathfrak{H}})_{t\in T})$ be a complete algebra of type $\tau = (T, a)$, $\sigma = (I, T, a, \omega)$ a heterogeneous algebra type admissible to τ. We put $A_i = A$ for every $i \in I$, $f_t^{\mathfrak{A}} = f_t^{\mathfrak{H}}$ for any $t \in T$, $\mathfrak{A} = ((A_i)_{i\in I}, (f_t^{\mathfrak{A}})_{t\in T}, \omega)$. Then $\mathfrak{A}$ is said to be a *heterogeneous algebra of type* σ *on* $\mathfrak{H}$.

By a *heterogeneous algebra* $\mathfrak{A}$ *on a complete algebra* $\mathfrak{H}$ we mean a heterogeneous algebra $\mathfrak{A}$ on $\mathfrak{H}$ of a type admissible to the type of $\mathfrak{H}$.

7. Definition. A heterogeneous algebra type $\sigma = (I, T, a, \omega)$ is said to be *trivial* if I has exactly one element, say 0, and $\omega(t) = \{0^{a(t)+1}\}$ for every $t \in T$.

Various heterogeneous algebras on the same complete algebra $\mathfrak{H}$ define various families of subalgebras. The family of subalgebras corresponding to the heterogeneous algebra of trivial type on $\mathfrak{H}$ coincides with the family of subalgebras of $\mathfrak{H}$ in the usual sense.

We have seen that a complete algebra $\mathfrak{H}$ defines various heterogeneous algebras on $\mathfrak{H}$ that determine various subalgebras. Conversely:

8. Proposition. *For any heterogeneous algebra* $\mathfrak{A} = ((A_i)_{i\in I}, (f_t^{\mathfrak{A}})_{t\in T}, \omega)$ *of type* (I, T, a, ω) *there exists a complete algebra* $\mathfrak{H} = (A, (f_t^{\mathfrak{H}})_{t\in T})$ *such that* $\mathfrak{A}$ *is a subalgebra of the heterogeneous algebra of type* (I, T, a, ω) *on* $\mathfrak{H}$.

Proof. We take an element ∞ not in $\bigcup_{i\in I} A_i$ and we put $A = \bigcup_{i\in I} A_i \cup \{\infty\}$. For any $t \in T$ and any $x_1, \ldots, x_{a(t)}$ in A, we define

$$f_t^{\mathfrak{H}}(x_1, \ldots, x_{a(t)}) = \begin{cases} f_t^{\mathfrak{A}}(x_1, \ldots, x_{a(t)}) & \text{if } f_t^{\mathfrak{A}}(x_1, \ldots, x_{a(t)}) \text{ is defined}, \\ \infty & \text{otherwise}. \end{cases}$$

Clearly, $\mathfrak{H} = (A, (f_t^{\mathfrak{H}})_{t\in T})$ is a complete algebra. If $f_t^{\mathfrak{C}} = f_t^{\mathfrak{H}}$ for every $t \in T$, provided $C_i = A$ for every $i \in I$, then $\mathfrak{C} = ((C_i)_{i\in I}, (f_t^{\mathfrak{C}})_{t\in T}, \omega)$ is a heterogeneous algebra of type (I, T, a, ω) on $\mathfrak{H}$ and $\mathfrak{A}$ is its subalgebra. □

3. ACCEPTOR AS AN EXAMPLE OF A HETEROGENEOUS ALGEBRA

An *acceptor* is an ordered quadruple $\mathfrak{N} = (S, V, f, J)$ where S, V are sets, J a subset of S, and f a function of the set $S \times V$ into 2^S. The elements in S are called *states*, the elements in J *initial states*, the elements in V are said to be *letters*, f is called the *transition function*. We denote by V^* the set of all words over V, by Λ the empty word. The binary operation of *catenation* is defined on the set V^*. The catenation of x, y in V^* is denoted by xy.

Let $\mathfrak{N} = (S, V, f, J)$ be an acceptor, $n \geqq 0$ an integer, $v_1, \ldots, v_n$ letters in V, $x = v_1 \ldots v_n$, and s a state in S. The word x is said to be *s-accepted* by $\mathfrak{N}$ if there exist states $s_0, s_1, \ldots, s_n$ in S such that $s_0 \in J$, $s_n = s$, and $s_{i+1} \in f(s_i, v_{i+1})$ for $i = 0, 1, \ldots, n - 1$.

Remark. Another set $F \subseteq S$ appears in the usual definition of an acceptor [14]; a word is said to be *accepted* if it is s-accepted for at least one s in F. Furthermore, finite acceptors are usually dealt with which means that the sets S, V are finite. To our aims, these restrictions have no importance and we omit them.

1. Definition. Let $\mathfrak{N} = (S, V, f, J)$ be an acceptor and suppose that $0 \notin V$.

We put $T = V \cup \{0\}$, $a(0) = 0$, $\omega(0) = J$, $a(v) = 1$, $\omega(v) = \{rs;\ s \in S,\ r \in f(s, v)\}$ for any $v \in V$. Furthermore, we set $A_s = V^*$ for any $s \in S$, $f_0^{\mathfrak{A}} = \Lambda$, $f_v^{\mathfrak{A}}(x) = xv$ for any $v \in V$ and any $x \in V^*$. Then $\mathfrak{A} = ((A_s)_{s \in S}, (f_t^{\mathfrak{A}})_{t \in T}, \omega)$ is a heterogeneous algebra.

We put $\mathfrak{A} = \not{h}(\mathfrak{N})$.

Remark. The operator $\not{h}$ assigns a heterogeneous algebra to any acceptor in a way which is different from the way described in [7], p. 70.

2. Proposition. *Let $\mathfrak{N} = (S, V, f, J)$ be an acceptor, suppose $\not{h}(\mathfrak{N}) = \mathfrak{A} = ((A_s)_{s \in S}, (f_t^{\mathfrak{A}})_{t \in T}, \omega)$. Let $(B_s)_{s \in S}$ be the least closed family in $\mathfrak{A}$.*

If $x \in V^*$ and $s \in S$, then the following assertions are equivalent.

(i) $x \in B_s$.

(ii) *The word x is s-accepted by $\mathfrak{N}$.*

Proof. For any $s \in S$, let C_s be the set of all words s-accepted by $\mathfrak{N}$. The following conditions (1) and (2) are satisfied.

(1) The family $(C_s)_{s \in S}$ is closed in $\mathfrak{A}$.

(2) $C_s \subseteq B_s$ holds for any $s \in S$.

The proof of (1) is immediate, (2) may be easily proved by induction on the length of a word.

By (1), (2), and the minimality of the closed family $(B_s)_{s \in S}$, we obtain $B_s = C_s$ for any $s \in S$ which implies the equivalence of (i) and (ii). □

Remark. This proposition is close to 3.8 and 4.4 of [13], cf. also [11].

3. Definition. Let $\mathfrak{A} = ((A_i)_{i\in I}, (f_t^{\mathfrak{A}})_{t\in T}, \omega)$ be a heterogeneous algebra. It will be called *good* if the following conditions are satisfied.

(i) There exists a set V such that $A_i = V^*$ for any $i \in I$ and $T = V \cup \{0\}$ where $0 \notin V$.

(ii) $a(0) = 0$, $f_0^{\mathfrak{A}} = \Lambda$ and $a(v) = 1$, $f_v^{\mathfrak{A}}(x) = xv$ for any $v \in V$ and any $x \in V^*$.

4. Definition. Let $\mathfrak{A} = ((A_i)_{i\in I}, (f_t^{\mathfrak{A}})_{t\in T}, \omega)$ be a good heterogeneous algebra. We have $T = V \cup \{0\}$ where $0 \notin V$.

Set $J = \omega(0)$, $f(i, v) = \{j \in I; ji \in \omega(v)\}$ for any $i \in I$ and $v \in V$. Finally, we put $\mathfrak{N} = (I, V, f, J)$, $\varphi(\mathfrak{A}) = \mathfrak{N}$.

Clearly, $\varphi(\mathfrak{A})$ is an acceptor for any good heterogeneous algebra $\mathfrak{A}$ and $\not{h}(\mathfrak{N})$ is a good heterogeneous algebra for any acceptor $\mathfrak{N}$.

5. Proposition. *$\varphi(\not{h}(\mathfrak{N})) = \mathfrak{N}$ for any acceptor $\mathfrak{N}$ and $\not{h}(\varphi(\mathfrak{A})) = \mathfrak{A}$ for any good heterogeneous algebra $\mathfrak{A}$.*

The proof follows immediately from 1, 3, and 4. □

Remark. This result is very close to 4.6 of [13].

4. FAMILY OF BLOCKS OF A COMPATIBLE RELATION AS AN EXAMPLE OF A HETEROGENEOUS ALGEBRA

We now give some other applications of heterogeneous algebras. Some more examples can be found in [7]; cf. also [10], [12].

The following notions appear in [2] and [6].

Let A be a set, ϱ a binary relation on A. A subset B of A is said to be a *block* of ϱ if it satisfies the following conditions. (i) $B \neq \emptyset$; (ii) $B \times B \subseteq \varrho$; (iii) $B \subseteq C \subseteq A$ and $C \times C \subseteq \varrho$ imply $B = C$.

Let $\boldsymbol{B} = (B_i)_{i\in I}$ be a family of type I of mutually different nonempty subset of a set A. We set

$$\varrho = \{xy;\ x \in A,\ y \in A, \text{ there exists } i \in I \text{ such that } x \in B_i,\ y \in B_i\}.$$

Then the relation ϱ is said to be ***B**-defined* on A.

Let $\boldsymbol{B} = (B_i)_{i\in I}$ be a family of type I of mutually different nonempty subset of a set A. Then $\boldsymbol{B}$ is said to be a *τ-family* if it satisfies the following conditions.

(a) If $i(0) \in I$ and $I(0) \subseteq I$, then $B_{i(0)} \subseteq \bigcup_{i\in I(0)} B_i$ implies $\bigcap_{i\in I(0)} B_i \subseteq B_{i(0)}$.

(b) If $M \subseteq A$ and $M \nsubseteq B_i$ holds for every $i \in I$, then there exists $D \subseteq M$ with exactly two elements such that $D \nsubseteq B_i$ holds for every $i \in I$.

By a slight modification of Theorem 1 in [6], we obtain

Proposition. *Let $\boldsymbol{B} = (B_i)_{i\in I}$ be an indexed family of mutually different nonempty subsets of a set A. Then the following conditions are equivalent.*

(i) *$\boldsymbol{B}$ is the system of all blocks of the $\boldsymbol{B}$-defined relation.*
(ii) *$\boldsymbol{B}$ is a τ-family.* □

Let $\mathfrak{A} = (A, (f_t^{\mathfrak{A}})_{t\in T})$ be a complete algebra, ϱ a binary relation on $\mathfrak{A}$. Then ϱ is said to be *compatible* with $\mathfrak{A}$ if $t \in T$, $a(t) > 0$, $x_1, \ldots, x_{a(t)}, x'_1, \ldots, x'_{a(t)}$ in A and $x_1x'_1 \in \varrho, \ldots, x_{a(t)}x'_{a(t)} \in \varrho$ imply $f_t^{\mathfrak{A}}(x_1, \ldots, x_{a(t)}) f_t^{\mathfrak{A}}(x'_1, \ldots, x'_{a(t)}) \in \varrho$.

1. Definition. Let (I, T, a, ω) be a heterogeneous algebra type. This type is said to be *full* if $\omega(t)$ is a complete $a(t)$-ary operation on I for any $t \in T$, i.e., if for any $i(1), \ldots, i(a(t))$ in I there exists exactly one $i(0) \in I$ such that $i(0)\, i(1) \ldots i(a(t)) \in \omega(t)$.

The applicability of these notions is demonstrated by the following

2. Proposition. *Let $\mathfrak{A} = (A, (f_t^{\mathfrak{A}})_{t\in T})$ be a complete algebra of type (T, a) and $\boldsymbol{B} = (B_i)_{i\in I}$ an indexed family of mutually different nonempty subsets of A which is a τ-family. Then the following conditions are equivalent.*

(i) *The $\boldsymbol{B}$-defined relation is compatible with $\mathfrak{A}$.*
(ii) *For any $t \in T$ there exists a complete $a(t)$-ary operation $\omega(t)$ on I such that the family $\boldsymbol{B}$ is closed in the heterogeneous algebra on $\mathfrak{A}$ of full type (I, T, a, ω) admissible to (T, a).*

Proof. Clearly, the condition (ii) is satisfied if and only if $\boldsymbol{B}$ is normal in terms of [2]. By Theorem 2 of [2], (i) implies (ii). The proof of the implication (c) ⇒ (b) in [2] includes the proof of the implication (ii) ⇒ (i). □

Remarks. This result is very close to Theorem 2 of [2] and to Theorem 3 of [6] where families of blocks of compatible relations are characterized. Various characterizations of a single block of a compatible relation may be found in [3] and [4]. Some conditions equivalent to the condition that every block of each compatible relation is a subalgebra are formulated in [5].

5. HETEROGENEOUS ALGEBRAS OF FULL TYPES

The definition of a heterogeneous algebra of full type is motivated by Proposition 4.2. We describe some properties of these algebras and prove that some well-known properties of acceptors are included.

First, we assign a full type to any heterogeneous algebra type.

1. Definition. Let $\tau = (I, T, a, \omega)$ be a heterogeneous algebra type.
We put $R = 2^I$. Let $t \in T$ be arbitrary.
If $a(t) = 0$, we set $\Omega(t) = \{\omega(t)\}$.

If $a(t) > 0$, and $i(1), \ldots, i(a(t))$ are arbitrary elements in I, we put

$$h_t(i(1). \ldots, i(a(t))) = \{i(0); i(0) \in I,\ i(0)\, i(1) \ldots i(a(t)) \in \omega(t)\}\ .$$

For arbitrary $r(1), \ldots, r(a(t))$ in R, we set

$$k_t(r(1), \ldots, r(a(t))) = \bigcup h_t(i(1), \ldots, i(a(t)))\,,$$

where the last union relates to all words $i(1) \ldots i(a(t))$ such that $i(1) \in r(1), \ldots, i(a(t)) \in r(a(t))$.

Finally, we put

$$\Omega(t) = \{k_t(r(1), \ldots, r(a(t)))\, r(1) \ldots r(a(t));\ r(1), \ldots, r(a(t)) \in R\}\ .$$

Then $\delta = (R, T, a, \Omega)$ is a full heterogeneous algebra type. We put $\mathscr{D}(\tau) = \delta$.

2. Definition. Let $\mathfrak{H} = (A, (f_t^{\mathfrak{H}})_{t \in T})$ be a complete algebra of type (T, a), $\tau = (I, T, a, \omega)$ a heterogeneous algebra type admissible to (T, a), $\delta = \mathscr{D}(\tau)$. (Clearly, δ is a type admissible to (T, a).) Let $\mathfrak{A}$ be a heterogeneous algebra of type τ on $\mathfrak{H}$, $\mathfrak{B}$ a heterogeneous algebra of type δ on $\mathfrak{H}$. Then we put $\mathfrak{B} = \mathscr{d}(\mathfrak{A})$.

3. Proposition. *Let $\mathfrak{H} = (A, (f_t^{\mathfrak{H}})_{t \in T})$ be a complete algebra, $\mathfrak{A} = ((A_i)_{i \in I}, (f_t^{\mathfrak{A}})_{t \in T}, \omega)$ a heterogeneous algebra on $\mathfrak{H}$, $\mathscr{d}(\mathfrak{A}) = \mathfrak{B} = ((B_r)_{r \in R}, (f_t^{\mathfrak{B}})_{t \in T}, \Omega)$. Then the following assertions hold.*

(i) *If a family $(C_i)_{i \in I}$ is closed in $\mathfrak{A}$, then the family $(D_r)_{r \in R}$ with $D_\emptyset = A$, $D_r = \bigcap_{i \in r} C_i$ for $r \neq \emptyset$, $r \in R$, is closed in $\mathfrak{B}$.*

(ii) *If a family $(D_r)_{r \in R}$ is closed in $\mathfrak{B}$, then the family $(C_i)_{i \in I}$ with $C_i = \bigcup_{i \in r} D_r$ for $i \in I$ is closed in $\mathfrak{A}$.*

Proof. (1) Let $(C_i)_{i \in I}$ be a family closed in $\mathfrak{A}$. Suppose $t \in T$.

If $a(t) = 0$, we have to prove that $f_t^{\mathfrak{H}} \in D_{\omega(t)}$. This is trivial if $\omega(t) = \emptyset$. If $\omega(t) \neq \emptyset$, then $f_t^{\mathfrak{H}} \in C_i$ for any $i \in \omega(t)$ which implies $f_t^{\mathfrak{H}} \in \bigcap_{i \in \omega(t)} C_i = D_{\omega(t)}$.

Let us have $a(t) > 0$, $r(1), \ldots, r(a(t))$ in R, $x_1 \in D_{r(1)}, \ldots, x_{a(t)} \in D_{r(a(t))}$.

The following cases may occur.

(a) There exists j, $1 \leqq j \leqq a(t)$, such that $r(j) = \emptyset$.

(b) $r(j) \neq \emptyset$ for $1 \leqq j \leqq a(t)$ and $k_t(r(1), \ldots, r(a(t))) = \emptyset$.

(c) $r(j) \neq \emptyset$ for $1 \leqq j \leqq a(t)$ and $k_t(r(1), \ldots, r(a(t))) \neq \emptyset$.

In case (a) we have, clearly, $k_t(r(1), \ldots, r(a(t))) = \emptyset$. Thus, in cases (a), (b) we obtain $f_t^{\mathfrak{H}}(x_1, \ldots, x_{a(t)}) \in A = D_\emptyset = D_{k_t(r(1), \ldots, g(a(t)))}$ because the operation $f_t^{\mathfrak{H}}$ is complete.

Suppose that (c) occurs. Let us have an arbitrary i in $k_t(r(1), \ldots, r(a(t)))$. By definition of k_t, there exist $i(1) \in r(1), \ldots, i(a(t)) \in r(a(t))$ such that $i \in h_t(i(1), \ldots \ldots, i(a(t)))$. Since $D_{r(j)} \subseteq C_{i(j)}$, we have $x_j \in C_{i(j)}$ for any j, $1 \leqq j \leqq a(t)$. This implies

$f_t^{\mathfrak{H}}(x_1, \ldots, x_{a(t)}) \in C_i$ for every $i \in k_t(r(1), \ldots, r(a(t)))$ which yields $f_t^{\mathfrak{H}}(x_1, \ldots, x_{a(t)}) \in$ $\in \bigcap_{i \in k_t(r(1),\ldots,r(a(t)))} C_i = D_{k_t(r(1),\ldots,r(a(t)))}$.

We have proved (i).

(2) Let $(D_r)_{r \in R}$ be a family closed in $\mathfrak{B}$. Suppose $t \in T$.

If $a(t) = 0$ and $i(0) \in \omega(t)$, then $\Omega(t) = \{\omega(t)\}$ implies that $f_t^{\mathfrak{H}} \in D_{\omega(t)} \subseteq \bigcup_{i(0) \in r} D_r =$ $= C_{i(0)}$.

Let us have $a(t) > 0$, $i(0)\, i(1) \ldots i(a(t))$ in $\omega(t)$ and $x_1 \in C_{i(1)}, \ldots, x_{a(t)} \in C_{i(a(t))}$. Then $i(0) \in h_t(i(1), \ldots, i(a(t)))$. Since $C_{i(j)} = \bigcup_{i(j) \in r} D_r$, there exists $r(j)$ such that $i(j) \in r(j)$ and $x_j \in D_{r(j)}$ for any j, $1 \leqq j \leqq r$. This implies that $h_t(i(1), \ldots, i(a(t))) \subseteq$ $\subseteq k_t(r(1), \ldots, r(a(t)))$ whence $i(0) \in k_t(r(1), \ldots, r(a(t)))$. Thus $f_t^{\mathfrak{H}}(x_1, \ldots, x_{a(t)}) \in$ $\in D_{k_t(r(1),\ldots,r(a(t)))} \subseteq \bigcup_{i(0) \in r} D_r = C_{i(0)}$.

We have proved (ii). □

4. Corollary. *Let $\mathfrak{H} = (A, (f_t^{\mathfrak{H}})_{t \in T})$ be a complete algebra, $\mathfrak{A} = ((A_i)_{i \in I}, (f_t^{\mathfrak{A}})_{t \in T}, \omega)$ a heterogeneous algebra on $\mathfrak{H}$, $\mathscr{d}(\mathfrak{A}) = \mathfrak{B} = ((B_r)_{r \in R}, (f_t^{\mathfrak{B}})_{t \in T}, \Omega)$. Let $\mathfrak{C} = ((C_i)_{i \in I}, (f_t^{\mathfrak{C}})_{t \in T}, \omega)$ be the least subalgebra of $\mathfrak{A}$, $\mathfrak{D} = ((D_r)_{r \in R}, (f_t^{\mathfrak{D}})_{t \in T}, \Omega)$ the laest subalgebra of $\mathfrak{B}$. Then $C_i = \bigcup_{i \in r} D_r$ for every $i \in I$.*

Proof. Let us put $E_i = \bigcup_{i \in r} D_r$ for every $i \in I$ and $F_r = \bigcap_{i \in r} C_i$ for every $r \in R$, $r \neq \emptyset$, and $F_\emptyset = A$. Then $(E_i)_{i \in I}$ is closed in $\mathfrak{A}$ and $(F_r)_{r \in R}$ in $\mathfrak{B}$ by 3, which implies $C_i \subseteq E_i$ for every $i \in I$ and $D_r \subseteq F_r$ for every $r \in R$. Thus, for every $i \in I$, we obtain $C_i \subseteq E_i = \bigcup_{i \in r} D_r \subseteq \bigcup_{i \in r} F_r = \bigcup_{i \in r} (\bigcap_{j \in r} C_j) = C_i$, which implies the assertion. □

Remark. The algebra $\mathscr{d}(\mathfrak{A})$ may be considered a deterministic version of $\mathfrak{A}$. Indeed, if $\mathfrak{N} = (S, V, f, J)$ is an acceptor, its deterministic version is $\mathfrak{D} = (2^S, V, g, \{J\})$ where $g(r, v) = \{ \bigcup_{s \in r} f(s, v)\}$ for any $r \in 2^S$ and any $v \in V$ (cf. [14]). We put $\mathfrak{D} = \mathscr{n}(\mathfrak{N})$.

5. Proposition. $\mathscr{n}(\mathfrak{N}) = \mathscr{q}(\mathscr{d}(\mathscr{p}(\mathfrak{N})))$ *for any acceptor* $\mathfrak{N}$.

The proof follows immediately from 3.1, 2, 3.3, 3.4. □

Thus, in the terminology of heterogeneous algebras, the operator $\mathscr{d}$ means transition to the deterministic version.

Remark. The results 3 and 4 are very close to 6.5 of [9].

Also the well-known equality of languages accepted by $\mathfrak{N}$ and $\mathscr{n}(\mathfrak{N})$ (cf. [14]) reflects in 4 and can be derived as a consequence of 4.

6. Example. Let us have $V = \{a, b\}$, $A = V^*$; $T = \{1, 2, 3, 4\}$, $a(1) = a(2) = 0$, $a(3) = a(4) = 1$, $f_1^{\mathfrak{H}} = \Lambda$, $f_2^{\mathfrak{H}} = b$, $f_3^{\mathfrak{H}}(x) = axb$, $f_4^{\mathfrak{H}}(x) = axa$ for any $x \in V^*$. Then $\mathfrak{H} = (A, (f_t^{\mathfrak{H}})_{t \in T})$ is a complete algebra of type (T, a).

We set $I = \{s, u\}$, $A_s = A_u = V^*$, $\omega(1) = \{s\}$, $\omega(2) = \{u\}$, $\omega(3) = \{ss\}$, $\omega(4) = \{su, uu\}$, $f_t^{\mathfrak{A}} = f_t^{\mathfrak{H}}$ for any $t \in T$, $\mathfrak{A} = ((A_i)_{i \in I}, (f_t^{\mathfrak{A}})_{t \in T}, \omega)$. Then $\mathfrak{A}$ is a heterogeneous algebra on $\mathfrak{H}$ of type (I, T, a, ω) admissible to (T, a).

By 1, we obtain $R = 2^I = \{\emptyset, \{s\}, \{u\}, I\}$; we set $O = \emptyset$, $S = \{s\}$, $U = \{u\}$. Further, we have $\Omega(1) = \{S\}$, $\Omega(2) = \{U\}$. Moreover, $h_3(s) = \{s\}$, $h_3(u) = \emptyset$ which implies $k_3(O) = O$, $k_3(S) = S$, $k_3(U) = O$, $k_3(I) = S$. Similarly, $h_4(s) = \emptyset$, $h_4(u) = \{s, u\}$ which entails $k_4(O) = O$, $k_4(S) = O$, $k_4(U) = I$, $k_4(I) = I$. Thus, $\Omega(3) = \{OO, SS, OU, SI\}$, $\Omega(4) = \{OO, OS, IU, II\}$. Putting $B_O = B_S = B_U = B_I = V^*$, $f_t^{\mathfrak{B}} = f_t^{\mathfrak{H}}$ for any $t \in T$, and $\mathfrak{B} = ((B_r)_{r \in R}, (f_t^{\mathfrak{B}})_{t \in T}, \Omega)$, we have $\mathfrak{B} = \mathscr{A}(\mathfrak{A})$.

Let $(C_i)_{i \in I}$ be the least closed family in $\mathfrak{A}$, $(D_r)_{r \in R}$ the least closed family in $\mathfrak{B}$. The components C_i, D_r can be constructed by using a slight generalization of 4.4 in [13]. Proposition 4.4 of [13] describes the components of the least subalgebra of a so called context-free algebra as sets of terminal words generated from non-terminal symbols of a generalized grammar with context-free productions. Generalized grammars corresponding to $\mathfrak{A}, \mathfrak{B}$ are (V, I, P) and (V, R, Q), respectively, where $P = \{(s, \Lambda), (u, b), (s, asb), (s, aua), (u, aua)\}$ and $Q = \{(S, \Lambda), (U, b), (O, aOb), (S, aSb), (O, aUb), (S, aIb), (O, aOa), (O, aSa), (I, aUa), (I, aIa)\}$. Then C_i is the set of words over V generated from $i \in I$ by means of the first grammar and D_r is the set of words generated from $r \in R$ by means of the second grammar. Clearly, $C_s = \{a^m b^m;\ m \geqq 0\} \cup \{a^{m+n} b a^n b^m;\ m \geqq 0,\ n \geqq 1\}$, $C_u = \{a^m b a^m;\ m \geqq 0\}$, $D_S = \{a^m b^m;\ m \geqq 0\} \cup \{a^{m+n} b a^n b^m;\ m \geqq 1,\ n \geqq 1\}$, $D_U = \{b\}$, $D_I = \{a^m b a^m;\ m \geqq 1\}$.

Clearly, $C_s = D_S \cup D_I$, $C_u = D_U \cup D_I$ which illustrates 4. It is easy to see that $D_r = \bigcap_{i \in r} C_i$ generally does not hold for $r \neq \emptyset$. We have $D_S \neq C_s = \bigcap_{i \in S} C_i$, $D_U \neq C_u = \bigcap_{i \in U} C_i$, $D_I = C_s \cap C_u = \bigcap_{i \in I} C_i$.

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